1. Back in 1996, Aharoni, Sonnenschein, Theisen, and Yankielowitz arXiv:hep-th/9611222 found a curious $SU(2) \times SU(2)$ SUSY gauge theory 'living' on a D3–brane probe located near an intersection of two orientifold planes. In this problem, we study the renormalization group flow and the IR fixed points of this gauge theory from a purely 4D QFT point of view.

   Ignoring the 'stringy' degrees of freedom, we have a $d = 4$, $\mathcal{N} = 1$ SUSY gauge theory with $G = SU(2) \times SU(2)$ and the following multiplets of chiral superfields:

$$A_1, A_2 \in (\mathbf{2}, \mathbf{2}), \qquad B_1, \ldots, B_8 \in (\mathbf{2}, \mathbf{1}), \qquad C_1, \ldots, C_8 \in (\mathbf{1}, \mathbf{2}). \tag{1}$$

   Note: in my notations I keep explicit flavor indices but suppress the gauge indices, so the net number of chiral superfields in this theory is

$$\big(2 \times 2 \times 2\big)_A \;+\; \big(8 \times 2 \times 1\big)_B \;+ \big(8 \times 1 \times 2\big)_C \;=\; 8 + 16 + 16 \;=\; 40.$$

   The theory has a superpotential

$$W \;=\; \lambda \sum_{i=1}^{4} B_i A_1 C_i \;+\; \lambda \sum_{i=5}^{8} B_i A_2 C_i\,. \tag{2}$$

   Again, the gauge indices are suppressed in this formula, but for each term here there is only one gauge-invariant way of contracting the gauge indices.

   (a) List global symmetries of this model and show that it has only 3 independent anomalous dimensions: γ_A (same for A_1 and A_2), γ_B (same for all B_i), and γ_C (same for all C_i). Allow for different gauge couplings $g_1 \neq g_2$ of the two $SU(2)$ factors.

   (b) Calculate the exact β_λ, β_{g_1}, and β_{g_2} in terms of the anomalous dimensions and show that all three beta-functions vanish when $\gamma_B = \gamma_C = -\frac{1}{2}\gamma_A$. Argue that this leads to a line of fixed points in the (λ, g_1, g_2) coupling space.

(c) Calculate the anomalous dimensions γ_A, γ_B, and γ_C to one-loop order and show that the fixed line lies at

$$g_1^2 \;=\; g_2^2 \;=\; \frac{16}{12}\,\lambda^2 \;+\; O(\lambda^4). \tag{3}$$

(d) Show that this fixed line is IR-attractive. That is, if we start with some other couplings in the UV and let the RG run to lower energies, then in the IR limit the couplings will end somewhere on the fixed line (3).

(e) Any IR-attractive fixed point gives rise to an SCFT (super-conformal field theory), and a line (surface, *etc.*) of such fixed points makes a whole family of non-trivial SCFTs. Argue that for the model in question, this family of SCFTs includes both weakly-coupled and strongly-coupled theories.

2. Next consider the RG flows across massive particle thresholds in SQCD with several flavors. Let's start with the regime where all flavors are much heavier than Λ_{SQCD}. In class I have argued that for this regime, the effective low-energy SYM theory has

$$\Lambda_{\text{SYM}}^{3N_c} \;=\; \Lambda_{\text{SQCD}}^{3N_c-N_f} \times \det(m) \times \binom{\text{a numeric}}{\text{constant}}. \tag{4}$$

(a) Your first task in this problem is to verify eq. (4) for the magnitude $|\Lambda_{\text{SYM}}|$. For simplicity, suppose that the matrices m, Z_A, and Z_B are diagonal and that the physical quark masses

$$M_f \;=\; \frac{|m_f|}{\sqrt{Z_f^A Z_f^B}} \tag{5}$$

have hierarchically different values, say $M_1 \gg M_2 \gg \cdots \gg M_{N_f}$. In this case, the RG flow has N_f well-separated thresholds.

(b) Now consider SQCD that has both heavy and light flavors. Let's integrate out the heavy flavors only, so the low-energy effective theory includes both the gauge fields and the light flavors. Run the RG flow to get the $|\Lambda_{\text{low}}|$ for this effective theory, then extend it to a holomorphic formula for the Λ_{low} in terms of the Λ_{high} and the masses of the heavy quarks.

(c) Optional exercise:

Generalize from SQCD to a SUSY gauge theory with any kind of a simple gauge group G and "quarks" and "antiquarks" in some generic multiplets $R_1 + R_2 + \cdots$ of G. Suppose that some of these "quarks" or "antiquarks" are heavy so we may integrate them out from the low-energy effective theory.

Show that the resulting low energy theory has

$$-b_{\text{low}} \times \log \Lambda_{\text{low}} \;=\; -b^{\text{high}} \times \Lambda_{\text{high}} \;+\; \sum_i \text{Index}(R_i) \times \log m_i\,, \tag{6}$$

where the sum is over the heavy multiplets only, and b_{high} and b_{low} are the one-loop beta-function coefficient of the respective high-energy and low-energy gauge theories.

Now consider the Higgs regime of SQCD with $N_f \leq N_c - 2$ light flavors. As I explained in class, in this regime we have large semi-classical squark VEVs parametrized by the 'mesonic' moduli $\langle \mathcal{M}_{ff'} \rangle = \langle B_f A_{f'} \rangle$. The effective low energy is the $SU(N_c - N_g)$ SYM coupled to these moduli, and its gauge coupling corresponds to

$$\Lambda_{\text{SYM}}^{3(N_c-N_f)} \;=\; \frac{\Lambda_{\text{SQCD}}^{3N_c-N_f}}{\det(\mathcal{M})} \times \begin{pmatrix}\text{a numeric}\\ \text{constant}\end{pmatrix}. \tag{7}$$

(d) Verify this formula for the magnitude $|\Lambda_{\text{SYM}}|$ by running the RG flow across all the massive vector thresholds. For simplicity, assume diagonal matrices $\langle A\rangle$ and $\langle B\rangle$ of squark VEVs (in some gauge) with hierarchically different eigenvalues,

$$\left\langle Q^{fc} \right\rangle \;=\; \delta^{fc} \times \phi_f \quad \text{and} \quad \left\langle \tilde{Q}_{fc} \right\rangle \;=\; \delta_{fc} \times \phi_f\,, \qquad \phi_1 \gg \phi_2 \gg \cdots \gg \phi_{N_f}\,, \tag{8}$$

so that the RG flow has well-separated thresholds.

(e) Optional exercise:

Generalize eq. (7) to any SUSY gauge theory G Higgsed down to a subgroup G' by VEVs of chiral superfields belonging to any multiplets of G.

In general, the massive vector superfields in $G - G'$ form several multiplets of G';

let's label such multiplets by v and let $\mathrm{Index}'(v)$ denote the index of such a multiplet WRT the unbroken G'. Show that

$$-b_{\mathrm{low}}\times\log\Lambda_{\mathrm{low}} \;=\; -b^{\mathrm{high}}\times\Lambda_{\mathrm{high}}\;-\;2\sum_{V}\mathrm{Index}'(v)\times\log\langle H_v\rangle\;+\;\text{a numeric constant} \tag{9}$$

where $\langle H_v\rangle$ is the VEV of the Higgs field that gives the vector fields in v their masses.

3. Finally, in lieu of the third problem, finish reading the 1982 Witten's paper *Constraints On Supersymmetry Breaking* that I have assigned last week.

   Skim over section 9 of the paper — the group theory there is hard to follow for non-experts. More importantly, the main result of section 9 is not quite right, as Witten himself had clarified in his 2000 paper arXiv:hep-th/0006010. Specifically, a pure super–Yang–Mills theory with gauge group G has Witten's index $I = C(G)$ — the Casimir of the adjoint multiplet — rather than $I = \mathrm{rank}(G) + 1$. For the $SU(N)$ and $Sp(N)$ groups both formulae give the same answer, but for the $SO(N)$ and the exceptional groups there is a difference.