Problem **2**(a):

In the absence of a superpotential, all chiral superfields are moduli except those eaten up by the supersymmetric Higgs mechanism. Generic VEVs of $N_f > N_c$ squarks and antisquarks break the $SU(N_c)$ gauge symmetry down to nothing, so all $N_c^2 - 1$ gauge fields become massive and hence $N_c^2 - 1$ chiral superfields are eaten up. The number of the remaining moduli fields is

$$\#\text{moduli} \;=\; 2N_f N_c \;-\; (N_c^2 - 1), \tag{S.1}$$

which for $N_f = N_c + 1$ happens to be $2(N_c+1)N_c - N_c^2 + 1 = N_c^2 + 2N_c + 1 = N_f^2$.

Problem **2**(b):

In matrix notations, the squarks Q form an $N_c \times N_f$ matrix while the antisquark matrix $\widetilde{Q}$ is $N_f \times N_c$. For $N_c < N_f$, the matrix product $\mathcal{M} = \widetilde{Q}Q$ has rank no greater than N_c, and since $\mathcal{M}$ itself is $N_f \times N_f$, its determinant must vanish, $\det(\mathcal{M}) = 0$.

Now consider the meson-baryon product

$$\mathcal{M}_{ff'}\mathcal{B}^{f'} \;=\; \widetilde{Q}_{f,c} \times Q^c_f Q^{c_1}_{f_1} \cdots Q^{c_N}_{f_N} \epsilon^{f' f_1 \cdots f_N} \times \frac{1}{N!}\epsilon_{c_1 \cdots c_N}\,. \tag{S.2}$$

The middle factor here is the antisymmetrized product of $N + 1 = N_f > N_c$ squark flavors. By Bose statistics, their colors have to be antisymmetrized as well, but we have only $N_c = N$ colors so that's not possible. Consequently,

$$Q^c_f Q^{c_1}_{f_1} \cdots Q^{c_N}_{f_N} \epsilon^{f' f_1 \cdots f_N} \;=\; 0 \quad \text{for any colors} \quad \Longrightarrow \quad \mathcal{M}_{ff'}\mathcal{B}^{f'} \;=\; 0. \tag{S.3}$$

Similarly, $\widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} = 0$.

To prove the last relation $\text{minor}(\mathcal{M})^{ff'} = \widetilde{\mathcal{B}}^f \mathcal{B}^{f'}$, consider the $N \times N$ sub-matrix $Q^{(\not{f}')}$ of the quark matrix which contains all the colors but only $N = N_f - 1$ flavors — all except the

f'. The determinant of this square sub-matrix is

$$\begin{aligned}
\det\left(Q^{(\cancel{f'})}\right) &= \epsilon_{c_1\cdots c_N} Q^1_{c_1}\cdots \cancel{Q^{f'}_{c}}\cdots Q^{c_N}_{f_N} \\
&= \frac{(-1)^{f'-1}}{N!}\,\epsilon^{f' f_1\cdots f_N}\epsilon_{c_1\cdots c_N} Q^1_{c_1}\cdots\cdots Q^{c_N}_{f_N} \\
&= (-1)^{f'-1}\times \mathcal{B}^{f'}
\end{aligned} \tag{S.4}$$

Similarly, the $N\times N$ square sub-matrix $\widetilde{Q}^{(\cancel{f})}$ of the antisquark matrix containing all flavors except f has determinant

$$\det\left(\widetilde{Q}^{(\cancel{f})}\right) \;=\; (-1)^{f-1}\times\widetilde{\mathcal{B}}^{f}. \tag{S.5}$$

Since the sub-matrices $\widetilde{Q}^{(\cancel{f})}$ and $Q^{(\cancel{f'})}$ include all the colors, their product

$$\mathcal{M}^{(\cancel{f})(\cancel{f'})} \;=\; \widetilde{Q}^{(\cancel{f})} Q^{(\cancel{f'})} \tag{S.6}$$

is an $N\times N$ sub-matrix of the meson matrix, including all antiquark flavors except f and all quark flavors except f'. By definition of the minor,

$$\begin{aligned}
\mathrm{minor}(\mathcal{M})^{ff'} &= (-1)^{f-f'}\times\det\left(\mathcal{M}^{(\cancel{f})(\cancel{f'})}\right) \\
&= (-1)^{f-f'}\times\det\left(\widetilde{Q}^{(\cancel{f})}\right)\times\det\left(Q^{(\cancel{f'})}\right) \\
&= \widetilde{\mathcal{B}}^{f}\times\mathcal{B}^{f'}.
\end{aligned} \tag{S.7}$$

Problem **2**(c):

Let's pick a flavor basis in which the meson matrix is diagonal, $\mathcal{M}=\mathrm{diag}(m_1,m_2,\ldots,m_{N_f})$. Since $\det(\mathcal{M})=0$, one eigenvalue must vanish, say $m_1=0$; the other eigenvalues could vanish too, but generically they don't, so let's assume $m_2,\ldots,m_{N_f}\neq 0$.

The condition $\mathcal{M}_{ff'}\mathcal{B}^{f'}$ makes the $\mathcal{B}^{f'}$ vector an eigenvector of the meson matrix corresponding to the zero eigenvalue. For the $\mathcal{M}=\mathrm{diag}(0,m_2,\ldots,m_{N_f})$, this means

$$\mathcal{B}^{f'} \;=\; \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{S.8}$$

Similarly, the condition $\widetilde{\mathcal{B}}^f \mathcal{M}_{ff'}$ restricts the antibaryon vector to

$$\widetilde{\mathcal{B}}^f \;=\; \Bigl(\tilde{b}, 0, \ldots, 0\Bigr). \tag{S.9}$$

Now consider the last condition (2). Given the restrictions we have already established, we automatically have

$$\operatorname{minor}(\mathcal{M})^{ff'} \;=\; 0 \;=\; \widetilde{\mathcal{B}}^f \times \mathcal{B}^{f'} \quad \text{for all } (f, f') \text{ pairs except } (1,1). \tag{S.10}$$

On the other hand, for $f = f' = 1$, we have one more constraint

$$\operatorname{minor}(\mathcal{M})^{1,1} \;=\; \prod_{f=2}^{N_f} m_f \;=\; \widetilde{\mathcal{B}}^1 \times \mathcal{B}^1 \;=\; \tilde{b}b. \tag{S.11}$$

Altogether, for a given meson matrix $\mathcal{M}$ with a zero eigenvalue, the constraints (2) fix all the baryonic and antibaryonic moduli except one: the ratio $\tilde{b}/b$.

Note that this result does not depend on the $\mathcal{M}$ matrix being actually diagonal, we used the diagonal form only to make our algebra simple. For a non-diagonal $\mathcal{M}$ with one zero eigenvalue, we have more complicated baryon and antibaryon vectors than (S.8) and (S.9) — they are the column eigenvector and the row eigenvector of $\mathcal{M}$ corresponding to the zero eigenvalue — but the net number of the un-fixed baryonic moduli remains the same: For a given meson matrix, the constraints (2) fix all the baryons and antibaryons in terms of just one free modulus.

So let's count the un-fixed moduli. There are N_f^2 mesons $\mathcal{M}_{ff'}$ subject to one constraint $\det(\mathcal{M}) = 0$, which gives us $N_f^2 - 1$ independent mesonic moduli. Given all these moduli, we need one more modulus to determine the baryons and the antibaryons. Thus, the net number of independent moduli is N_f^2 — which is precisely the number of the chiral multiplets left un-eaten by the Higgs mechanism, *cf.* part (a).

Problem $\mathbf{2}$(d):

In any SUSY vacuum of the effective theory we must have $\partial W/\partial(\text{any chiral SF}) = 0$. In particular,

$$\begin{aligned} \frac{\partial W}{\partial \widetilde{\mathcal{B}}^f} &= C\mathcal{M}_{ff'}\mathcal{B}^{f'} = 0, \\ \frac{\partial W}{\partial \mathcal{B}^{f'}} &= C\widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} = 0, \\ \frac{\partial W}{\partial \mathcal{M}_{ff'}} &= C\left(\widetilde{\mathcal{B}}^f \times \mathcal{B}^{f'} - \operatorname{minor}(\mathcal{M})^{ff'}\right) = 0, \end{aligned} \tag{S.12}$$

which reproduces three out of four constraints (2). The remaining constraint $\det(\mathcal{M})$ follows from these three: For any $N_f \times N_f$ matrix,

$$\sum_{ff'} \operatorname{minor}(\mathcal{M})^{ff'}\mathcal{M}_{ff'} = N_f \det(\mathcal{M}), \tag{S.13}$$

but given the chiral ring equations (S.12),

$$\sum_{ff'} \operatorname{minor}(\mathcal{M})^{ff'}\mathcal{M}_{ff'} = \sum_{ff'} \widetilde{\mathcal{B}}^f \mathcal{B}^{f'}\mathcal{M}_{ff'} = 0, \tag{S.14}$$

hence $\det(\mathcal{M}) = 0$.

Problem $\mathbf{2}(\star)$:

Without loss of generality, we may use the $SU(N_f)_L \times SU(N_f)_R$ flavor symmetries of the effective theory to choose a flavor basis where the $\langle\mathcal{M}\rangle_{ff'}$ matrix is diagonal. As we saw in part (c), this means

$$\langle\mathcal{M}\rangle = \operatorname{diag}(0, m_2, \ldots, m_{N_f}), \qquad \langle\mathcal{B}\rangle = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \left\langle\widetilde{\mathcal{B}}\right\rangle = (\tilde{b}, 0, \ldots, 0). \tag{S.15}$$

Now let's expand the moduli superfields around this point:

$$\mathcal{M}_{ff'} = \langle\mathcal{M}\rangle_{ff'} + \delta\mathcal{M}_{ff'}(y,\theta), \quad \mathcal{B}^f = \langle\mathcal{B}\rangle^f + \delta\mathcal{B}^f(y,\theta), \quad \widetilde{\mathcal{B}}^f = \left\langle\widetilde{\mathcal{B}}\right\rangle^f + \delta\widetilde{\mathcal{B}}^f(y,\theta) \tag{S.16}$$

without imposing any constraints on the $\delta\mathcal{M}, \delta\mathcal{B}, \delta\widetilde{\mathcal{B}}$ superfields. Instead, some of these $2N_f^2 + 2N_f$ superfields get their masses from the superpotential (4).

Indeed, expanding the determinant of the $\langle\mathcal{M}\rangle + \delta\mathcal{M}$ matrix, we obtain

$$\begin{aligned}\det\bigl(\mathcal{M}+\delta\mathcal{M}\bigr) \;&=\; 0 \;+\; \prod_{i>1} m_i \times \delta\mathcal{M}_{11}\\ &\quad+\sum_{j>1}\prod_{i\neq 1,j} m_i \times \bigl(\delta\mathcal{M}_{11}\delta\mathcal{M}_{jj} \;-\; \delta\mathcal{M}_{1j}\delta\mathcal{M}_{j1}\bigr) \;+\; O(\delta\mathcal{M}^3),\end{aligned} \tag{S.17}$$

while

$$\begin{aligned}\mathcal{B}^f\mathcal{M}_{ff'}\widetilde{\mathcal{B}}^{f'} \;&=\; 0 \;+\; b\tilde{b}\times\delta\mathcal{M}_{11}\\ &\quad+\sum_{j>1} m_j\times\delta\mathcal{B}^j\delta\widetilde{\mathcal{B}}^j \;+\; b\times\sum_j\delta\mathcal{M}_{1j}\delta\widetilde{\mathcal{B}}^j \;+\; \tilde{b}\times\sum_j\delta\mathcal{B}^j\delta M_{j1}\\ &\quad+\;\text{cubic terms.}\end{aligned} \tag{S.18}$$

Combining these expansions and using $b\times\tilde{b} = m_2\times\cdots m_{N_f}$, we obtain

$$\begin{aligned}W \;&=\; C\times\sum_{j=2}^{N_f}\Bigl(\delta\mathcal{M}_{1j},\delta\mathcal{B}^j\Bigr)\begin{pmatrix}\frac{b\tilde{b}}{m_j} & b\\ \tilde{b} & m_j\end{pmatrix}\begin{pmatrix}\delta\mathcal{M}_{j1}\\ \delta\widetilde{\mathcal{B}}^j\end{pmatrix}\\ &\quad+\;C\left(b\times\delta\widetilde{\mathcal{B}}^1 \;+\; \tilde{b}\delta\times\mathcal{B}^1 \;-\;\sum_{i>1}\frac{b\tilde{b}}{m_i}\times\delta\mathcal{M}_{ii}\right)\times\delta\mathcal{M}_{11}\\ &\quad+\;\text{cubic and higher-order terms.}\end{aligned} \tag{S.19}$$

Each term on the first line here seems to give mass to 4 chiral superfields, namely $\delta\mathcal{M}_{j1}$, $\delta\mathcal{M}_{1j}$, $\delta\mathcal{B}^j$, and $\delta\widetilde{\mathcal{B}}^j$. However, the 2×2 matrix

$$\begin{pmatrix}\frac{b\tilde{b}}{m_j} & b\\ \tilde{b} & m_j\end{pmatrix}$$

has rank $= 1$ (for generic b, $\tilde{b}$, and m_j), so only 2 combinations of these fields become massive while the other 2 remain massless. Combining $N_f - 1$ terms on the first line of (S.19) makes for $2(N_f-1)$ massive fields, and two more fields become massive thanks to the term on the second line, namely the $\delta\mathcal{M}_{11}$ and *one combination* of the $\delta\mathcal{B}^1$, $\delta\widetilde{\mathcal{B}}^1$, and $\delta\mathcal{M}_{ii}$ (for $i\neq 1$). Altogether, $2N_f$ combinations of the $\delta\mathcal{M}_{ff'}$, $\delta\mathcal{B}^f$, and $\delta\widetilde{\mathcal{B}}^f$ fields become massive while the remaining N_f^2 combinations remain massless.

The above counting of massive fields works in the generic case of $b \neq 0$, $\tilde{b} \neq 0$, and all $m_j \neq 0$ (for $j > 1$), or in the basis-independent terms, for $\langle\mathcal{B}\rangle \neq 0$, $\left\langle\widetilde{\mathcal{B}}\right\rangle \neq 0$, and $\mathrm{rank}(\langle\mathcal{M}\rangle) = N_f - 1$. But at some special points of the moduli space, some of the mass terms may vanish, so we end up with more than N_f^2 massless fields.

For example, suppose $\langle\mathcal{B}\rangle = \left\langle\widetilde{\mathcal{B}}\right\rangle = 0$ while $\mathrm{rank}(\langle\mathcal{M}\rangle) \leq N_f - 3$, which in the eigenbasis of $\langle\mathcal{M}\rangle$ means $b = \tilde{b} = 0$ and $m_2 = m_3 = 0$. Consequently, the second line of eq. (S.19) vanishes, while on the first line we get non-zero mass terms only for $m_j \neq 0$. Altogether, $2\,\mathrm{rank}(\langle\mathcal{M}\rangle)$ fields become massive, while $N_f^2 + 2N_f - 2\,\mathrm{rank}(\langle\mathcal{M}\rangle)$ fields remain massless.

In particular, at the special point $\langle\mathcal{B}\rangle = \left\langle\widetilde{\mathcal{B}}\right\rangle = \langle\mathcal{M}\rangle = 0$ — which classically corresponds to no squark VEVs at all — the superpotential (4) contains no mass terms at all, so all the $N_f^2 + 2N_f$ moduli fields are massless.

Problem **2**(e):

Here is the table of the $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ quantum numbers of all the quarks, antiquarks, mesons, baryons, and antibaryons:

fields \ QN	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$
Q	$\mathbf{N_f}$	$\mathbf{1}$	$+1$	$+1$
$\widetilde{Q}$	$\mathbf{1}$	$\mathbf{N_f}$	-1	$+1$
$\mathcal{M}$	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$+2$
$\mathcal{B}$	$\overline{\mathbf{N_f}}$	$\mathbf{1}$	$+N_c$	$+N_c$
$\widetilde{\mathcal{B}}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	$-N_c$	$+N_c$
$\Lambda^{3N_c-N_f}$	$\mathbf{1}$	$\mathbf{1}$	0	$+2N_f$

(S.20)

On the last line here, $\Lambda^{3N_c-N_f} \propto \exp(2\pi i\tau_w)$ is neutral with respect to the anomaly-free symmetries $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ but has a non-zero charge $+2N_f$ under the anomalous axial symmetry. This charge follows from the shift of the vacuum angle needed to cancel the axial anomaly:

$$\psi^\alpha \to e^{ia}\psi^\alpha, \quad \tilde{\psi}^\alpha \to e^{ia}\tilde{\psi}^\alpha, \quad \Theta \to \Theta + 2a \quad \Longrightarrow \quad \Lambda^{3N_c-N_f} \to e^{2N_f ia}\Lambda^{3N_c-N_f}. \tag{S.21}$$

As to the pure-R symmetry,

- All scalars, elementary or composite, have R-charge zero.
- All their fermionic superpartners have R-charge -1.
- The gauginos have R-charge $+1$.
- The instanton factor $\Lambda^{3N_c-N_f} \propto \exp(2\pi i\tau_w)$ has R-charge $-2N_f + 2N_c = -2$.

Problem **2**(f):

The effective superpotential must have R-charge +2 and be neutral/invariant under all the other flavor symmetries. Since all the moduli scalars $\mathcal{M}_{ff'}$, $\mathcal{B}^{f'}$, $\widetilde{\mathcal{B}}^f$ are R-neutral, the R–charge of the effective superpotential must come from the Λ, thus

$$W_{\text{eff}}\big(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}\big) \;=\; \Lambda^{N_f-3N_c}\times H(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}}) \tag{S.22}$$

for some holomorphic function H of the moduli scalars. (Note: $\Lambda^{N_f-3N_c} = \Lambda^{1-2N_c}$.) Next, the only holomorphic $SU(N_f)_L \times SU(N_f)_R$ invariant combinations of the moduli fields $\mathcal{M}_{ff'}$, $\widetilde{\mathcal{B}}^f$, and $\mathcal{B}^{f'}$ are $X = \widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'}$ and $Y = \det(M)$, hence H depends on the moduli only via these two invariants, thus

$$W_{\text{eff}}\big(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}\big) \;=\; \Lambda^{N_f-3N_c}\times F\Big(\Big(X=\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'}\Big),\big(Y=\det(\mathcal{M})\big)\Big) \tag{6}$$

for some holomorphic function $F(X,Y)$ of two arguments. Note that the two $SU(N_f)_L \times SU(N_f)_R$ invariants X and Y have zero net quark numbers, so we don't get additional restrictions from requiring the $U(1)_V$ invariance of the superpotential.

It remains to consider the axial symmetry $U(1)_A$. Under this symmetry both X and Y invariants have charge $A = +2N_f$ while the overall factor $\Lambda^{N_f-3N_c}$ has charge $A = -2N_f$. Consequently, to assure the $U(1)_A$ invariance of the superpotential (6), the holomorphic function

$F(X, Y)$ must be a homogeneous function of degree 1,

$$F(\alpha X, \alpha Y) \;=\; \alpha \times F(X, Y). \tag{S.23}$$

In principle, there is an infinite number of such functions. Indeed, for any holomorphic function $f(z)$ of a single argument,

$$F(X, Y) \;=\; X \times f(Y/X) \quad \text{is homogeneous of degree} = 1. \tag{S.24}$$

However, in the context of superpotential (6) we have an additional physical constraint: For large values of the moduli, the Higgsed SQCD is approximately semi-classical, so the effective superpotential should not have any singularities in this regime. Moreover, large moduli does not mean large $Y = \det(\mathcal{M})$ or large $X = \widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} \mathcal{B}^{f'}$ — to the contrary, we expect these invariants to stay small if not exactly zero due to classical constraints. Therefore, the holomorphic homogeneous function $F(X, Y)$ should not have any singularities at all, and the only such functions are polynomials of appropriate degree. In out case,

$$F(X, Y) \;=\; aX \,+\, bY \tag{S.25}$$

for some numeric constants a and b, and hence

$$W_{\text{eff}}\big(\mathcal{M}, \mathcal{B}, \widetilde{\mathcal{B}}; \Lambda^{3N_c - N_f}\big) \;=\; \frac{a}{\Lambda^{3N_c - N_f}} \times \widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} \mathcal{B}^{f'} \,+\, \frac{b}{\Lambda^{3N_c - N_f}} \times \det(\mathcal{M}), \quad \text{exactly.} \tag{S.26}$$

The classical superpotential (4) indeed has this form for $b = -a$.

Note that the powers of Λ here are completely fixed by the R–symmetry, which eliminates any instanton corrections carrying higher powers of Λ. Consequently, in the absence of tree-level masses or Yukawa couplings, *the semiclassical effective superpotential (4) is exact and does not suffer any quantum corrections whatsoever.*

In other words, *the chiral ring of SQCD with* $N_f = N_c + 1$ *does not suffer from any quantum corrections.*

Problem **3**(a):

Let's call the mixed RA charge R'. A quick look at the table (S.20) and the list of pure-R charges below immediately identifies the charge generating the symmetry (8) as

$$R' \;=\; R \;+\; \frac{1}{N_f}\,A. \tag{S.27}$$

The instanton parameter $\Lambda^{3N_c-N_f}$ has $A=+2N_f$ and $R=-2$, hence $R'=0$, which indicates that the R' charge is anomaly free.

Alternatively, you may calculate the color anomaly of the R' charge directly from eq. (8) for the fermion charges:

$$\begin{aligned} \text{anomaly} \;&=\; R'(\lambda)\times 2\,\text{Index}(\lambda) \;+\; R'(\psi)\times 2\,\text{Index}(\psi) \;+\; R'(\tilde\psi)\times 2\,\text{Index}(\tilde\psi) \\ &=\; (+1)\times 2N_c \;-\; \frac{N_c}{N_f}\times N_f \;-\; \frac{N_c}{N_f}\times N_f \\ &=\; +2N_c \;-\; N_c \;-\; N_c \;=\; 0. \end{aligned} \tag{S.28}$$

Problem **3**(b):

fields \ QN	$SU(N_f)_L$	$SU(N_f)_R$	V charge	R' charge	#colors
λ	$\mathbf{1}$	$\mathbf{1}$	0	$+1$	N_c^2-1
ψ_Q	$\mathbf{N_f}$	$\mathbf{1}$	$+1$	$\frac{1}{N_f}-1$	N_c
$\tilde\psi_Q$	$\mathbf{1}$	$\mathbf{N_f}$	-1	$\frac{1}{N_f}-1$	N_c
Ψ_M	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$\frac{2}{N_f}-1$	1
Ψ_B	$\overline{\mathbf{N_f}}$	$\mathbf{1}$	$+N_c$	$\frac{N_c}{N_f}-1$	1
$\widetilde\Psi_B$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	$-N_c$	$\frac{N_c}{N_f}-1$	1

(S.29)

Problem $\mathbf{2}$(c):

Let's calculate the non-trivial flavor anomalies.

- Three $SU(N_f)_L$ generators. By group theory,

$$\operatorname{tr}(T_L^a\{T_L^b,T_L^c\}) \;=\; A_L\times d^{abc} \tag{S.30}$$

where the coefficient A_L counts the number of fundamental $\mathbf{N_f}$ multiplets of the $SU(N_f)_L$ minus the number of anti-fundamental $\overline{\mathbf{N_f}}$ multiplets. (In theories with non-trivial tensor multiplets, they also count with non-trivial coefficients, but fortunately we don't have them here.) For the elementary fermions

$$A_L^{\rm elem} \;=\; A_L(\lambda)\,+\,A_L(\psi_Q)\,+\,A_L(\tilde{\psi}_Q) \;=\; 0\,+\,N_c\,+\,0 \;=\; N_c\,. \tag{S.31}$$

For the composite fermions

$$A_L^{\rm comp} \;=\; A_L(\Psi_M)\;+\;A_L(\Psi_B)\;+\;A_L(\widetilde{\Psi}_B) \;=\; N_f\;-\;1\;+\;0\;=\;N_c\,. \tag{S.32}$$

Anomalies match.

- Three $SU(N_f)_R$ generators. Again

$$\operatorname{tr}(T_L^a\{T_R^b,T_R^c\}) \;=\; A_R\times d^{abc} \quad\text{where } A_R \;=\; \#\mathbf{N_f}\;-\;\#\overline{\mathbf{N_f}}, \tag{S.33}$$

but now we count multiplets of the $SU(N_f)_R$. For the elementary fermions

$$A_R^{\rm elem} \;=\; A_R(\lambda)\,+\,A_R(\psi_Q)\,+\,A_R(\tilde{\psi}_Q) \;=\; 0\,+\,0\,+\,N_c \;=\; N_c \tag{S.34}$$

while for the composite fermions

$$A_R^{\rm comp} \;=\; A_R(\Psi_M)\;+\;A_R(\Psi_B)\;+\;A_R(\widetilde{\Psi}_B) \;=\; N_f\;+\;0\;-\;1\;=\;N_c\,. \tag{S.35}$$

The anomalies match.

- Two $SU(N_f)_L$ generators and one V charge. By group theory,

$$\mathrm{tr}(T_L^a T_L^b B) \ = \ \delta^{ab} \times A_{LLV} \quad \text{where } A_{LLV} \ = \sum_{\text{fermions}} V(\psi) \times \operatorname*{Index}_{SU(N_f)_L}(\psi) \times \#\psi \tag{S.36}$$

where $\#\psi$ counts the color and $SU(N_f)_R$ multiplicities of the fermionic species ψ. For the elementary fermions, $A_{LLV}(\lambda) = A_{LLV}(\tilde{\psi}_Q) = 0$, hence

$$A_{LLV}^{\text{elem}} \ = \ A_{LLV}(\psi_Q) \ = \ (+1) \times \frac{1}{2} \times N_c \ = \ +\frac{N_c}{2}. \tag{S.37}$$

For the composite fermions, $A_{LLV}(\Psi_M) = 0$ because M has $V = 0$ and $A_{LLV}(\widetilde{\Psi}_B) = 0$ because $\widetilde{\mathcal{B}}$ is a singlet of $SU(N_f)_L$, thus

$$A_{LLV}^{\text{comp}} \ = \ A_{LLV}(\Psi_B) \ = \ (+N_c) \times \frac{1}{2} \times 1 \ = \ +\frac{N_c}{2}. \tag{S.38}$$

The anomalies match.

- Two $SU(N_f)_R$ generators and one V charge. Similarly, for the elementary fermions

$$A_{RRV}^{\text{elem}} \ = \ 0 \ + \ 0 \ + \ A_{RRV}(\tilde{\psi}_Q) \ = \ (-1) \times \frac{1}{2} \times N_c \ = \ -\frac{N_c}{2} \tag{S.39}$$

while for the composite fermions

$$A_{RRV}^{\text{comp}} \ = \ 0 \ + \ 0 \ + \ A_{RRV}(\widetilde{\Psi}_B)\ (-N_c) \times \frac{1}{2} \times 1 \ = \ -\frac{N_c}{2}\,. \tag{S.40}$$

Anomalies match.

- Two $SU(N_f)_L$ generators and one R' charge. Again, among the elementary fermions, only the quarks contribute to this anomaly (because the gauginos and the antiquarks are

$SU(N_f)_L$ singlets), thus

$$A^{\text{elem}}_{LLR'} \;=\; A_{LLR'}(\psi_Q) \;=\; R'(\psi_Q)\times \operatorname*{Index}_L(\psi_Q)\times \#\psi_Q \;=\; \left(\frac{1}{N_f}-1\right)\times\frac{1}{2}\times N_c \;=\; -\frac{N_c^2}{2N_f}\,. \tag{S.41}$$

On the composite side,

$$\begin{aligned}
A_{LLR'}(\Psi_M) &= R'(\Psi_M)\times \operatorname*{Index}_L(\Psi_M)\times \#\Psi_M \;=\; \left(\frac{2}{N_f}-1\right)\times\frac{1}{2}\times N_f \;=\; \frac{2-N_f}{2}\,,\\
A_{LLR'}(\Psi_B) &= R'(\Psi_B)\times \operatorname*{Index}_L(\Psi_B)\times \#\Psi_B \;=\; \left(\frac{N_c}{N_f}-1\right)\times\frac{1}{2}\times 1 \;=\; -\frac{1}{2N_f}\,,\\
A_{LLR'}(\widetilde{\Psi}_B) &= 0 \quad \text{because } \operatorname*{Index}_L(\widetilde{\Psi}_B)=0,\\
\text{net } A^{\text{comp}}_{LLR'} &= \frac{2-N_f}{2} - \frac{1}{2N_f} \;=\; -\frac{(N_f-1)^2}{2N_f} \;=\; -\frac{N_c^2}{2N_f}\,,
\end{aligned} \tag{S.42}$$

and the anomalies match.

- Two $SU(N_f)_R$ generators and one R' charge. Similarly to the LLR' anomaly, on the elementary side $A_{RRR'}(\lambda) = A_{RRR'}(\psi_Q) = 0$ and

$$A^{\text{elem}}_{RRR'} \;=\; A_{RRR'}(\tilde{\psi}_Q) \;=\; \left(\frac{1}{N_f}-1\right)\times\frac{1}{2}\times N_c \;=\; -\frac{N_c^2}{2N_f}\,, \tag{S.43}$$

while on the composite side

$$\begin{aligned}
A_{RRR'}(\Psi_M) &= R'(\Psi_M)\times \operatorname*{Index}_R(\Psi_M)\times \#\Psi_M \;=\; \left(\frac{2}{N_f}-1\right)\times\frac{1}{2}\times N_f \;=\; \frac{2-N_f}{2}\,,\\
A_{LLR'}(\Psi_B) &= 0 \quad \text{because } \operatorname*{Index}_R(\widetilde{\Psi}_B)=0,\\
A_{RRR'}(\widetilde{\Psi}_B) &= R'(\Psi_B)\times \operatorname*{Index}_R(\Psi_B)\times \#\Psi_B \;=\; \left(\frac{N_c}{N_f}-1\right)\times\frac{1}{2}\times 1 \;=\; -\frac{1}{2N_f}\,,\\
\text{net } A^{\text{comp}}_{RRR'} &= \frac{2-N_f}{2} - \frac{1}{2N_f} \;=\; -\frac{(N_f-1)^2}{2N_f} \;=\; -\frac{N_c^2}{2N_f}\,,
\end{aligned} \tag{S.44}$$

and the anomalies match.

It remains to calculate and compare the purely abelian anomalies involving the net quark number V and the R' charge. By charge conjugation symmetry, the anomalies involving odd powers of V and even powers of R' are automatically zero, so we need to check only the remaining $\mathrm{tr}(VVR')$, $\mathrm{tr}(R'R'R')$, and $\mathrm{tr}(R')$ anomalies.

- Two V charges and one R' charge. On the elementary side, the gauginos do not contribute to this anomaly while the quarks and the antiquarks contribute

$$A_{VVR'}(\psi_Q) \;=\; A_{VVR'}(\tilde{\psi}_Q) \;=\; V^2\times R'\times\# \;=\; (\pm1)^2\times\left(\frac{1}{N_f}-1\right)\times N_cN_f \;=\; -N_c^2, \tag{S.45}$$

thus net $A^{\mathrm{elem}}_{VVR'} = -2N_c$. On the composite side, the mesons do not contribute (because of $V=0$) while the baryons and the antibaryons contribute

$$A_{VVR'}(\Psi_B) \;=\; A_{VVR'}(\widetilde{\Psi}_B) \;=\; V^2\times R'\times\# \;=\; (\pm N_c)^2\times\left(\frac{N_c}{N_f}-1\right)\times N_f \;=\; -N_c^2$$

so the net composite anomaly $A^{\mathrm{comp}}_{VVR'} = -2N_c^2$ matches the net elementary anomaly.

- Three R' charges. This time all fermionic species contribute to this anomaly. On the elementary side,

$$\begin{aligned}
A_{R'R'R'}(\lambda) \;&=\; R'^3(\lambda)\times\#\lambda \;=\; (+1)^3\times(N_c^2-1) \;=\; N_c^2-1,\\
A_{R'R'R'}(\psi_Q) \;&=\; R'^3(\psi_Q)\times\#\psi_Q \;=\; \left(\frac{1}{N_f}-1\right)^3\times N_cN_f \;=\; -\frac{N_c^4}{N_f^2}\,,\\
A_{R'R'R'}(\tilde{\psi}_Q) \;&=\; R'^3(\tilde{\psi}_Q)\times\#\tilde{\psi}_Q \;=\; \left(\frac{1}{N_f}-1\right)^3\times N_cN_f \;=\; -\frac{N_c^4}{N_f^2}\,,\\
\text{net } A^{\mathrm{elem}}_{R'R'R'} \;&=\; N_c^2\,-\,1\,-\,\frac{2N_c^4}{N_f^2} \;=\; -N_f^2\,+\,6N_f\,-\,12\,+\,\frac{8}{N_f}\,-\,\frac{2}{N_f^2}\,,
\end{aligned} \tag{S.46}$$

while on the composite side

$$\begin{aligned}
A_{R'R'R'}(\Psi_M) \;&=\; R'^3(\Psi_M)\times\#\Psi_M \;=\; \left(\frac{2}{N_f}-1\right)^3\times N_f^2 \;=\; -\frac{(N_f-2)^3}{N_f}\,,\\
A_{R'R'R'}(\Psi_B) \;&=\; R'^3(\Psi_B)\times\#\Psi_B \;=\; \left(\frac{N_c}{N_f}-1\right)^3\times N_f \;=\; -\frac{1}{N_f^2}\,,\\
A_{R'R'R'}(\widetilde{\Psi}_B) \;&=\; R'^3(\widetilde{\Psi}_B)\times\#\widetilde{\Psi}_B \;=\; \left(\frac{N_c}{N_f}-1\right)^3\times N_f \;=\; -\frac{1}{N_f^2}\,,\\
\text{net } A^{\mathrm{comp}}_{R'R'R'} \;&=\; -\frac{(N_f-2)^3}{N_f}\;-\;\frac{2}{N_f^2} \;=\; -N_f^2\,+\,6N_f\,-\,12\,+\,\frac{8}{N_f}\,-\,\frac{2}{N_f^2}\,,
\end{aligned} \tag{S.47}$$

and the net anomalies match.

- The trace anomaly, a single R' charge. For the elementary fermions,

$$\begin{aligned} A_{R'}(\lambda) &= R'(\lambda)\times\#\lambda = (+1)\times(N_c^2-1) = N_c^2-1, \\ A_{R'}(\psi_Q) &= R'(\psi_Q)\times\#\psi_Q = \left(\frac{1}{N_f}-1\right)\times N_cN_f = -N_c^2\,, \\ A_{R'}(\tilde{\psi}_Q) &= R(\tilde{\psi}_Q)\times\#\tilde{\psi}_Q = \left(\frac{1}{N_f}-1\right)\times N_cN_f = -N_c^2\,, \\ \text{net } A^{\text{comp}}_{R'R'R'} &= -N_c^2-1, \end{aligned} \tag{S.48}$$

while on the composite side

$$\begin{aligned} A_{R'}(\Psi_M) &= R'(\Psi_M)\times\#\Psi_M = \left(\frac{2}{N_f}-1\right)\times N_f^2 = -(N_f-2)N_f\,, \\ A_{R'}(\Psi_B) &= R'(\Psi_B)\times\#\Psi_B = \left(\frac{N_c}{N_f}-1\right)\times N_f = -1\,, \\ A_{R'}(\widetilde{\Psi}_B) &= R'(\widetilde{\Psi}_B)\times\#\widetilde{\Psi}_B = \left(\frac{N_c}{N_f}-1\right)\times N_f = -1\,, \\ \text{net } A^{\text{comp}}_{R'R'R'} &= -N_f^2+2N_f-2 = -N_c^2-1, \end{aligned} \tag{S.49}$$

and the trace anomalies match too.

This completes our check of the 't Hooft's anomaly matching conditions.