1. First, an exercise in off-shell chiral rings. Consider a $U(N)$ gauge theory with a single adjoint multiplet Φ of chiral fields; there are no quarks or antiquarks or any other superfields besides the $\Phi(y,\theta)$, $\overline{\Phi}(\bar{y},\bar{\theta})$, and $V(x,\theta,\bar{\theta})$.

The chiral ring of this theory is made out of gauge-invariant combinations of the scalar and gaugino fields. In matrix notations, all such gauge invariant combinations are traces — or products of traces — of matrix products of the ϕ and λ_α matrices,

$$T_k \;=\; \mathrm{tr}\big(\phi^k\big),\quad P_k^\alpha \;=\; \mathrm{tr}\big(\phi^k\lambda^\alpha\big),\quad R_{k,\ell}^{\alpha\beta} \;=\; \mathrm{tr}\big(\phi^k\lambda^\alpha\phi^\ell\lambda^\beta\big),\quad etc., \tag{1}$$

or in other words, the traces (1) *generate* the chiral ring.

Actually, many of the traces (1) are equivalent to each other as members of the chiral ring because their difference equals to an (anti)commutator of the supercharge $\overline{Q}^{\dot\alpha}$ with some gauge-invariant field or field product. Your task is to show that the only independent generators are the T_k, P_k^α, and $R_k = R_{k,0}^{12}$, and there are no others.

(a) Show that for any matrix product X of ϕ and λ^α matrices,

$$\big[\overline{Q}^{\dot\alpha},\mathrm{tr}(X\mathcal{D}_\mu\phi)\big\} \;=\; \pm i\bar{\sigma}_\mu^{\dot\alpha\beta}\times\mathrm{tr}\big(X[\lambda_\beta,\phi]\big) \tag{2}$$

where $\mathcal{D}_\mu$ is the gauge-covariant derivative,

$$\mathcal{D}_\mu\phi(x) \;=\; \partial_\mu\phi(x) \;+\; i\mathcal{A}_\mu(x)\phi(x) \;-\; i\phi(x)\mathcal{A}_\mu(x). \tag{3}$$

Since $\mathrm{tr}(X\mathcal{D}_\mu\phi)$ is a gauge-invariant local operator, eq. (2) means that in the chiral ring

$$\mathrm{tr}\big(X[\lambda_\beta,\phi]\big) \;\overset{\text{c.r.}}{\equiv}\; 0 \quad\longrightarrow\quad \mathrm{tr}\big(X\lambda^\beta\phi\big) \;\overset{\text{c.r.}}{\equiv}\; \mathrm{tr}\big(X\phi\lambda^\beta\big). \tag{4}$$

This allows us to re-order the ϕ and the λ^α matrices in the traces (1), thus

$$\mathrm{tr}\big(\phi\cdots\phi\lambda^\alpha\phi\cdots\phi\lambda^\beta\cdots\text{more matrices}\cdots\lambda^\gamma\big) \;\overset{\text{c.r.}}{\equiv}\; \mathrm{tr}\big(\phi\cdots\phi\times\lambda^\alpha\lambda^\beta\cdots\lambda^\gamma\big). \tag{5}$$

In particular, $R_{k,\ell}^{\alpha\beta} \overset{\text{c.r.}}{\equiv} R_{k+\ell,0}^{\alpha\beta}$.

(b) Now show that

$$\left[\overline{Q}^{\dot\alpha}\,\mathrm{tr}(X\mathcal{D}_\mu\lambda_\gamma)\right\} \;=\; \pm i\bar\sigma_\mu^{\dot\alpha\beta}\times\mathrm{tr}\big(X\{\lambda_\beta,\lambda_\gamma\}\big) \qquad (6)$$

and hence

$$\mathrm{tr}(X\lambda_\beta\lambda_\gamma) \;\overset{\text{c.r.}}{\equiv}\; -\,\mathrm{tr}(X\lambda_\gamma\lambda_\beta). \qquad (7)$$

In particular,

$$\mathrm{tr}\big(\phi^{k+\ell}\lambda^\beta\lambda^\gamma\big) \;\overset{\text{c.r.}}{\equiv}\; \tfrac{1}{2}\epsilon^{\beta\gamma}\times\mathrm{tr}\big(\phi^{k+\ell}\lambda^\alpha\lambda_\alpha\big), \qquad (8)$$

thus

$$R_{k,\ell}^{\alpha\beta} \;\overset{\text{c.r.}}{\equiv}\; \tfrac{1}{2}\epsilon^{\beta\gamma}\times R_{k+\ell}\,. \qquad (9)$$

(c) Finally, show that all traces including 3 or more gaugino fields λ^α are equivalent to zero,

$$\mathrm{tr}\big(\phi^k\lambda^\alpha\lambda^\beta\lambda^\gamma\cdots\big) \;\overset{\text{c.r.}}{\equiv}\; 0. \qquad (10)$$

Thus, the only independent generators of the chiral ring are the

$$T_k \;=\; \mathrm{tr}\big(\phi^k\big),\quad P_k^\alpha \;=\; \mathrm{tr}\big(\phi^k\lambda^\alpha\big),\quad \text{and}\quad R_k \;=\; \mathrm{tr}\big(\phi^k\lambda^\alpha\lambda_\alpha\big). \qquad (11)$$

Moreover, for finite N, the traces involving more then N ϕ matrices are polynomial functions of the traces with fewer ϕ's. For example, for $N=2$

$$\begin{aligned} 2T_3 &= 3T_2T_1 - T_1^3, \\ 2P_3^\alpha &= P_2^\alpha\times T_1 + P_1^\alpha\times T_2 - P_0^\alpha\times(T_1^2-T_2)T_1\,, \\ 2R_3 &= R_2\times T_1 + R_1\times T_2 - R_0\times(T_1^1-T_2)T_1\,, \end{aligned} \qquad (12)$$

and similar (but more complicated) relations for $k=4,5,\ldots$. In general, such relations are corrected by the instanton effects, but that goes beyond the scope of this exercise.

- ⋆ Optional reading assignment:

  If you are interested in chiral rings and chiral-ring equations, read arXiv:hep-th/0211170, *Chiral Rings and Anomalies in Supersymmetric Gauge Theory* by Freddy Cachazo, Michael Douglas, Nathan Seiberg, and Edward Witten.

  Note: this paper covers many related subjects, and the math is not easy to follow. So take your time, don't try to finish the whole paper this week. Meanwhile, make sure to do the rest of this homework on time!

2. And now a required reading assignment: Philip Argyres's 2001 lecture notes, sections §3.3–5 about SQCD with $N_f \geq N_c$, conformal and superconformal symmetry, and Seiberg duality.

   Focus on §3.4 about the conformal and superconformal symmetries and their representations, as I am going to explain these subject this week. But read the other 2 sections as well, since I shall start explaining the Seiberg duality as soon as I explain the superconformal field theories.

3. Finally, an exercise in conformal symmetry and radial quantization. I shall explain the radial quantization and the state-operator correspondence in CFT on Thursday 10/16, so you can do this problem after the Thursday lecture. Meanwhile, do the rest of this homework set!

   Suppose a CFT (supersymmetric or otherwise) has an abelian gauge symmetry and let $|F^{\mu\nu}\rangle$ be the primary state (in the Hilbert space on the S^3) corresponding to the $F^{\mu\nu}$ operator at $x = 0$. Consider the descendant states

$$|J^\nu_{\text{el}}\rangle \;=\; P_\mu\,|F^{\mu\nu}\rangle \quad \text{and} \quad \left|J^\nu_{\text{mag}}\right\rangle \;=\; P_\mu\left|\tilde{F}^{\mu\nu}\right\rangle \;=\; \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}P_\mu\,|F_{\kappa\lambda}\rangle\,. \tag{13}$$

   Use $P^\dagger_\mu = K_\mu$ (in radial quantization) and the conformal algebra to calculate the Hilbert norms $\left\langle J^\nu_{\text{el}}|J^\nu_{\text{el}}\right\rangle$ and $\left\langle J^\nu_{\text{mag}}|J^\nu_{\text{mag}}\right\rangle$ of these states in terms of the scaling dimension Δ of the $F^{\mu\nu}$ field. Show that both norms are zero if $\Delta = 2$ and both are positive if $\Delta > 2$.

   Use this result to argue that either the abelian gauge field is free (in the IR limit where the theory is conformal) or else it must couple to both electric and magnetic charges.