

Problem 1(a):

Let's start with the continuous global symmetries

$$G_{\text{global}}^{\text{classical}} = \left(SU(4) \times U(1)_X \times U(1)_Y \right)_1 \times \left(SU(4) \times U(1)_X \times U(1)_Y \right)_2 \times U(1)_R \quad (\text{S.1})$$

which act on the A , B , and C fields according to the following charge table

QN fields	$SU(4)_1$	$U(1)_{X1}$	$U(1)_{Y1}$	$SU(4)_2$	$U(1)_{X2}$	$U(1)_{Y2}$
A_1	1	+2	0	1	0	0
$B_1, \dots, B_4$	4	-1	+1	1	0	0
$C_1, \dots, C_4$	$\bar{\mathbf{4}}$	-1	-1	1	0	0
A_2	1	0	0	1	+2	0
$B_5, \dots, B_8$	1	0	0	4	-1	+1
$C_5, \dots, C_8$	1	0	0	$\bar{\mathbf{4}}$	-1	-1

(S.2)

Here QN — the quantum numbers — refer to the abelian charges or to the non-abelian multiplets (in boldface). As to the R symmetry,

$$\begin{aligned}
 &\text{all scalars } \phi \text{ have } R = +\frac{2}{3}, \\
 &\text{all matter fermions } \psi_\alpha \text{ have } R = -\frac{1}{3}, \\
 &\text{all gauginos } \lambda_\alpha \text{ have } R = +1, \\
 &\text{all vectors } V_\mu \text{ have } R = 0.
 \end{aligned} \quad (\text{S.3})$$

In the quantum theory we must check the abelian classical symmetries for the gauge anomalies

$$\sum_{\text{fermions}} (\text{global charge}) \times \text{Index} \left[\text{gauge } SU(2)_1 \text{ or } SU(2)_2 \right]. \quad (\text{S.4})$$

It is easy to see that the Y_1 and Y_2 charges are anomalous WRT both $SU(2)$ factors but the combination $Y = Y_1 - Y_2$ is anomaly-free. Also, the X_1 , X_2 , and R charges are non-anomalous.

Thus, the quantum theory has continuous global symmetry

$$G_{\text{global}}^{\text{quantum}} = \left(SU(4) \times U(1)_X \right)_1 \times \left(SU(4) \times U(1)_X \right)_2 \times U(1)_Y \times U(1)_R. \quad (\text{S.5})$$

As to the discrete symmetries, we have the anomaly-free $\mathbf{Z}_8$ subgroup of the $U(1)_{Y1+Y2}$ which multiplies all the B_i fields by $e^{2\pi i/8}$ and all the C_i by $e^{-2\pi i/8}$. More importantly, we have a $\mathbf{Z}_2$ symmetry that swaps fields appearing in the first term of the superpotential (2) with fields appearing in the second term,

$$\mathbf{Z}_2 : \quad A_1 \leftrightarrow A_2, \quad B_i \leftrightarrow B_{i\pm 4}, \quad C_i \leftrightarrow C_{i\pm 4}. \quad (\text{S.6})$$

This symmetry assures that the A_1 and the A_2 fields have the same anomalous dimension γ_A . Also, together with the $SU(4)_1 \times SU(4)_2$ it makes sure that all eight B_i fields have the same anomalous dimension γ_B and all eight C_i fields have the same anomalous dimension γ_C .

Problem 1(b):

Thanks to SUSY, there is no counterterm δ_λ for the Yukawa coupling. Consequently, its renormalization follows solely from the wave-function renormalization of the chiral superfields, thus

$$\beta_\lambda = \lambda \times (\gamma_A + \gamma_B + \gamma_C). \quad (\text{S.7})$$

As to the gauge couplings, their beta-function are given by the Novikov–Shifman–Vainshtein–Zaharov equations

$$\begin{aligned} \beta_{g1} &= \frac{g_1^3}{16\pi^2 - 4g_1^2} \times \left[-3 \times 2 + 2 \times 2 \times \frac{1}{2} \times (1 - 2\gamma_A) + 4 \times 2 \times \frac{1}{2} \times (1 - 2\gamma_B) \right] \\ &= -\frac{g_1^3}{4\pi^2 - g_1^2} \times (\gamma_A + 2\gamma_B) \end{aligned} \quad (\text{S.8})$$

and similarly

$$\begin{aligned} \beta_{g2} &= \frac{g_2^3}{16\pi^2 - 4g_2^2} \times \left[-3 \times 2 + 2 \times 2 \times \frac{1}{2} \times (1 - 2\gamma_A) + 4 \times 2 \times \frac{1}{2} \times (1 - 2\gamma_C) \right] \\ &= -\frac{g_2^3}{4\pi^2 - g_2^2} \times (\gamma_A + 2\gamma_C). \end{aligned} \quad (\text{S.9})$$

Equations (S.7), (S.8), and (S.9) give us exact beta-functions (true to all orders of the perturbation theory) in terms of the anomalous dimensions. Hence, the fixed points where all three

beta-functions vanish, $\beta_\lambda = \beta_{g1} = \beta_{g2} = 0$ lie wherever

$$\gamma_A + \gamma_B + \gamma_C = \gamma_A + 2\gamma_B = \gamma_A + 2\gamma_C = 0, \quad (\text{S.10})$$

or equivalently

$$\gamma_B(\lambda, g_1, g_2) = \gamma_C(\lambda, g_1, g_2) = -\frac{1}{2}\gamma_A(\lambda, g_1, g_2). \quad (\text{S.11})$$

Note that the three eqs. (S.10) are not independent, hence only two eqs. (S.11).

Viewed as equations for the couplings λ , g_1 and g_2 — on which the anomalous dimensions depend — the two eqs. (S.11) define a line in the 4D coupling space. All point on this line are fixed points of the renormalization group.

Preamble to problem 1(c):

A few weeks ago we saw in class that in the Wess–Zumino model the chiral superfield Φ has anomalous dimension

$$\gamma^{\text{WZ}} = +\frac{|\lambda|^2}{32\pi^2} + O(|\lambda^4|) \quad (\text{S.12})$$

where λ is the canonically normalized Yukawa coupling. This formula has a straightforward generalization to a theory of several chiral superfields Φ_i with Yukawa couplings λ_{ijk} , namely

$$\gamma^{\text{Yukawas}}[\Phi_i] = +\frac{1}{32\pi^2} \sum_{jk} |\lambda_{ijk}|^2 + \text{higher loops}. \quad (\text{S.13})$$

In this formula, we should sum over all quantum numbers hiding behind the indices j and k , such as the species (*e.g.*, A , B , or C), the ‘flavor’ (*e.g.*, i of A_i , B_i , or C_i) and the color — or whatever the gauge index may be appropriate for the field in question (*e.g.*, the colors of both $SU(2)$ groups for the A_i fields). Also, if $j \neq k$ we should include both (i, k) and (k, j) pairs into the sum — which gives us an overall factor of 2.

Now consider the effect of the gauge couplings. In SQED, we saw in class that

$$\gamma^{\text{SQED}}[\Phi_i] = -\frac{e^2 q_i^2}{8\pi^2} + O(e^4). \quad (\text{S.14})$$

Generalization to the non-abelian gauge theories is quite straightforward: For a simple gauge

group G ,

$$\gamma^{\text{gauge}}[\Phi_i] = -\frac{g^2}{8\pi^2} \times C_2(\Phi_i) + \text{higher loops} \quad (\text{S.15})$$

where C_2 is a quadratic Casimir of a gauge multiplet to which the Φ_i fields belongs (*e.g.*, $C_2 = I(I+1)$ for an $SU(2)$ multiplet of isospin I), and g is normalized conventionally (as in the Standard Model or GUTs, not as in the *Superspace* book). More generally, for a gauge symmetry $G = \prod_\nu G_\nu$ which has several simple (or abelian) factors G_ν with respective couplings g_ν , we have

$$\gamma^{\text{gauge}}[\Phi_i] = -\frac{1}{8\pi^2} \sum_\nu g_\nu^2 \times C_2^{(G_\nu)}(\Phi_i) + \text{higher loops}. \quad (\text{S.16})$$

Finally, in a generic theory that has both gauge and Yukawa couplings, the one-loop anomalous dimensions are

$$\gamma_{1\text{loop}}^{\text{net}}[\Phi_i] = \gamma_{1\text{loop}}^{\text{Yukawa}}[\Phi_i] + \gamma_{1\text{loop}}^{\text{gauge}}[\Phi_i] = +\frac{1}{32\pi^2} \sum_{jk} |\lambda_{ijk}|^2 - \frac{1}{8\pi^2} \sum_\nu g_\nu^2 \times C_2^{(G_\nu)}(\Phi_i). \quad (\text{S.17})$$

Problem 1(c):

Now let's apply eqs. (S.17) for the theory in question. The gauge group is $SU(2) \times SU(2)$, and WRT each factor, the A, B, C fields are either doublets ($C_2 = \frac{3}{4}$) or singlets ($C_2 = 0$). Specifically, $A_i \in (\mathbf{2}, \mathbf{2})$, $B_i \in (\mathbf{2}, \mathbf{1})$, $C_i \in (\mathbf{1}, \mathbf{2})$, hence the gauge couplings' contributions to the one-loop anomalous dimensions are

$$\gamma_{1\text{loop}}^{\text{gauge}}[A] = -\frac{3}{32\pi^2} \times (g_1^2 + g_2^2), \quad \gamma_{1\text{loop}}^{\text{gauge}}[B] = -\frac{3}{32\pi^2} \times g_1^2, \quad \gamma_{1\text{loop}}^{\text{gauge}}[C] = -\frac{3}{32\pi^2} \times g_2^2. \quad (\text{S.18})$$

Now consider the Yukawa couplings' contributions (S.13). For the superpotential (2) all non-zero couplings have the same value $= \lambda$, hence

$$\gamma_{1\text{loop}}^{\text{Yukawa}}[\Phi_i] = +\frac{|\lambda|^2}{32\pi^2} \times \#(j, k) \text{ pairs for which } \lambda_{ijk} \neq 0. \quad (\text{S.19})$$

Let's start with Φ_i being an A field; for the sake of definiteness, let $\Phi_i = A_1$ of some fixed $SU(2) \times SU(2)$ colors (c_1, c_2) . Then we get a non-zero Yukawa coupling for $\Phi_j = B_f$ of flavor

$f = 1, 2, 3, 4$ and color c_1 and $\Phi_k = C_f$ (same f) of color c_2 . Summing over $f = 1, 2, 3, 4$ gives us four (j, k) pairs, times two for the $j \leftrightarrow k$ exchange. Altogether, there are eight (j, k) pairs, hence

$$\gamma_{1\text{loop}}^{\text{Yukawa}}[A] = +\frac{|\lambda|^2}{32\pi^2} \times 8. \quad (\text{S.20})$$

Now let Φ_i be a B field, say $\Phi_i = B_1$ of color c_1 . Then we get $\lambda_{ijk} \neq 0$ for $\Phi_j = C_1$ (note same flavor) and some color $c_2 = 1, 2$ and $\Phi_k = A_1$ of colors (c_1, c_2) . All the flavors here are fixed, and so is the $SU(2)_1$ color c_1 , but the $SU(2)_2$ color c_2 is not, hence summing over the $c_2 = 1, 2$ gives us two (j, k) pairs. Multiplying by two for the $j \leftrightarrow k$ exchange, we get four pairs, thus

$$\gamma_{1\text{loop}}^{\text{Yukawa}}[B] = +\frac{|\lambda|^2}{32\pi^2} \times 4. \quad (\text{S.21})$$

Similarly,

$$\gamma_{1\text{loop}}^{\text{Yukawa}}[C] = +\frac{|\lambda|^2}{32\pi^2} \times 4. \quad (\text{S.22})$$

Combining these Yukawa contributions with the gauge contributions (S.18), we arrive at the net one-loop anomalous dimensions

$$\begin{aligned} \gamma_{1\text{loop}}^{\text{net}}[A] &= \frac{8|\lambda|^2 - 3g_1^2 - 3g_2^2}{32\pi^2}, \\ \gamma_{1\text{loop}}^{\text{net}}[B] &= \frac{4|\lambda|^2 - 3g_1^2}{32\pi^2}, \\ \gamma_{1\text{loop}}^{\text{net}}[C] &= \frac{4|\lambda|^2 - 3g_2^2}{32\pi^2}. \end{aligned} \quad (\text{S.23})$$

It remains to plug in these anomalous dimensions into the fixed-point equations

$$\gamma_B(\lambda, g_1, g_2) = \gamma_C(\lambda, g_1, g_2) = -\frac{1}{2}\gamma_A(\lambda, g_1, g_2). \quad (\text{S.11})$$

Clearly, the first equation $\gamma_B = \gamma_C$ requires $g_1 = g_2$. In fact, by symmetry between the two $SU(2)$ gauge groups, this condition is exact to all orders of the perturbation theory. As to the

second equation (S.11),

$$32\pi^2(\gamma_A + \gamma_B + \gamma_C) = 16|\lambda|^2 - 6g_1^2 - 6g_2^2 + \text{higher loops}, \quad (\text{S.24})$$

hence the fixed points lie along the line

$$g_1^2 = g_2^2 = \frac{16}{12}|\lambda|^2 + O(|\lambda|^4/16\pi^2). \quad (3)$$

Quod erat demonstrandum.

Problem 1(d):

Let's take a closer look at the beta-functions

$$\beta_{g1} = -\frac{g_1^3}{4\pi^2 - g_1^2} \times (\gamma_A + 2\gamma_B), \quad (\text{S.8})$$

$$\beta_{g2} = -\frac{g_2^3}{4\pi^2 - g_1^2} \times (\gamma_A + 2\gamma_C), \quad (\text{S.9})$$

$$\beta_\lambda = +\lambda \times (\gamma_A + \gamma_B + \gamma_C), \quad (\text{S.7})$$

and the one-loop anomalous dimensions (S.23). Suppose we start at some point in the coupling space where $g_1 > g_2$. In this case, eqs. (S.23) yield $\gamma_B < \gamma_C$, hence according to eqs. (S.8) and (S.9) $\beta_1 > \beta_2$. This means that the difference $g_1 - g_2$ increases in the UV direction but decreases in the IR direction. Likewise, had we started with $g_2 > g_1$, the difference $g_2 - g_1$ would also decrease in the IR direction. Thus, regardless of our starting point in the coupling space, the RG flow would bring the gauge couplings g_1 and g_2 closer to each other in the IR, and ultimately — in the very deep IR — we would end up with $g_1 = g_2$.

Now the other direction in which we may deviate from the fixed line (3). Let start with the Yukawa coupling too weak compared to the gauge couplings, *i.e.* $g_1^2 = g_2^2 > \frac{4}{3}|\lambda|^2$. In this case,

$$\gamma_A + 2\gamma_B = \gamma_A + 2\gamma_C = \gamma_A + \gamma_B + \gamma_C < 0 \quad (\text{S.25})$$

and hence $\beta_{g1} = \beta_{g2} > 0$ while $\beta_\lambda < 0$, which means that in the IR direction the Yukawa coupling becomes stronger while the gauge couplings become weaker. Likewise, if we start the

RG flow with the Yukawa coupling too strong relative to the gauge couplings, *i.e.* $g_1^2 = g_2^2 < \frac{4}{3}|\lambda|^2$, then

$$\gamma_A + 2\gamma_B = \gamma_A + 2\gamma_C = \gamma_A + \gamma_B + \gamma_C > 0 \quad (\text{S.26})$$

and hence $\beta_{g1} = \beta_{g2} < 0$ while $\beta_\lambda > 0$: this time, the Yukawa coupling becomes weaker in the IR direction while the gauge couplings become stronger. In either case, the gauge and Yukawa coupling converge towards each other — or rather towards the fixed line (3) — in the IR direction. And that’s what makes the fixed line (3) IR-attractive.

Problem 1(e):

The line (3) or IR-attractive fixed points gives rise to a whole family of non-trivial SCFTs. At one end of the fixed line (3), all 3 couplings g_1 , g_2 , and λ are weak and all the anomalous dimensions are small. But apart from this weakly-coupled region, the fixed line (3) extends to strong, or at least $O(1)$ couplings. In this regime, the perturbation theory breaks down and we no longer have accurate formulae for the anomalous dimensions, or even for the $g_1 = g_2$ as a function of $|\lambda|$ — the fixed line in the 3D coupling space probably isn’t straight (outside the weak-couplings region), and we don’t know its precise shape. But it certainly does go somewhere in the strongly-coupled region, so our family of the SCFTs includes both weakly-coupled and strongly-coupled theories.

Problem 2(a–b), preamble:

For an hierarchical pattern of quark masses, we may integrate out one heavy flavor at a time. So consider an effective theory $\#k$ for the energy range $M_f \gg E \gg M_{f+1}$: From this EFT, k of the heaviest flavors have already been integrated out while $N_f - k$ lighter flavors remain in the theory. Let τ_k be the Wilsonian gauge coupling of that EFT and Λ_k its dimensional transmutant,

$$\Lambda_k^{3N_c - N_f + k} = (\text{UV cutoff})^{3N_c - N_f + k} \times \exp(2\pi i \tau_k). \quad (\text{S.27})$$

In particular, for $k = N_f$ the EFT $\#N_f$ is the low-energy SYM without any flavors at all. Also, we may identify the original SQCD as EFT $\#0$.

In a moment, I am going to show that integrating out one quark flavor at a time leads to

$$\Lambda_{k+1}^{3N_c-N_f+k+1} = \Lambda_k^{3N_c-N_f+k} \times m_{k+1} \times \left(\frac{\text{numeric}}{\text{factor}} \right). \quad (\text{S.28})$$

Once we establish this formula, we may use it N_f times to get

$$\Lambda_{k=N_f}^{3N_c} = \Lambda_{k=0}^{3N_c-N_f} \times \prod_{f=1}^{N_f} m_f \times \left(\frac{\text{numeric}}{\text{factor}} \right) \quad (\text{S.29})$$

and hence

$$\Lambda_{\text{SYM}}^{3N_c} = \Lambda_{\text{orig}}^{3N_c-N_f} \times \det(m) \times \left(\frac{\text{numeric}}{\text{factor}} \right), \quad (4)$$

which verifies part (a). Also, for part (b), we integrate the k heavy flavors and then stop, so applying eq. (S.28) k times we get

$$\Lambda_k^{3N_c-N_f+k} = \Lambda_{\text{orig}}^{3N_c-N_f} \times \prod_{f=1}^k m_f \times \left(\frac{\text{numeric}}{\text{factor}} \right). \quad (\text{S.30})$$

Problem 2(a–b), deriving eq. (S.28):

Let's start with the magnitudes of the two sides of eq. (S.28). In the EFT# k for $E \gtrsim M_f$, the RG flow is governed by the NSVZ (Novikov–Shifman–Vainshtein–Zaharov) equation

$$\frac{8\pi^2}{g^2(E)} - N_c \log \frac{1}{g^2(E)} + \sum_{f=k+1}^{N_f} \frac{1}{2} (\log Z_f^A(E) + \log Z_f^B(E)) = (3N_c - N_f + k) \times \log \frac{E}{|\Lambda_k|}. \quad (\text{S.31})$$

Likewise, in the EFT# $(k+1)$ for $E \lesssim M_f$, the RG flow is governed by the similar NSVZ equation for a theory with one less flavor,

$$\frac{8\pi^2}{g^2(E)} - N_c \log \frac{1}{g^2(E)} + \sum_{f=k+2}^{N_f} \frac{1}{2} (\log Z_f^A(E) + \log Z_f^B(E)) = (3N_c - N_f + k + 1) \times \log \frac{E}{|\Lambda_{k+1}|}. \quad (\text{S.32})$$

The two formulae (S.31) and (S.32) should produce the same physical gauge coupling at the threshold $E = M_{k+1}$. Or rather, because of some threshold corrections, at $E = M' = M_{k+1} \times \text{an}$

$O(1)$ numeric constant. Thus, taking the difference between eqs. (S.31) and (S.32) for $E = M'$, we get

$$\frac{1}{2}(\log Z_{k+1}^A(M') + \log Z_{k+1}^B(M')) = (3N_c - N_f + k) \times \log \frac{M'}{|\Lambda_k|} - (3N_c - N_f + k + 1) \times \log \frac{M'}{|\Lambda_{k+1}|} \quad (\text{S.33})$$

and hence

$$(3N_c - N_f + k + 1) \log(|\Lambda_{k+1}|) - (3N_c - N_f + k) \log(|\Lambda_k|) = \log(M') + \frac{1}{2} \log(Z_{k+1}^A) + \frac{1}{2} \log(Z_{k+1}^B). \quad (\text{S.34})$$

But the RHS of this formula amounts to

$$\log(M_{k+1}) + \left(\begin{smallmatrix} \text{numeric} \\ \text{constant} \end{smallmatrix} \right) + \log \sqrt{Z_{k+1}^A Z_{k+1}^B} = \log(|m_{k+1}|) + \left(\begin{smallmatrix} \text{numeric} \\ \text{constant} \end{smallmatrix} \right), \quad (\text{S.35})$$

therefore

$$(3N_c - N_f + k + 1) \log(|\Lambda_{k+1}|) = (3N_c - N_f + k) \log(|\Lambda_k|) + \log(|m_{k+1}|) + \left(\begin{smallmatrix} \text{numeric} \\ \text{constant} \end{smallmatrix} \right) \quad (\text{S.36})$$

and hence

$$|\Lambda_{k+1}|^{3N_c - N_f + k + 1} = |\Lambda_k|^{3N_c - N_f + k} \times |m_{k+1}| \times \left(\begin{smallmatrix} \text{numeric} \\ \text{factor} \end{smallmatrix} \right). \quad (\text{S.37})$$

This verifies the magnitude part of eq. (S.28).

As to the phase part of eq. (S.28), it should follow by holomorphy, but we may verify it directly in terms of the instanton angles Θ_k and Θ_{k+1} of the two effective theories. Note that the CP violating phase of an effective QCD or SQCD is not its instanton angle but rather

$$\bar{\Theta} = \Theta + \text{phase det} \left(\begin{smallmatrix} \text{mass matrix for} \\ \text{the EFTs flavors} \end{smallmatrix} \right), \quad (\text{S.38})$$

thus

$$\begin{aligned} \bar{\Theta}_k &= \Theta_k + \sum_{f=k+1}^{N_f} \text{phase}(m_f), \\ \bar{\Theta}_{k+1} &= \Theta_{k+1} + \sum_{f=k+2}^{N_f} \text{phase}(m_f). \end{aligned} \quad (\text{S.39})$$

But the CP violating phase should not change when we integrate out a massive flavor, thus

$$\overline{\Theta}_{k+1} = \overline{\Theta}_k \quad (\text{S.40})$$

and hence

$$\Theta_{k+1} = \Theta_k + \text{phase}(m_{k+1}). \quad (\text{S.41})$$

Finally,

$$\Theta_k = 2\pi \text{Re } \tau_k = \text{phase}(\Lambda_k^{3N_c - N_f + k}) \quad (\text{S.42})$$

and likewise

$$\Theta_{k+1} = 2\pi \text{Re } \tau_{k+1} = \text{phase}(\Lambda_{k+1}^{3N_c - N_f + k + 1}). \quad (\text{S.43})$$

Consequently, eq. (S.41) becomes

$$\text{phase}(\Lambda_{k+1}^{3N_c - N_f + k + 1}) = \text{phase}(\Lambda_k^{3N_c - N_f + k}) + \text{phase}(m_{k+1}), \quad (\text{S.44})$$

which is precisely the phase part of eq. (S.28).

Problem 2(d), preamble:

For the hierarchical squark VEVs (8), we may integrate out one set of massive vector superfields at a time. So similarly to parts (a–b), let EFT# k be the effective theory at energies $M_k \gtrsim E \gtrsim M_{k+1}$, where this time

$$M_k = \phi_k \times g \sqrt[4]{Z_k^A Z_k^B} \quad (\text{S.45})$$

denotes the physical mass of the vector superfields stemming from the k^{th} eigenvalue ϕ_k . At each threshold, the lower-energy EFT has one color and one flavor less than the higher-energy

EFT, the EFT# k has $N_c - k$ colors and $N_f - k$ flavors, hence one-loop beta-function coefficient

$$b_k = -3(N_c - k) + (N_f - k) = -3N_c + N_f + 2k. \quad (\text{S.46})$$

In a moment, we shall see that each step of integrating out the massive particles leads to

$$\Lambda_{k+1}^{3N_c - N_f - 2(k+1)} = \frac{\Lambda_k^{3N_c - N_f - 2k}}{\mathcal{M}_{k+1, k+1}} \times \left(\frac{\text{numeric}}{\text{factor}} \right). \quad (\text{S.47})$$

Given this formula, it easy to see that after k steps starting from the original SQCD

$$\Lambda_k^{3N_c - N_f - 2k} = \Lambda_{\text{orig}}^{3N_c - N_f} \times \prod_{f=1}^k \frac{1}{\mathcal{M}_{ff}} \times \left(\frac{\text{numeric}}{\text{factor}} \right). \quad (\text{S.48})$$

In particular, once we use up all N_f flavors of squark VEVs, we end up with the low-energy SYM with $G = SU(N_c - N_f)$ and

$$\begin{aligned} \Lambda_{\text{SYM}}^{3(N_c - N_f)} &= \Lambda_{\text{orig}}^{3N_c - N_f} \times \prod_f^{\text{all}} \frac{1}{\mathcal{M}_{ff}} \times \left(\frac{\text{numeric}}{\text{factor}} \right) \\ &= \frac{\Lambda_{\text{orig}}^{3N_c - N_f}}{\det(\mathcal{M})} \times \left(\frac{\text{numeric}}{\text{factor}} \right). \end{aligned} \quad (\text{S.49})$$

exactly as in eq. (7).

Problem 2(d), verifying eq. (S.47):

Let's start with the magnitudes on two sides of eq. (S.47). The RG flow of the gauge coupling at energies above the M_{k+1} threshold is governed by the NSVZ equation

$$\begin{aligned} \frac{8\pi^2}{g^2(E)} - (N_c - k) \log \frac{1}{g^2(E)} + \sum_{f=k+1}^f \left(\frac{1}{2} \log Z_f^A + \frac{1}{2} \log Z_f^B \right) &= \\ = (3N_c - N_f - 2k) \log \frac{E}{|\Lambda_k|}, \end{aligned} \quad (\text{S.50})$$

while the RG flow below the threshold is governed by a similar NSVZ equation

$$\begin{aligned} \frac{8\pi^2}{g^2(E)} - (N_c - k - 1) \log \frac{1}{g^2(E)} + \sum_{f=k+2}^f \left(\frac{1}{2} \log Z_f^A + \frac{1}{2} \log Z_f^B \right) = \\ = (3N_c - N_f - 2k - 2) \log \frac{E}{|\Lambda_{k+1}|}. \end{aligned} \quad (\text{S.51})$$

At the threshold $E = M_k$ — or rather in its vicinity at energy

$$M' = M_{k+1} \times \left(\frac{\text{numeric}}{\text{factor}} \right) = \sqrt{|\mathcal{M}_{k+1,k+1}|} \times g \sqrt[4]{Z_{k+1}^A Z_{k+1}^B} \times \left(\frac{\text{numeric}}{\text{factor}} \right) \quad (\text{S.52})$$

— both EFT $\#k$ and EFT $\#(k+1)$ should have the same gauge coupling $g(E = M')$. Thus, subtracting eq. (S.51) from eq. (S.50), we get

$$\begin{aligned} -\log \frac{1}{g^2} + \frac{1}{2} \log(Z_{k+1}^A) + \frac{1}{2} \log(Z_{k+1}^B) = \\ = (3N_c - N_f - 2k) \log \frac{M'}{|\Lambda_k|} - (3N_c - N_f - 2k - 2) \log \frac{M'}{|\Lambda_{k+1}|} \end{aligned} \quad (\text{S.53})$$

and hence

$$\begin{aligned} (3N_c - N_f - 2k - 2) \log(|\Lambda_{k+1}|) - (3N_c - N_f - k) \log(|\Lambda_k|) = \\ = -2 \log(M') - \log \frac{1}{g^2} + \frac{1}{2} \log(Z_{k+1}^A) + \frac{1}{2} \log(Z_{k+1}^B) \\ = \log \frac{g^2 \sqrt{Z_{k+1}^A Z_{k+1}^B}}{M'^2} \\ = \log \frac{1}{|\mathcal{M}_{k+1,k+1}|} + \left(\frac{\text{numeric}}{\text{constant}} \right). \end{aligned} \quad (\text{S.54})$$

Exponentiating both sides of this formula, we immediately arrive at

$$|\Lambda_{k+1}|^{3N_c - N_f - 2k - 2} = \frac{|\Lambda_k|^{3N_c - N_f - 2k}}{|\mathcal{M}_{k+1,k+1}|} \times \left(\frac{\text{numeric}}{\text{factor}} \right), \quad (\text{S.55})$$

which is precisely the magnitude aspect of eq. (S.47).

As to the phase aspect of eq. (S.47), it follows by holomorphy from the magnitude aspect, but there is also an independent argument in terms of the Θ angles. In general, integrating out massive fermions leads to

$$\Theta_{k+1} = \Theta_k + \sum_{\text{mult}} 2R(\text{mult}) \times \text{phase}(m_{\text{mult}}) \quad (\text{S.56})$$

where the sum is over multiplets of Dirac fermions that are present in the higher-energy EFT# k but not in the lower-energy EFT# $(k+1)$, and the index is WRT to the lower-energy gauge group. For the problem at hand, the fermions in EFT# k but not in EFT# $(k+1)$ belong to the massive vector superfields, and WRT to the lower-energy gauge group $SU(N_c - k - 1)$ they form a fundamental multiple, an antifundamental multiplet, and a singlet. Consequently, eq. (S.56) becomes

$$\Theta_{k+1} = \Theta_k + \text{phase}(m_{\text{fund}}) + \text{phase}(m_{\text{antifund}}). \quad (\text{S.57})$$

In the SUSY Higgs mechanism, both masses stem from the Yukawa couplings of the LH gluinos, quarks, and antiquarks to the conjugate $\langle A^* \rangle$ and $\langle B^* \rangle$ scalars. For the situation at hand, this means to the ϕ_{k+1}^* , hence

$$\text{phase}(m_{\text{fund}}) = \text{phase}(m_{\text{antifund}}) = -\text{phase}(\phi_{k+1}) = -\frac{1}{2} \text{phase}(\mathcal{M}_{k+1,k+1}). \quad (\text{S.58})$$

Thus,

$$\Theta_{k+1} = \Theta_k - \text{phase}(\mathcal{M}_{k+1,k+1}) \quad (\text{S.59})$$

and therefore

$$\text{phase}(\Lambda_{k+1}^{3N_c - N_f - 2(k+1)}) = \text{phase}(\Lambda_k^{3N_c - N_f - 2k}) - \text{phase}(\mathcal{M}_{k+1,k+1}), \quad (\text{S.60})$$

exactly as in the phases of two sides of eq. (S.47).