This whole homework is about Ward–Takahashi identities in supersymmetric QED (superfield formulation). It has 6 problems, or rather 5 problems and one reading assignment.

Update: Problem 6 is postponed to the next homework set#5.

1. Let's start with a simple exercise about the electric current superfield

$$J \;=\; e\overline{A}\exp(+2eV)V \;-\; e\overline{B}\exp(-2eV)B. \tag{1}$$

   (a) Derive the classical equation of motion for the electron superfields $A, \overline{A}, B, \overline{B}$ and show that they lead to the current conservation equations

$$D^2 J \;=\; 0, \quad \overline{D}^2 J \;=\; 0. \tag{2}$$

   (b) Focus on the single-θ, single-$\bar{\theta}$ component of the current superfield,

$$J(x,\theta,\bar{\theta}) \;=\; (\theta\sigma^\mu\bar{\theta}) \times j_\mu(x) \;+\; \text{other components} \tag{3}$$

   and show that $j_\mu(x)$ is the ordinary electric current.

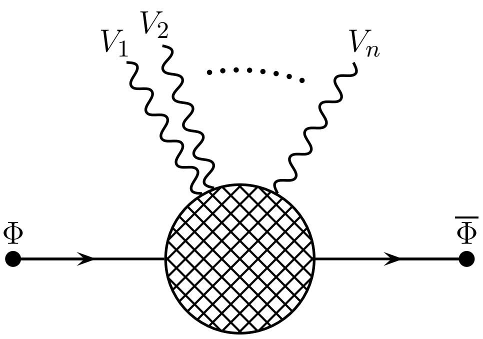
   (c) Verify that eqs. (2) include the ordinary current conservation, $\partial_\mu j^\mu = 0$.

2. Next, a reading assignment: my notes on the diagrammatic proof of Ward–Takahashi identities in the ordinary QED.

The rest of this homework set deals with the actual SQED Ward–Takahashi identities in the superfield form. For simplicity, take the electron field to be massless, hence no AB or $\overline{BA}$ propagators.

3. Let's start with the amplitudes $\mathcal{S}_n$ involving a charged chiral superfield $\Phi = A$ or B, its conjugate $\overline{\Phi} = \overline{A}$ or $\overline{B}$, and n vector superfields $V_1, \ldots, V_n$. Diagrammatically

$$(\pm 2e)^n \mathcal{S}_n(V_1, \ldots, V_n) \;=\; \tag{4}$$



To be precise, these amplitudes are amputated with respect to the vector fields $V_1, \ldots, V_n$ but not the charged fields Φ and $\overline{\Phi}$; in other words, they include the external lines for the Φ and $\overline{\Phi}$ but not for the vectors. On the other hand, these amplitudes include the external vector fields V_i themselves but not the external Φ or $\overline{\Phi}$, and there is no overall $\int d^4\theta$, just the operator between the Φ and the $\overline{\Phi}$,

$$\text{amplitude} \;=\; (\pm 2\mathrm{e})^{\mathrm{n}} \int \mathrm{d}^4\theta\, \Phi \times \boxed{\mathcal{S}_n(V_1, \ldots, V_n)} \times \overline{\Phi} \tag{5}$$

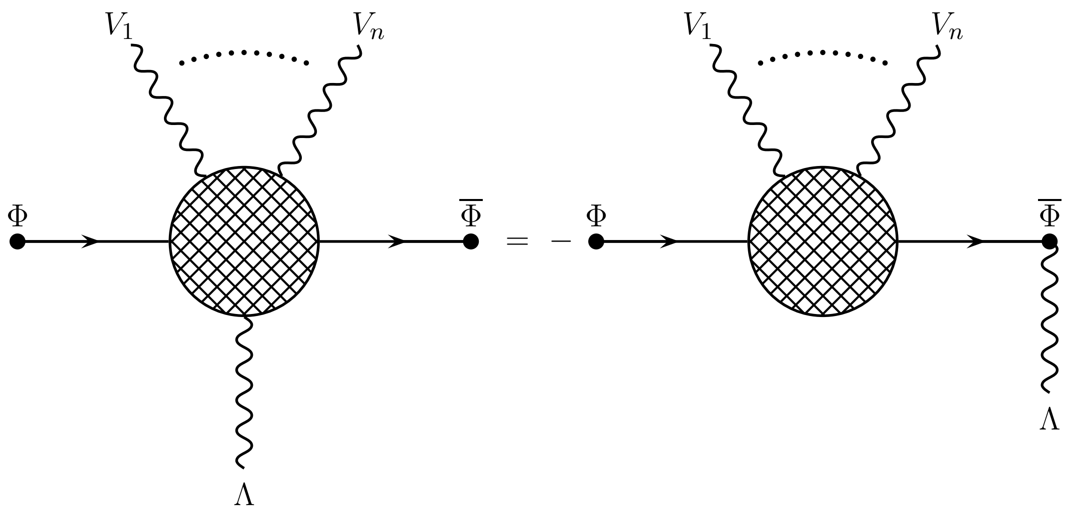
This part only!

Eq. (23) of my notes on WT identities in SQED has tree-level examples of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_2$.

The Ward–Takahashi identities for this class of amplitudes says that **if** any of the vector fields happen to be chiral or antichiral, $V_i = \Lambda(y, \theta)$ or $V_i = \overline{\Lambda}(\bar{y}, \bar{\theta})$, **then**

$$\begin{aligned} \mathcal{S}_{n+1}(V_1, \ldots, V_i = \Lambda, \ldots, V_{n+1}) \;&=\; -\mathcal{S}_n(V_1, \ldots \cancel{V_i} \ldots, V_{n+1}) \times \Lambda, \\ \mathcal{S}_{n+1}(V_1, \ldots, V_i = \overline{\Lambda}, \ldots, V_{n+1}) \;&=\; -\overline{\Lambda} \times \mathcal{S}_n(V_1, \ldots \cancel{V_i} \ldots, V_{n+1}), \end{aligned} \tag{6}$$

or graphically (suppressing powers of $\pm 2e$)



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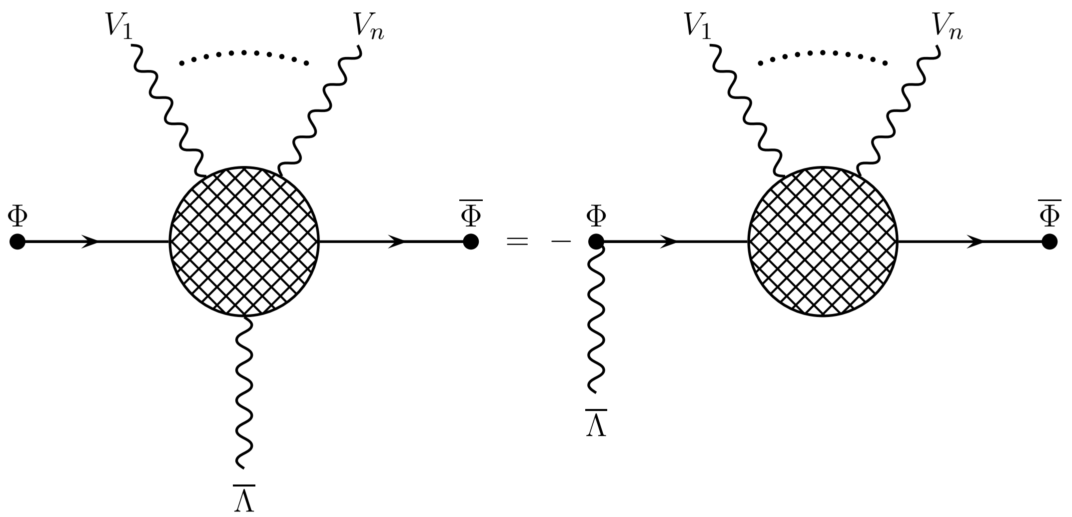
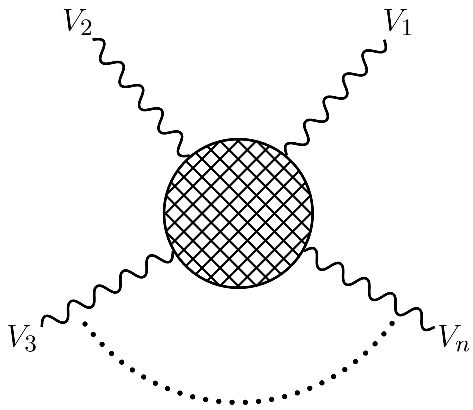


(8)

Your task in this problem is to prove the relations (6) at the tree level.

Note: this does not work diagram-by-diagram. Instead, you have to sum over all the places the $(n+1)^{\text{st}}$ "photon" $V_{n+1} = \Lambda$ or $V_{n+1} = \overline{\Lambda}$ can be inserted into an amplitude that already has n other photons.

4. Next, consider the n–vector amputated amplitudes without any external Φ or $\overline{\Phi}$ lines,



$$= \; i(2q)^n \int d^4\theta \, \mathcal{V}_n(V_1,\ldots,V_n). \tag{9}$$

A very important Ward–Takahashi identity says that all these amplitudes vanish when any one of the vectors V_i is chiral or antichiral,

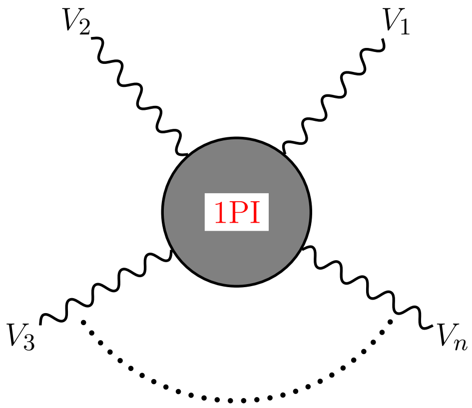
$$\int d^4\theta \, \mathcal{V}(V_1,\ldots,V_n) \;=\; 0 \quad \text{when any } V_i = \Lambda \text{ or } V_i = \overline{\Lambda} \tag{10}$$

Your task in this problem is to prove this identity at the one-loop level.

Note: the proof involves cancellations between diagrams where that bad vector $V_n = \Lambda$ or $V_n = \overline{\Lambda}$ is inserted into different places in the charged loop relative to the other $n-1$ vectors. *Assume that all the loop-momentum integrals either converge or else may be regulated in a way that does not affect the vertices or the chiral propagators.* This assumption allows us to cancel diagrams graphically without worrying about shifting the loop momenta $q^\mu \to q^\mu + p^\mu$ in divergent $\int d^4q$ integrals.

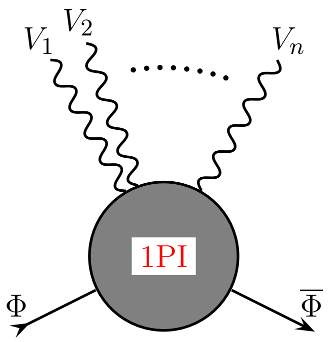
5. Now use the results of problems (3) and (4) as lemmas to prove that the WT relations (6) and (10) hold true to all orders of the perturbation theory.

6. Finally, let's re-express the WT relations (6) and (10) in terms of completely amputated and one-particle irreducible amplitudes



$$= \; i(\pm 2e)^n \int d^4\theta \, \mathcal{V}_n^{1\mathrm{PI}}(V_1, \ldots, V_n) \tag{11}$$

and



$$= \; i(\pm 2e)^n \int d^4\theta \, \Phi \, \Gamma_n(V_1, \ldots, V_n) \, \overline{\Phi} \tag{12}$$

(a) Use eqs. (10) for the non-1PI amplitudes to show that when any one of the vector superfields V_i happens to be chiral or antichiral, all the 1PI amplitudes (11) must vanish.

As we saw in class on 9/11, it is this family of WT identities which makes SQED a renormalizable theory.

(b) Likewise, use eqs. (6) to show that when any of the vector superfields — say, V_n — happens to be chiral or antichiral, the 1PI amplitudes (12) must obey recursive relations:

for $n > 1$,

$$\Gamma_1(V = \Lambda) \; = \; \Lambda \times (1 + \Gamma_0), \qquad \Gamma_1(V = \overline{\Lambda}) \; = \; (1 + \Gamma_0) \times \overline{\Lambda}, \tag{13}$$

while for $n > 1$

$$\begin{aligned} \Gamma_n(V_1, \ldots, V_{n-1}, V_n = \Lambda) &= \Lambda \times \Gamma_{n-1}(V_1, \ldots, V_{n-1}), \\ \Gamma_n(V_1, \ldots, V_{n-1}, V_n = \overline{\Lambda}) &= \Gamma_{n-1}(V_1, \ldots, V_{n-1}) \times \overline{\Lambda}. \end{aligned} \tag{14}$$

Note that the $1+\Gamma_0$ combination in eq. (13) is related to the *dressed* chiral propagator

$$\equiv \mathcal{S}_0 = \frac{1}{1+\Gamma_0(p)} \times \frac{iD^2\overline{D}^2}{16p^2}\,. \tag{15}$$

As we saw in class on 9/11, these relations lead to the SQED Ward identities

$$\text{all } \delta_1^{(n)} = \delta_2\,. \tag{16}$$