Problem **6**(a):

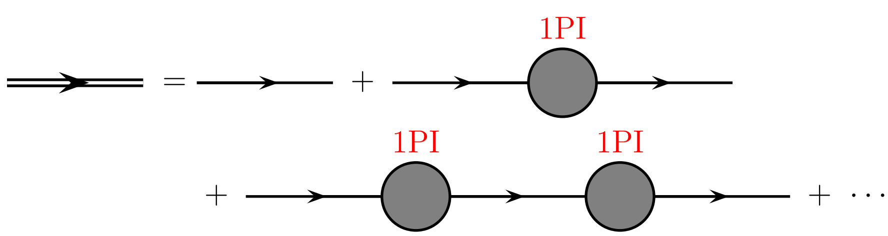
Let's start with eqs. (10) for the 1PI purely photonic amplitudes. In a diagram without external charged lines Φ or $\overline{\Phi}$, all charged propagators form closed loops, so cutting any one such propagator cannot severe the loop. This means that the one-particle-irreducibility of a diagram depends only on the internal photons connecting the charged loops to each other, but not on the charged loops themselves or on the external photons. So if we add a new external photon to an 1PI diagram, we would always get a 1PI diagram regardless of where we attach the new photon, OOH, no such attachment would change a non-1PI diagram into a 1PI diagram.

In problems (4) and (5) we have proved the Ward identities (10) not just for the *net* n-vector amplitudes but also for for partial sums over much smaller sets of diagrams, namely over all possible attachments of a bad photon to any particular $n-1$ diagram. Since all such sets are either all-1PI or none-1PI, summing only over the 1PI sets immediately gives us eqs. (11) for the *net 1PI* n-vector amplitudes $\mathcal{V}_n^{\rm 1PI}(V_1,\ldots,V_N)$.

Problem **6**(b):

Beyond the tree level, the un-amputated $\mathcal{S}_0$ amplitude is the *dressed* chiral propagator

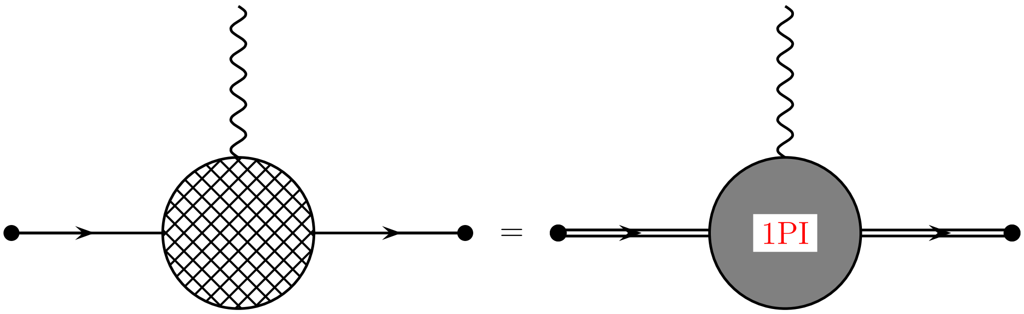
$i\mathcal{S}_0(p)\ \equiv$



$$= i\Pi_a\ +\ i\Pi_a\times i\Gamma_0\times i\Pi_a\ +\ i\Pi_a\times i\Gamma_0\times i\Pi_a\times i\Gamma_0\times i\Pi_a\ +\ \cdots$$
$$= \frac{i\Pi_a}{1+\Pi_a\Gamma_0(p)\Pi_a}\ \equiv\ \frac{1}{1+\hat\Gamma_0(p)}\times\frac{iD^2\overline{D}^2}{16p^2+i0} \tag{15}$$

where $\hat\Gamma_0=\Pi_a\Gamma_0\Pi_a$ is indistinguishable from the 1PI 2–point bubble Γ_0 itself when sandwiched between the chiral Φ on the left and the antichiral $\overline{\Phi}$ on the right.

The un-amputated 1–vector amplitude $i\mathcal{S}_1(V)$ comprises one 1PI *dressed vertex* $i\Gamma_1(V)$ and two dressed charged propagators,



(S.1)

$$i.\,e.,\quad i\mathcal{S}_1(p_1,p_2;V) \;=\; i\mathcal{S}_0(p_1)\times i\Gamma_1(p_1,p_2;V)\times i\mathcal{S}_0(p_2). \tag{S.2}$$

When the vector field V happens to be chiral, eq. (6) tells us that

$$\mathcal{S}_1(p_1,p_2;V=\Lambda) \;=\; -\mathcal{S}_0(p_1)\times\Lambda. \tag{S.3}$$

Combining this formula with eq. (S.2) and eq. (15) for the $\mathcal{S}_0$, we have

$$\frac{\Pi_a}{1+\hat{\Gamma}_0(p_1)}\times\Gamma_1(p_1,p_2;V=\Lambda)\times\frac{\Pi_a}{1+\hat{\Gamma}_0(p_2)} \;=\; +\frac{\Pi_a}{1+\hat{\Gamma}_0(p_1)}\times\Lambda \tag{S.4}$$

and consequently

$$\Pi_a\times\Gamma_1(p_1,p_2;V=\Lambda)\times\Pi_a \;=\; \Lambda\times(1+\hat{\Gamma}_0(p_2)). \tag{S.5}$$

In the context of $\int d^4\theta\,\overline{\Phi}\Gamma_1(V)\Phi$, this formula is equivalent to

$$\Gamma_1(p_1,p_2;V=\Lambda) \;=\; \Lambda\times(1+\Gamma_0(p_2)). \tag{13a}$$

Similarly, when the vector field V happens to be antichiral, we have

$$\mathcal{S}_1(p_1,p_2;V=\overline{\Lambda}) \;=\; -\overline{\Lambda}\times\mathcal{S}_0(p_2) \tag{S.6}$$

and consequently

$$\frac{\Pi_a}{1+\hat{\Gamma}_0(p_1)}\times\Gamma_1(p_1,p_2;V=\overline{\Lambda})\times\frac{\Pi_a}{1+\hat{\Gamma}_0(p_2)} \;=\; +\overline{\Lambda}\times\frac{\Pi_a}{1+\hat{\Gamma}_0(p_2)} \tag{S.7}$$

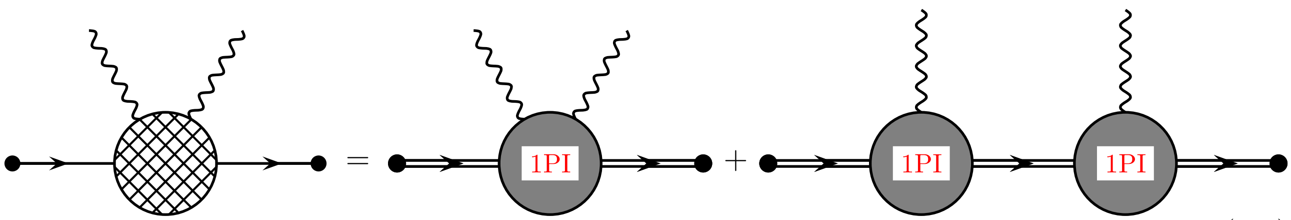
$$\Downarrow$$

$$\Pi_a \times \Gamma_1(p_1, p_2; V = \overline{\Lambda}) \times \Pi_a \;=\; (1 + \hat{\Gamma}_0(p_1)) \times \overline{\Lambda}, \tag{S.8}$$

which is equivalent to

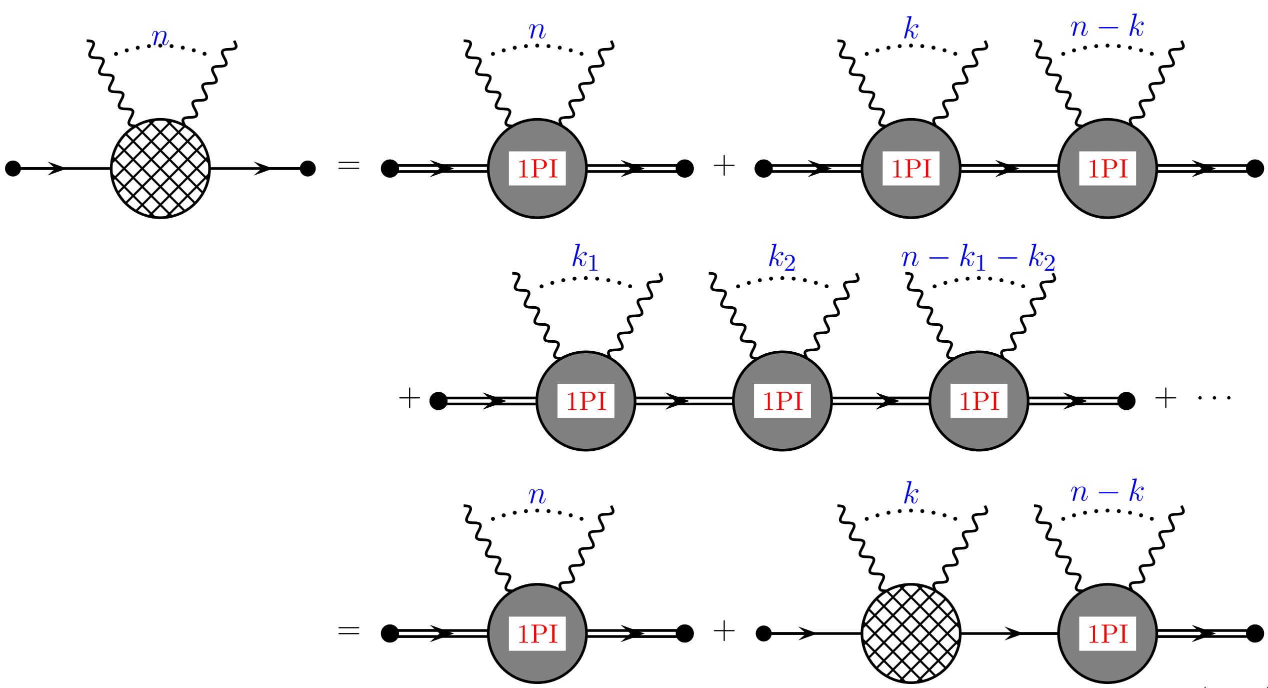
$$\Gamma_1(p_1, p_2; V = \overline{\Lambda}) \;=\; +\Gamma_0(p_1) \times \overline{\Lambda}. \tag{13b}$$

Finally, let's prove eqs. (14) for multi-vector amplitudes. The two-vector un-amputated amplitude comprises two kinds of diagrams,



(S.9)

while for $n > 2$ vectors there are even more possibilities:



(S.10)

The last line in this graphic summary gives a recursive formula for the $\mathcal{S}_n$ in terms of the 1PI amplitudes and the $\mathcal{S}_k$ with $k < n$. Specifically,

$$\begin{aligned}\mathcal{S}_n(V_1,\ldots,V_n) \;&=\; -\mathcal{S}_0\times\Gamma_n(V_0,\ldots,V_n)\times\mathcal{S}_0 \\ &\qquad -\sum_{k=1}^{n-1}\sum_{\text{partitions}}\mathcal{S}_k(k\,\text{vectors})\times\Gamma_{n-k}(n-k\,\text{vectors})\times\mathcal{S}_0\end{aligned} \tag{S.11}$$

where the second sum is over which k vectors out of $V_1,\ldots,V_n$ appear in the $\mathcal{S}_k$ while the rest appear in the Γ_{n-k}; the minus sign on the RHS follow from the i factors in each amplitude, dressed propagators, or dressed vertex.

Eqs. (S.11) give rise to an induction-in-n proof that eqs. (6) imply eqs. (14) for the 1PI amplitudes. To prove the induction base for $n=2$, consider what happens to the two-vector amplitude

$$\mathcal{S}_2(V_1,V_2) \;=\; -\mathcal{S}_0\times\Gamma_2(V_1,V_2)\times\mathcal{S}_0 \;-\; \mathcal{S}_0\times\Gamma_1(V_1)\times\mathcal{S}_1(V_2) \;-\; \mathcal{S}_0\times\Gamma_1(V_2)\times\mathcal{S}_1(V_1) \tag{S.12}$$

when one of the vectors happens to be chiral, say $V_2=\Lambda$. In this case, in the second term on the RHS we have $\mathcal{S}_1(V_2=\Lambda)=-\mathcal{S}_0\times\Lambda$ while for the third term eqs. (13a) gives us $\Gamma_1(V_2=\Lambda)=+\Lambda\times(1+\Gamma_0)$. At the same time, on the LHS we have

$$\mathcal{S}(V_1,V_2=\Lambda) \;=\; -\mathcal{S}_1(V_1)\times\Lambda \;=\; +\mathcal{S}_0\times\Gamma_1(V_1)\times\mathcal{S}_0\times\Lambda. \tag{S.13}$$

Plugging all these formulae into eq. (S.12), we obtain

$$\begin{aligned}+\cancel{\mathcal{S}_0\times\Gamma_1(V_1)\times\mathcal{S}_0\times\Lambda} \;=\; &-\mathcal{S}_0\times\Gamma_2(V_1,V_2=\Lambda)\times\mathcal{S}_0 \\ &+\;\cancel{\mathcal{S}_0\times\Gamma_1(V_1)\times\mathcal{S}_0\times\Lambda} \\ &-\;\mathcal{S}_0\times\Lambda(1+\Gamma_0)\times\mathcal{S}_1(V_1)\end{aligned} \tag{S.14}$$

where the second term on the RHS cancels against the LHS. The remaining terms give us

$$\mathcal{S}_0\times\Gamma_2(V_1,V_2=\Lambda)\times\mathcal{S}_0 \;=\; -\mathcal{S}_0\times\Lambda(1+\Gamma_0)\times\mathcal{S}_1(V_1) \;=\; +\mathcal{S}_0\times\Lambda(1+\Gamma_0)\times\mathcal{S}_0\Gamma_1(V_1)\mathcal{S}_0 \tag{S.15}$$

where in the context of $\int d^4\theta\,\Phi\Gamma\overline{\Phi}$,

$$\begin{aligned}\Gamma_2(V_1,V_2=\Lambda) \;&\cong\; \Lambda\times(1+\Gamma_0)\mathcal{S}_0\times\Gamma_1(V_1) \\ &\cong\; \Lambda\times(1+\hat{\Gamma}_0)\mathcal{S}_0\times\Gamma_1(V_1) &&\langle\!\langle\,\text{since }\Lambda\text{ is chiral just like }\Phi\,\rangle\!\rangle \\ &=\; \Lambda\times\Pi_a\times\Gamma_1(V_1) &&\langle\!\langle\,\text{in light of eq. (15)}\,\rangle\!\rangle \\ &\cong\; \Lambda\times\Gamma_1(V_1)\end{aligned} \tag{S.16}$$

in accordance with eq. (14).

That was the induction base for $n = 2$. Now, we need to prove that **if** eqs. (14) holds true for all $n' < n$ **then** they also hold true for n. Our starting point is the recursion formula (S.11) for the n-vector amplitude $\mathcal{S}(V_1, \ldots, V_n)$. Suppose $V_n = \Lambda$, and let's split the sum over the partitions of n vectors (into k arguments of the S_k and $n - k$ arguments of the Γ_{n-k}) into two sums according to whether the $V_n = \Lambda$ is an argument of the S_k or the Γ_{n-k}, thus

$$\begin{aligned} \mathcal{S}_n(V_1, \ldots, V_{n-1}; \Lambda) \;=\; & -\mathcal{S}_0 \times \Gamma_n(V_0, \ldots, V_{n-1}; \Lambda) \times \mathcal{S}_0 \\ & - \sum_{k=1}^{n-1} \sum_{\text{selections}} \mathcal{S}_k(V, \ldots, V) \times \Gamma_{n-k}(V, \ldots, V; \Lambda) \times \mathcal{S}_0 \\ & - \sum_{k=1}^{n-1} \sum_{\text{selections}} \mathcal{S}_k(V, \ldots, V; \Lambda) \times \Gamma_{n-k}(V, \ldots, V) \times \mathcal{S}_0 \,. \end{aligned} \tag{S.17}$$

By the induction assumption, on the second line here

$$\mathcal{S}_k(V, \ldots, V) \times \Gamma_{n-k}(V, \ldots, V; \Lambda) \times \mathcal{S}_0 \;=\; +\mathcal{S}_k(V, \ldots, V) \times \Lambda \times \Gamma_{n-k-1}(V, \ldots, V) \times \mathcal{S}_0, \tag{S.18}$$

while on the third line of (S.17) we may use eq. (6) to rewrite

$$\mathcal{S}_k(V, \ldots, V; \Lambda) \times \Gamma_{n-k}(V, \ldots, V) \times \mathcal{S}_0 = \; -\mathcal{S}_{k-1}(V, \ldots, V) \times \Lambda \times \Gamma_{n-k}(V, \ldots, V) \times \mathcal{S}_0 \,. \tag{S.19}$$

Due to opposite signs in eqs. (S.18) and (S.19), these terms cancel each other for appropriate partitions of the $(V_1, \ldots, V_{n-1}; \Lambda)$ into arguments of the $\mathcal{S}_k$ and Γ_{n-k}. In the context of the big sums in eq. (S.17), the only terms that survive this cancellation are the $k = n - 1$ term on the second line and the $k = 1$ term on the third line, thus

$$\begin{aligned} \mathcal{S}_n(V_1, \ldots, V_{n-1}; \Lambda) \;=\; & -\mathcal{S}_0 \times \Gamma_n(V_0, \ldots, V_{n-1}; \Lambda) \times \mathcal{S}_0 \\ & - \; \mathcal{S}_{n-1}(V_1, \ldots, V_{n-1}) \times \Gamma_1(\Lambda) \times \mathcal{S}_0 \\ & - \; \mathcal{S}_1(\Lambda) \times \Gamma_{n-1}(V_1, \ldots, V_{n-1}) \times \mathcal{S}_0 \,. \end{aligned} \tag{S.20}$$

Applying eqs. (6) to the LHS here and to the $\mathcal{S}_1(\Lambda)$ in the third term on the RHS, and also applying eq. (13) to the second term on the RHS, we rewrite eq. (S.20) as

$$\begin{aligned}-\mathcal{S}_{n-1}(V_1,\ldots,V_{n-1})\times\Lambda \;&=\; -\mathcal{S}_0\times\Gamma_n(V_0,\ldots,V_{n-1};\Lambda)\times\mathcal{S}_0\\ &\quad-\;\mathcal{S}_{n-1}(V_1,\ldots,V_{n-1})\times\Lambda\times(1+\Gamma_0)\mathcal{S}_0\\ &\quad+\;\mathcal{S}_0\times\Lambda\times\Gamma_{n-1}(V_1,\ldots,V_{n-1})\times\mathcal{S}_0\end{aligned}\tag{S.21}$$

where the LHS and the second term on the RHS cancel each other *in the context of an operator between* Φ *and* $\overline{\Phi}$,

$$\mathcal{S}_{n-1}(V_1,\ldots,V_{n-1})\times\Lambda\;\cong\;\mathcal{S}_{n-1}(V_1,\ldots,V_{n-1})\times\Lambda\times(1+\Gamma_0)\mathcal{S}_0\,.\tag{S.22}$$

Consequently,

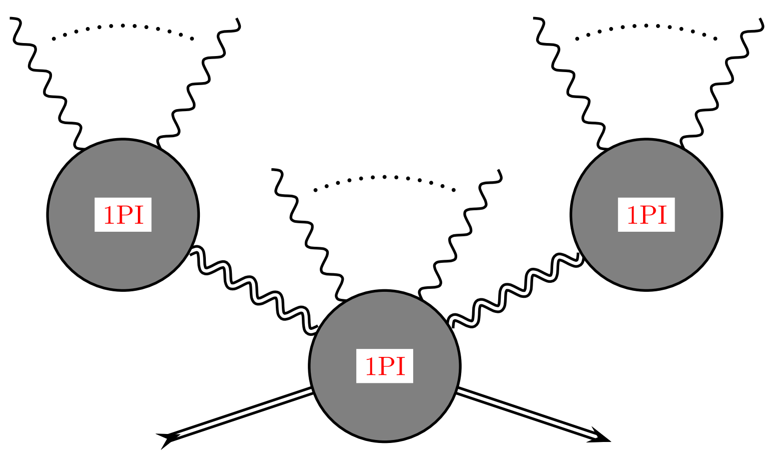
$$\mathcal{S}_0\times\Gamma_n(V_0,\ldots,V_{n-1};\Lambda)\times\mathcal{S}_0\;\cong\;+\mathcal{S}_0\times\Lambda\times\Gamma_{n-1}(V_1,\ldots,V_{n-1})\times\mathcal{S}_0\tag{S.23}$$

and hence in the context of $\int d^4\theta\,\Phi\Gamma\overline{\Phi}$,

$$\Gamma_n(V_0,\ldots,V_{n-1};\Lambda)\;\cong\;\Lambda\times\Gamma_{n-1}(V_1,\ldots,V_{n-1}).\tag{16a}$$

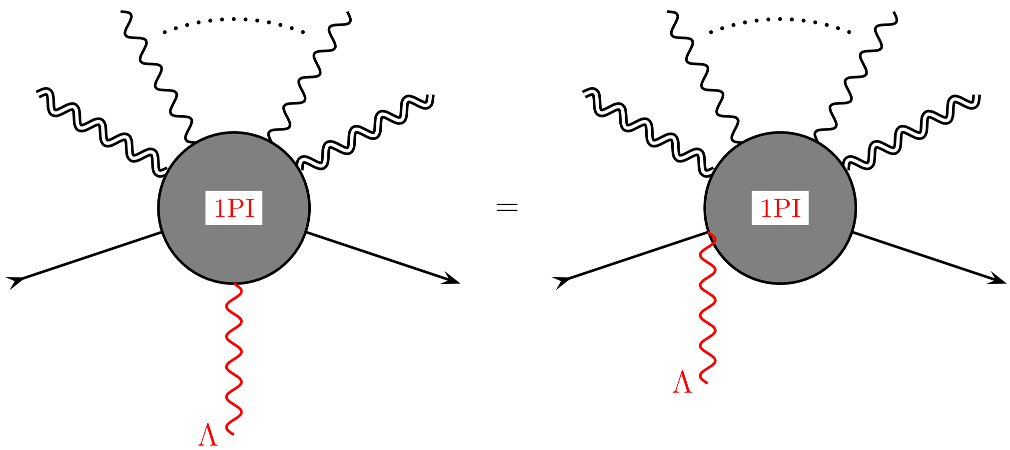
This completes the proof-by-induction of eqs. (14a) for the $V_n=\Lambda$. The antichiral case $V_n=\overline{\Lambda}$ is left out as an exercise for the students.

PS: Strictly speaking, the gray bubbles labeled '1PI' in eq. (S.10) are only one-*charged*–particle irreducible, but they might be severed by cutting a vector propagator, for example



(S.24)

However, when one adds a bad photon $V_{n+1}=\Lambda$ or $\overline{\Lambda}$ to a diagram like this, the bad photon has to be attached to the lower bubble; thanks to eqs. (10), attachments to the upper bubbles cancel out. Consequently, as far as eqs. (14) for the diagrams like (S.24) are concerned, we may disregard the upper bubbles with all the vectors attached to them and focus on the lower bubble only, thus



(S.25)

In other words, once we have proved the Ward identities (16) for the one-*charged*–particle irreducible amplitudes, the completely-1PI amplitudes must also satisfy the same identities (16).