ES/1

Kähler Geometry.

start with a real Riemann manifold, dim = 2n.

$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \qquad \mu,\nu = 1,\dots,2n$

@ each x, want to make tangent space into complex

split v^μ into w^i & $\bar{w}^{\bar{i}}$ $\quad i=1,\dots,n$

→ Almost complex structure

tensor $I^\mu{}_\nu \qquad I^\mu{}_\nu I^\nu{}_\lambda = -\delta^\mu_\lambda$

as a matrix $I^2 = -1$

eigenvalues $\pm i$

eigenspaces w^i ~~for a~~ $(Iw)^\flat = +i w^\flat$

$\bar{w}^{\bar{i}} \qquad (I\bar{w})^\flat = -i\bar{w}^\flat$

metric $g_{\mu\nu}$ must be hermitean wrt I

~~$(Iw_1)\cdot$~~ $(Iv_1)\cdot(Iv_2) = v_1\cdot v_2$

$\Rightarrow g_{\mu\nu} I^\mu{}_\alpha I^\nu{}_\beta = g_{\alpha\beta}$

consequence: $I_{\mu\nu} = g_{\mu\alpha} I^\alpha{}_\nu = -I_{\nu\mu}$

→ Kähler 2-form $\omega = I_{\mu\nu}\, dx^\mu \wedge dx^\nu$.

E-5/2

Theorem: if $d\omega = 0$

Then I is invariant under parallel transport

$$\nabla_\alpha I^{\mu\beta}{}_{\gamma} = 0.$$

→ I ~~conc~~ is complex structure.

∃ al~~m~~ost complex structure

→ ∃ global complex coordinates

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or rather ∃ atlas of coordinate patches such that in every patch

$z^1, \dots z^n$ + cc $\bar{z}^1 \dots \bar{z}^n$

+ in these coordinates

$$I^i{}_k = +i \quad I^{\bar{i}}{}_{\bar{k}} = -i$$

$$I^i{}_{\bar{k}} = I^{\bar{i}}{}_k = 0.$$

metric is hermitian

$$ds^2 = g_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}}$$

$$g_{ij} = g_{\bar{i}\bar{j}} = 0.$$

most importantly: on every overlap of 2 coord patches

$z^i_{(1)}$, ~~$z_{(2)}$~~ $z^i_{(2)}$

The $z^i_{(2)}$ are holomorphic functions of the $z^i_{(1)}$ + vice versa

~~Ca~~ Example: Complex sphere $\mathbb{C}^*$

stereographic projection

~~ma~~ coordinates z for everything besides the S pole.

S pole $z \to \infty$.

$$ds^2 = \frac{1}{(1+|z|^2)^2} dz\, d\bar{z}$$

Another coord patch: everything besides the N pole.

similar stereographic projection.

$$ds^2 = \frac{1}{(1+|w|^2)^2} dw\, d\bar{w}$$

between the poles $w = \frac{1}{z}$

holomorphic

Kähler Theorem: If $\exists$ complex structure compatible with metric.

Then $\frac{\partial}{\partial z^i} g_{j\bar{k}} = \frac{\partial}{\partial z^j} g_{i\bar{k}}$

$$\frac{\partial}{\partial \bar{z}^k} g_{j\bar{l}} = \frac{\partial}{\partial \bar{z}^l} g_{j\bar{k}}$$

$\Rightarrow$ Locally, within a coord patch

$$g_{i\bar{j}}(z,\bar{z}) = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j} K(z,\bar{z})$$

for some real analytic $K(z,\bar{z})$.

E5/4

Diff coord patches may have diff. Kähler functions $K_1 \neq K_2$.

But they are related Kähler transform. in the overlap of 2 patches

$$K_2 - K_1 = \underbrace{h(z) + h^*(\bar{z})}_{\text{Harmonic function}}$$

$h(z)$: holomorphic.

$$g_{(1)i\bar{j}} = \frac{\partial^2}{\partial z_1^i \partial \bar{z}_1^{\bar{j}}} (K_1) = \frac{\partial^2}{\partial z_1^i \partial \bar{z}_1^{\bar{j}}} K_2$$

def'e for $g_{(2)i\bar{j}}$

$$g^{(2)}_{i\bar{j}} = \left(\frac{\partial z_1}{\partial z_2}\right)_i^{\ k} \left(\frac{\partial \bar{z}_1}{\partial \bar{z}_2}\right)_{\bar{j}}^{\ \bar{l}} g^{(1)}_{k\bar{l}}$$

↳ automatic

Example: Complex sphere

$$K_1 = \log(1+|z|^2).$$

$$\frac{\partial K}{\partial z} = \frac{\bar{z}}{1+|z|^2}, \quad \frac{\partial^2 K}{\partial z \partial \bar{z}} = \frac{1}{1+|z|^2} - \frac{z\bar{z}}{(\ )^2} = \frac{1}{(1+|z|^2)^2}$$

$$g^{(1)}_{z\bar{z}} = \frac{1}{(1+|z|^2)^2}$$

$$w = \frac{1}{z} \qquad K_2 = \log(1+|w|^2)$$

$$K_2 = \log\left(\frac{|z|^2+1}{|z|^2}\right) = \log(1+|z|^2) - \log z - \log \bar{z}$$

$$= K_1 + h(z) + h^*(\bar{z}).$$

E-5/6

Riemann Tensor.

$\{ijk\} = \{\bar{i}\bar{j}\bar{k}\} = 0.$

$\{ij\bar{k}\} = \frac{1}{2}(\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}} + \underbrace{\partial_{\bar{k}} g_{ij}}_{0}) = 0,$

$\{i\bar{j}\bar{k}\} = 0.$

$\to$ only non-zero $\{ijk\} \neq 0$, $\{\bar{i}\bar{j}\bar{k}\} \neq 0$

$\to$ only Christoffels $\Gamma_i{}^k{}_k$, $\Gamma_{\bar{i}}{}^{\bar{i}}{}_{\bar{k}}$

$$\boxed{R_{[i\bar{j}][k\bar{e}]} = \partial_i \partial_{\bar{j}} \partial_k \partial_{\bar{e}} K - (\partial_i \partial_k \partial_{\bar{m}} K) \cdot g^{\bar{m}n} (\partial_n \partial_{\bar{j}} \partial_{\bar{e}} K)}$$

$\boxed{\text{other } R_{\mu\nu,\sigma\rho} = 0}$

$R_{i\bar{j}k\bar{e}}$: seperately symmetric in $i \leftrightarrow k$ and/or $\bar{j} \leftrightarrow \bar{e}$

$\hookrightarrow$ often written as $R_{ik\bar{j}\bar{e}}$

Ricci Tensor $R_{i\bar{e}} = g^{\bar{j}k} R_{i\bar{j}k\bar{e}}$

$\boxed{R_{i\bar{e}} = \partial_i \partial_{\bar{e}} \log\det(g)}$

Ricci flat Kahler manifold has

$\det(g) = e^{h} e^{\bar{h}}$

for a holomorphic $h(z)$.

E5/6

$\Gamma_\alpha{}^\mu{}_\nu$ is connection of local $SO(d)$ symmetry. $(\Gamma_\alpha)^\mu{}_\nu$

$R_{\alpha\beta}$ $\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma_\alpha{}^\mu{}_\nu V^\nu$

$[\nabla_\alpha, \nabla_\beta] V^\mu = R_{\alpha\beta,}{}^\mu{}_\nu V^\nu$

$R_{\alpha\beta}{}^\mu{}_\nu$: curvature tensor $(R_{\alpha\beta})^\mu{}_\nu$ of $SO(d)$.

Holonomy group = group of parallel transports of vectors, tensors, etc.

Generic Riemann manifold

$Hol = SO(d)$.

special cases: Hol is subgroup of $SO(d)$.

Kähler manifold

$Hol \subset U(n) \subset SO(2n)$

generic Kähler $Hol = U(n)$.

$(R_{i\bar{j}})^{\bar{k}}{}_{\bar{l}}$ or $(R_{i\bar{j}})^k{}_l$

but not $(R_{i\bar{j}})^k{}_{\bar{l}}$ or v.v.

$\rightarrow Hol \subset U(n)$.

Theorem:

$\mathrm{Hol} \subset U(n)$ if & only if the manifold is ~~Köler~~ Kähler

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Kähler and Ricci-flat

$\mathrm{Hol} \subset SU(n)$

SUSY context

NLΣM, Kahler geometry

$$\mathcal{L} = \int d^4\theta \, K(\phi, \bar{\phi})$$

↳ Kahler potential.

$$\mathcal{L} = g_{i\bar{j}}(\phi,\bar{\phi})\left(\partial_\mu \phi^i \partial_\mu \phi^{*\bar{j}} + \frac{i}{2} \bar{\psi}^{\bar{j}} \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} \psi^i\right)$$

$$+ \sigma^\mu_{\alpha\dot{\beta}} \{i\bar{j}k\} \partial_\mu \phi^{*\bar{j}} \psi^{\alpha i} \bar{\psi}^{\dot{\alpha} j} + h.c.$$

$$+ R_{ij\bar{k}\bar{l}} (\psi^i \psi^j_\alpha)(\bar{\psi}^{\bar{k}\dot{\alpha}} \bar{\psi}^{\bar{l}}_{\dot{\alpha}})$$

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$\mathcal{N}=2$ SUSY

Hypermultiplets

↳ need HyperKahler geometry.

in real terms, it has 3 complex structures I, J, K

$$I^2 = J^2 = K^2 = -1$$

$$IJK = -1$$

$\nabla_\alpha I = 0$, $\nabla_\alpha J = 0$, $\nabla_\alpha K = 0$

metric hermitian w.r.t all 3.

Holonomy group $\dim = 2N = 4k$ multiple of 4

Holonomy group: $Hol \subset USp(2k) \subset SU(2k)$

sometimes $USp(2k)$ is called $Sp(k)$

HyperKahler → Ricci flat.

$5d$ SUGRA

$\mathcal{N}=1$ → Kähler NLSM

$\mathcal{N}=2$ → Quaternionic version of HyperKahler.

parallel transport mixes up

I, J, K.

$Hol = SU(2) \times USp(2k)$.

↳ mixes I, J, K

mixes 2 gravitini