SQCD

Basics

As a prototypical supersymmetric gauge theory, consider SQCD — the supersymmetrized QCD — with N colors and 1 flavor. The vector fields $\mathcal{A}^a_\mu(x)$ for $a = 1, \ldots, (N^2 - 1)$ are accompanied by the gauginos — LH Weyl fermions λ^a_α and their hermitian conjugates $\bar{\lambda}^a_{\dot{\alpha}}$ — and auxiliary fields $\mathcal{D}^a$. In the superfield formalism, they are packaged into real superfields $V^a(x, \theta, \bar{\theta})$, which form an adjoint multiplet of the $SU(N)$ gauge group. Or in matrix notations, there is a traceless hermitian matrix of vector superfields

$$\mathcal{V}^i_{\ j}(x, \theta, \bar{\theta}) \;=\; \sum_a gV^a(x, \theta, \bar{\theta}) \times \left(t^a\right)^i_{\ j} \tag{1}$$

where $t^a = \frac{1}{2}\lambda^a_{\text{Gell-Mann}}$ are the $SU(N)$ generators in the fundamental representation of the group.

Each Dirac quark $\Psi^i(x)$ (together with its conjugate $\overline{\Psi}_i(x)$) is equivalent to 2 LH Weyl fermions (plus their conjugates): the LH quark and the LH antiquark. Each of these Weyl fermions comes with a complex scalar squark and an auxiliary field, and in the superfield formalism, all these component fields are packaged into $2N$ chiral superfields $A^i(y, \theta)$ and $B_i(y, \theta)$, plus their anti-chiral hermitian conjugates $\overline{A}_i(\bar{y}, \bar{\theta})$ and $\overline{B}^i(\bar{y}, \bar{\theta})$. The A^i form the fundamental multiplet $\mathbf{N}$ of the $SU(N)$ gauge group while the B_i form the conjugate multiplet $\overline{\mathbf{N}}$, so under the ordinary gauge transforms $U(x) \in SU(N)$, these superfields transform as

$$A'^i(y, \theta) \;=\; U^i_{\ j}(x) A^j(y, \theta), \qquad B'_i(y, \theta) \;=\; U^{*\,j}_{i}(x) B_j(y, \theta). \tag{2}$$

Let's assemble the A^i into a column vector of length N while the B_i form a row vector of the same length; then in matrix notations eqs. (2) become

$$A'(y, \theta) \;=\; U(x) \times A(y, \theta), \qquad B'(y, \theta) \;=\; B(y, \theta) \times U^\dagger(x) \;=\; B(y, \theta) \times U^{-i}(x). \tag{3}$$

These are the ordinary — *i.e.*, non-supersymmetric — gauge transforms. Under the supersymmetric gauge transforms, we generalize these formulae from x-dependent but not

θ-dependent unitary matrices $U(x)$ to more general matrices $U(y,\theta)$ of chiral superfields. To determine the nature of such matrices, we start by writing the general $SU(N)$ matrix $U(x)$ as $\exp(i\Lambda(x))$ for some traceless hermitian x-dependent matrix $\Lambda(x)$ and hence

$$U(x) \;=\; \exp\Bigl(i\sum\nolimits_a \Lambda^a(x)\times t^a\Bigr) \tag{4}$$

where $\Lambda^a(x)$ are ordinary real scalar fields. Now let's promote these real scalar fields $\Lambda^a(x)$ to arbitrary chiral superfields $\Lambda^a(y,\theta)$. Consequently, we get a general traceless matrix of chiral superfields

$$\Lambda(y,\theta) \;=\; \sum_a \Lambda^a(y,\theta)\times t^a \tag{5}$$

and hence

$$U(y,\theta) \;=\; \exp\bigl(i\Lambda(y,\theta)\bigr). \tag{6}$$

Note that the lowest components (for $\theta=0$, $y=x$) of the chiral superfields Λ^a are complex rather than real scalar fields, hence the lowest component of the Λ matrix is traceless but not hermitian and the lowest component of the U matrix has a unit determinant but it's not unitary. Instead it's a general $SL(N,\mathbf{C})$ matrix!

So here is the bottom line for the supersymmetric gauge transforms of the quark superfields $A^i(y,\theta)$ and $B_i(y,\theta)$: In matrix notations

$$A'(y,\theta) \;=\; \exp\bigl(+i\Lambda(y,\theta)\bigr)\times A(y,\theta), \qquad B'(y,\theta) \;=\; B(y,\theta)\times\exp\bigl(-i\Lambda(y,\theta)\bigr), \tag{7}$$

where $\Lambda(y,\theta)=\sum_a \Lambda^a(y,\theta)\times t^a$ for the most general chiral superfields $\Lambda^a(y,\theta)$. Also, the hermitian conjugates $\overline{\Lambda}^a(\bar y,\bar\theta)$ of these $\Lambda^a(y,\theta)$ superfields govern the gauge transforms of the antichiral quark superfields $\overline{A}_i(\bar y,\bar\theta)$ and $\overline{B}^i(\bar y,\bar\theta)$: In matrix notations — where the $\overline{A}_i$ form a row vector while the $\overline{B}^i$ form a column vector, —

$$\overline{A}'(\bar y,\bar\theta) \;=\; \overline{A}(\bar y,\bar\theta)\times\exp\bigl(-i\overline{\Lambda}(\bar y,\bar\theta)\bigr), \qquad \overline{B}'(\bar y,\bar\theta) \;=\; \exp\bigl(+i\overline{\Lambda}(\bar y,\bar\theta)\bigr)\times\overline{B}(\bar y,\bar\theta), \tag{8}$$

for $\overline{\Lambda}(\bar y,\bar\theta)=\sum_a \overline{\Lambda}^a(\bar y,\bar\theta)\times t^a$.

Next, consider the gauge-invariant kinetic terms in the Lagrangian for the quark superfields, By analogy with SQED, we write these terms as

$$\mathcal{L}_{\text{kin}} = \int d^4\theta \Big(\overline{A} \times \exp(+2\mathcal{V}) \times A + B \times \exp(-2\mathcal{V}) \times \overline{B} \Big) \tag{9}$$

where

$$\mathcal{V} = \sum_a g V^a(x,\theta,\bar{\theta}) \times t^a \tag{10}$$

combines the vector superfields V^a of SQCD into a hermitian-matrix-valued superfield. To keep both terms of (9) invariant under the supersymmetric gauge transforms (7) and (8) of the quark superfields, the vector superfields should transform such that

$$\begin{aligned} \exp(+2\mathcal{V}') &= \exp(+i\overline{\Lambda}) \times \exp(+2\mathcal{V}) \times \exp(-i\Lambda), \\ \exp(-2\mathcal{V}') &= \exp(+i\Lambda) \times \exp(-2\mathcal{V}) \times \exp(-i\overline{\Lambda}); \end{aligned} \tag{11}$$

note that these two matrix conditions are equivalent to each other rather than independent. Indeed, if the vector field transforms as in eq. (11) while the quark superfields transform as in eqs. (7) and (8), then

$$\begin{aligned} \overline{A}' \times \exp(+2\mathcal{V}") \times A &= \overline{A} \exp(-i\overline{\Lambda}) \times \exp(+i\overline{\Lambda}) \exp(+2\mathcal{V}) \exp(-i\Lambda) \times \exp(i\Lambda) A \\ &= \overline{A} \times \exp(+2\mathcal{V}) \times A, \\ B' \times \exp(-2\mathcal{V}') \times \overline{B}' &= B' \exp(-i\Lambda) \times \exp(+i\Lambda) \exp(-2\mathcal{V}) \exp(-i\overline{\Lambda}) \times \exp(i\overline{\Lambda}) \overline{B} \\ &= B \times \exp(-2\mathcal{V}) \times \overline{B}. \end{aligned} \tag{12}$$

In terms of the vector superfield $\mathcal{V}$ itself rather than its exponential, the non-abelian gauge transform (11) amounts to

$$\begin{aligned} \mathcal{V}' = \mathcal{V} &+ \tfrac{i}{2}\big(\overline{\Lambda} - \Lambda\big) + \tfrac{1}{4}\big[\overline{\Lambda}, \Lambda\big] + \tfrac{i}{2}\big[\overline{\Lambda} + \Lambda, \mathcal{V}\big] \\ &+ \text{ multiple commutators.} \end{aligned} \tag{13}$$

Despite the non-abelian commutator terms in this formula, the leading $\frac{i}{2}(\overline{\Lambda} - \Lambda)$ term here allows to bring any $\mathcal{V}$ superfield to the Wess–Zumino gauge:

$$\mathcal{V}(x,\theta,\bar{\theta}) = (\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}) \mathcal{A}_\mu(x) + \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \tfrac{1}{2} \theta^2 \bar{\theta}^2 \mathcal{D}, \tag{14}$$

where $\mathcal{A}_\mu$, λ_α, $\bar{\lambda}_{\dot{\alpha}}$, and $\mathcal{D}$ are matrix-valued component fields.

Proof: Let's start by eliminating the lowest component of the vector superfield

$$C(x) \;=\; \mathcal{V}|_{\theta=\bar{\theta}=0} \tag{15}$$

by letting $\Lambda_1(x) = -iC(x)$, $\overline{\Lambda}_1(x) = +iC$ without any higher components. Consequently,

$$\exp(2\mathcal{V}')|_{\theta=\bar{\theta}=0} \;=\; e^{-C} \times e^{+2C} \times e^{-C} \;=\; 1 \quad\Longrightarrow\quad C' \;=\; 0. \tag{16}$$

Note that this gauge transform changes all components of the $\mathcal{V}$ superfield, but that's OK as long as it eliminates the lowest component. Thus, at the end of the first stage we have

$$V(x,\theta\bar{\theta}) \;=\; \theta^\alpha \chi_\alpha(x) \;+\; \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) \;+\; \cdots$$

where $\cdots$ stand for terms with higher powers of θ or $\bar{\theta}$. So let the second-stage gauge transform be parametrized by

$$\Lambda_2(y,\theta) \;=\; -2i\theta^\alpha \chi_\alpha(y), \qquad \overline{\Lambda}_2(\bar{y},\bar{\theta}) \;=\; +2i\bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(y), \tag{17}$$

which eliminates all the terms of the first-order in θ or $\bar{\theta}$ without re-introducing the C term

$$\exp(2\mathcal{V}'') \;=\; \exp(i\overline{\Lambda}) \times \exp(2\mathcal{V}') \times \exp(-i\Lambda) \;=\; 1 \;+\; \text{terms at least quadratic in}(\theta,\bar{\theta}), \tag{18}$$

hence

$$V''(x,\theta,\bar{\theta}) \;=\; \theta^2 f(x) \;+\; \bar{\theta}^2 f^*(x) \;+\; (\theta\sigma^\mu\bar{\theta})\mathcal{A}_\mu(x) \;+\; \text{higher-order terms.} \tag{19}$$

Again, the f, f^*, and $\mathcal{A}_\mu$ can be different than the similar components of the V', but that's OK as long as all the zero-order and first-order components are eliminated.

Finally, we perform the third-stage gauge transform using $\Lambda_3 = -2i\theta^2 f(x)$, $\overline{\Lambda}_3 = +2i\bar{\theta}^2 f^*(x)$, which eliminates the f and f^* of the vector superfield $\mathcal{V}'''$ and brings it to the Wess-Zumino gauge (14). *Quod erat demonstrandum.*

Next, consider the tension superfields. For an abelian vector superfield V, the tension superfields are simply

$$\text{chiral } W_\alpha \;=\; -\tfrac{1}{4}\overline{D}^2 D_\alpha V \quad \text{and antichiral } \overline{W}_{\dot{\alpha}} \;=\; -\tfrac{1}{4}D^2\overline{D}_{\dot{\alpha}} V. \tag{20}$$

Their non-abelian analogies for the matrix-valued vector superfield $\mathcal{V}$ are

$$\text{chiral } \mathcal{W}_\alpha \;=\; -\tfrac{1}{8}\overline{D}^2\bigl(e^{-2\mathcal{V}} D_\alpha e^{+2\mathcal{V}}\bigr) \quad \text{and antichiral } \overline{\mathcal{W}}_{\dot{\alpha}} \;=\; +\tfrac{1}{8}D^2\bigl(e^{+2\mathcal{V}}\overline{D}_{\dot{\alpha}} e^{-2\mathcal{V}}\bigr). \tag{21}$$

However, unlike the abelian tensions (20), their non-abelian counterparts (21) are not gauge invariant. Instead, they gauge-transform as adjoint multiplets of chiral / antichiral superfields,

$$\begin{aligned}
\mathcal{W}'_\alpha(y,\theta) &= \exp\bigl(+i\Lambda(y,\theta)\bigr) \times \mathcal{W}_\alpha(y,\theta) \times \exp\bigl(-i\Lambda(y,\theta)\bigr), \\
\overline{\mathcal{W}}'_{\dot\alpha}(\bar y,\bar\theta) &= \exp\bigl(+i\overline{\Lambda}(\bar y,\bar\theta)\bigr) \times \overline{\mathcal{W}}_{\dot\alpha}(\bar y,\bar\theta) \times \exp\bigl(-i\overline{\Lambda}(\bar y,\bar\theta)\bigr).
\end{aligned} \tag{22}$$

Indeed, consider

$$\Gamma_\alpha \stackrel{\rm def}{=} e^{-2\mathcal{V}} D_\alpha e^{+2\mathcal{V}}. \tag{23}$$

Under gauge transforms (11) of the vector superfield $\mathcal{V}$, this Γ_α transforms to

$$\begin{aligned}
\Gamma'_\alpha &= e^{-2\mathcal{V}'} D_\alpha e^{+2\mathcal{V}} = e^{+i\Lambda} e^{-2\mathcal{V}} e^{-i\overline{\Lambda}} \times D_\alpha\bigl(e^{+i\overline{\Lambda}} e^{+2\mathcal{V}} e^{-i\Lambda}\bigr) \\
&= e^{+i\Lambda} e^{-2\mathcal{V}} \times \bigl(e^{-i\overline{\Lambda}} \times D_\alpha e^{+i\overline{\Lambda}}\bigr) \times e^{+2\mathcal{V}} e^{-i\Lambda} \\
&\qquad + e^{+i\Lambda} \times \bigl(e^{-2\mathcal{V}} D_\alpha e^{+2\mathcal{V}}\bigr) \times e^{-i\Lambda} + e^{+i\Lambda} D_\alpha e^{-i\Lambda} \\
&= 0 \quad \langle\!\langle\, \text{because } D_\alpha \overline{\Lambda} = 0 \,\rangle\!\rangle \\
&\qquad + e^{+i\Lambda} \times \Gamma_\alpha \times e^{-i\Lambda} + e^{+i\Lambda} D_\alpha e^{-i\Lambda}.
\end{aligned} \tag{24}$$

Next, since $\Lambda(y,\theta)$ is chiral, $\overline{D}_{\dot\beta}\Lambda = 0$, we have

$$\overline{D}^2\bigl(e^{+i\Lambda} \times \Gamma_\alpha \times e^{-i\Lambda}\bigr) = e^{+i\Lambda} \times \bigl(\overline{D}^2 \Gamma_\alpha\bigr) \times e^{-i\Lambda}, \tag{25}$$

while

$$\begin{aligned}
\overline{D}^2\bigl(e^{+i\Lambda} D_\alpha e^{-i\Lambda}\bigr) &= e^{+i\Lambda} \times \bigl[\overline{D}^2, D_\alpha\bigr] e^{-i\Lambda} \\
&= e^{+i\Lambda} \times \bigl(-4i\partial_\mu \sigma^\mu_{\alpha\dot\beta} \overline{D}^{\dot\beta}\bigr) e^{-i\Lambda} \\
&= 0.
\end{aligned} \tag{26}$$

Thus, taking $\overline{D}^2$ of both sides of eq. (24), we get

$$\overline{D}^2 \Gamma'_\alpha = e^{+i\Lambda} \times \bigl(\overline{D}^2 \Gamma_\alpha\bigr) \times e^{-i\Lambda} \tag{27}$$

and hence

$$\mathcal{W}'_\alpha = e^{+i\Lambda} \times \mathcal{W}_\alpha \times e^{-i\Lambda}. \tag{28}$$

Likewise, let

$$\overline{\Gamma}_{\dot\alpha} \stackrel{\text{def}}{=} e^{+2\mathcal{V}}\overline{D}_{\dot\alpha}e^{-2V}. \tag{29}$$

Then under gauge transform (11) of the vector superfield this $\overline{\Gamma}_{\dot\alpha}$ transforms to

$$\overline{\Gamma}'_{\dot\alpha} = 0 + e^{+i\overline{\Lambda}} \times \overline{\Gamma}_{\dot\alpha} \times e^{-i\overline{\Lambda}} + e^{+i\overline{\Lambda}}\overline{D}_{\dot\alpha}e^{-i\overline{\Lambda}} \tag{30}$$

and hence

$$D^2\overline{\Gamma}'_{\dot\alpha} = 0 + e^{+i\overline{\Lambda}} \times \left(D^2\overline{\Gamma}_{\dot\alpha}\right) \times e^{-i\overline{\Lambda}} + 0, \tag{31}$$

thus

$$\overline{\mathcal{W}}'_{\dot\alpha} = e^{+i\overline{\Lambda}} \times \overline{\mathcal{W}}_{\dot\alpha} \times e^{-i\overline{\Lambda}}. \tag{32}$$

Quod erat demonstrandum.

However, while the non-abelian tension superfields themselves are not gauge invariant, we may form gauge-invariant quadratic combinations of these superfields; in matrix form, these invariant combinations are the traces

$$\operatorname{tr}\left(\mathcal{W}^\alpha\mathcal{W}_\alpha\right) \quad \text{and} \quad \operatorname{tr}\left(\overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha}\right). \tag{33}$$

The gauge-invariant kinetic terms for the gauge fields and their superpartners — indeed, the entire super–Yang–Mills (SYM) Lagrangian — obtains from the $d^2\theta$ / $d^2\bar\theta$ integrals of these invariant traces as

$$\mathcal{L}_{\text{SYM}} = \frac{i\tau}{8\pi}\int d^2\theta\, \operatorname{tr}\left(\mathcal{W}^\alpha\mathcal{W}_\alpha\right) - \frac{i\tau^*}{8\pi}\int d^2\bar\theta\, \operatorname{tr}\left(\overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha}\right) \tag{34}$$

where

$$\tau \stackrel{\text{def}}{=} \frac{4\pi i}{g^2} + \frac{\Theta}{2\pi} \tag{35}$$

is the *complex gauge coupling* of the theory combining the ordinary gauge coupling g and the instanton angle Θ. As we shall learn later in class, this combination is particularly convenient for the electric-magnetic dualities of gauge theories. In other contexts, a more convenient combination is

$$f \;=\; -2\pi i\times\tau \;=\; \frac{8\pi^2}{g^2}\;-\;i\Theta, \tag{36}$$

but either way, the (inverse) gauge coupling and the instanton angle are combined in a single complex parameter τ or f.

To work out the SYM Lagrangian (34) in terms of the component fields, let's put the vector superfield $\mathcal{V}$ in the Wess–Zumino gauge. (Because otherwise, the unphysical components of the $\mathcal{V}$ would not completely decouple from the non-abelian terms in the Lagrangian.) In this gauge

$$\begin{aligned}\mathcal{W}_\alpha(y,\theta) \;&=\; \lambda_\alpha(y)\;+\;\theta_\alpha\mathcal{D}(y)\;+\;\tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta}\theta_\beta\mathcal{F}_{\mu\nu}(y)\;-\;i\theta^2\sigma^\mu_{\alpha\dot\beta}D_\mu\bar\lambda^{\dot\beta},\\ \overline{\mathcal{W}}^{\dot\alpha}(\bar y,\bar\theta)\;&=\;\bar\lambda^{\dot\alpha}(\bar y)\;+\;\bar\theta^{\dot\alpha}\mathcal{D}(\bar y)\;-\;\tfrac{i}{2}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}_{\ \dot\beta}\bar\theta^{\dot\beta}\mathcal{F}_{\mu\nu}(y)\;+\;i\bar\theta^2\bar\sigma^{\mu,\dot\alpha\beta}D_\mu\lambda_\beta(\bar y),\end{aligned} \tag{37}$$

for the non-abelian

$$\begin{aligned}\mathcal{F}_{\mu\nu}\;&=\;\partial_\mu\mathcal{A}_\nu\;-\;\partial_\nu\mathcal{A}_\mu\;+\;i[\mathcal{A}_\mu,\mathcal{A}_\nu],\\ D_\mu\lambda_\beta\;&=\;\partial_\mu\lambda_\beta\;+\;i[\mathcal{A}_\mu,\lambda_\beta],\\ D_\mu\bar\lambda^{\dot\beta}\;&=\;\partial_\mu\bar\lambda^{\dot\beta}\;+\;i[\mathcal{A}_\mu,\bar\lambda^{\dot\beta}].\end{aligned} \tag{38}$$

Consequently,

$$\begin{aligned}\mathcal{L}_{\mathrm{SYM}}\;=\;&\frac{1}{g^2}\,\mathrm{tr}\left(-\tfrac12\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\;+\;i\bar\lambda_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\beta}\overset{\leftrightarrow}{D}_\mu\lambda_\beta\;+\;\mathcal{D}^2\right)\\ &+\;\frac{\Theta}{16\pi^2}\,\mathrm{tr}\left(\epsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}\right)+\;\frac{i\Theta}{16\pi^2}\,\partial_\mu\,\mathrm{tr}\big(\bar\lambda_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\beta}\lambda_\beta\big).\end{aligned} \tag{39}$$

Finally, let me write down the complete Lagrangian of SQCD with N colors and one flavor:

$$\begin{aligned}\mathcal{L}_{\mathrm{SQCD}}\;=\;&\int\! d^4\theta\Big(\overline{A}\exp(+2\mathcal{V})A\;+\;B\exp(-2\mathcal{V})\overline{B}\Big)\\ &+\int\! d^2\theta\Big(\frac{i\tau}{8\pi}\,\mathrm{tr}\big(\mathcal{W}^\alpha\mathcal{W}_\alpha\big)\;+\;mBA\Big)\;+\int\! d^2\bar\theta\Big(\frac{-i\tau^*}{8\pi}\,\mathrm{tr}\big(\overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha}\big)\;+\;m^*\overline{AB}\Big)\end{aligned} \tag{40}$$

where m is the quark mass.

Higgs Regime

For $m = 0$, the classical scalar potential of SQCD has flat directions, and the for non-zero squark VEVs $\langle A\rangle$ and $\langle B\rangle$ along these flat directions lead to partial Higgsing of the $SU(N)$ gauge symmetry down to $SU(N-1)$. In a later class, we shall learn that these classical flat directions remain flat to all orders of the perturbation theory, but the non-perturbative effects spoil the flatness. But for the moment, let us stick to the classical scalar potential for the squark fields A^i and B_i.⋆ In the Wess–Zumino gauge,

$$V_{\text{scalar}} \;=\; \frac{1}{2g^2}\sum_a \bigl(\mathcal{D}^a\bigr)^2 \;=\; \frac{g^2}{2}\sum_a \bigl(A^\dagger t^a A \,-\, Bt^a B^\dagger\bigr)^2. \tag{41}$$

This non-negative potential vanishes when

$$A^\dagger t^a A \;=\; Bt^a B^\dagger \quad \forall\,\text{Gell-Mann matrices } 2t^a, \tag{42}$$

which happens if and only if A and $B^\dagger$ are the same column vectors up to an overall phase,

$$B^{*i} \;=\; A^i \times e^{i\,\text{phase}}, \quad \text{same phase for all } i = 1, \ldots, N. \tag{43}$$

Thus, the classical vacuum space of SQCD is parametrized by scalar VEVs $\langle A^i\rangle$ and $\langle B_i\rangle$ obeying the relations (43). Naively, such VEVs have $2N + 1$ independent real parameters, but that's before we take the gauge redundancy into account.

Note that the Wess–Zumino gauge condition does not fix the ordinary gauge transforms but only their superpartners, so the scalar VEVs related by the ordinary gauge symmetries are physically equivalent to each other. In particular, any set of $\langle A^i\rangle$ is gauge-equivalent to the

$$\langle A\rangle \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a \end{pmatrix} \tag{44}$$

⋆ By abuse of notations, we use the same letters A and B for the chiral superfields and their scalar components.

for some complex a. In this gauge, eqs. (43) call for

$$\left\langle B^{\dagger}\right\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b^* \end{pmatrix} \quad \Longrightarrow \quad \langle B\rangle = \begin{pmatrix} 0 & \cdots & 0 & b \end{pmatrix} \tag{45}$$

where $|b| = |a|$. These VEVs Higgs the $SU(N)$ gauge symmetry down to its $SU(N-1)$ subgroup acting on the first $N-1$ colors. Furthermore, a and b have opposite charges WRT $U(1)$ subgroup of $SU(N)$ that commutes with the unbroken $SU(N-1)$, so we may use this $U(1)$ symmetry to make $a = b = \text{renamed} = \Phi$, thus

$$\langle A\rangle = \Phi \times \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad \langle B\rangle = \Phi \times \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{46}$$

for an arbitrary complex *modulus* Φ. Note that the vacuum space of the one-flavor SQCD does not have any other moduli besides Φ, so the physical moduli space has complex dimension 1 or real dimension 2.

Now let's count the degrees of freedom of the Higgsed down SQCD. The original $SU(N)$ gauge symmetry of the theory is Higgsed down to $SU(N-1)$, so the theory has $(N-1)^2-1$ massless vector supermultiplets and $2N-1$ massive vector supermultiplets. The theory also starts with $2N$ scalar supermultiplets A^i and B_i, but the SUSY Higgs mechanism eats a whole scalar supermultiplet for each vector supermultiplet that becomes massive, so $2N-1$ out of $2N$ scalar supermultiplets are eaten and there is only one un-eaten scalar supermultiplet, namely the modulus b and its superpartner.

Each massive vector supermultiplet comprises a massive vector, a Dirac fermion (or equivalently two Weyl fermions), and a real scalar. In the component field formulation, the vector masses arise via the ordinary Higgs mechanism, the fermion masses stem from the Yukawa couplings $\Phi^*\lambda\psi + \text{H.c.}$ to the squark VEVs, and the real scalar's masses stem from the non-flat directions of the scalar potential. You should work out all these masses by yourselves as a part of your homework set#2. And at the end of this exercise, you should see that thanks to the unbroken SUSY, all members of the same vector supermultiplet V^a get the same mass — despite seemingly different origins of these masses.

But in these notes, I am going to focus on the manifestly supersymmetric picture of the Higgs mechanism and masses of vector multiplets. So instead of using a SUSY-breaking Wess–Zumino gauge, let's use a supersymmetric unitary gauge in which

$$A^i(y,\theta)\ =\ \Phi(y,\theta)\times\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\quad\text{and}\quad B_i(y,\theta)\ =\ \Phi(y,\theta)\times\begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix}\tag{47}$$

as chiral superfields, for any (y,θ). Note that the supersymmetrized gauge transforms of chiral superfields

$$A^i(y,\theta)\ \to\ U^i_{\ j}(y,\theta)\times A^j(y,\theta),\qquad B_i(y,\theta)\ \to\ B_j(y,\theta)\times\left(U^{-1}(y,\theta)\right)^j_{\ i}\tag{48}$$

are parametrized by the $SL(N,\mathbf{C})$ — but generally non-unitary — matrices

$$U^i_{\ j}(y,\theta)\ =\ \exp\left(i\sum\nolimits_a\Lambda^a(y,\theta)t^a\right)^i_{\ j}.\tag{49}$$

It is this non-unitarity which allows bringing both any non-zero column vector A^i and any non-zero row vector B_i to the form (47) at the same time. Indeed, this can be done by the following 3-stage gauge transform:

1. As a first stage, we bring the column vector of the $A^i(y,\theta)$ chiral fields to the desired form (47),

$$A'(y,\theta)\ =\ U_1(y,\theta)A(y,\theta)\ =\ \begin{pmatrix}0\\ \vdots\\ 0\\ A'^N(y,\theta)\end{pmatrix};\tag{50}$$

this is always possible for any initial array of $A^i(y,\theta)$.

2. At the second stage, we would like to turn the row-vector B — or rather $B' = BU_1^{-1}$ in the $N^{\rm th}$ direction without disturbing the A' column vector, so we are limited to the $U_2(y,\theta)$ matrices such that

$$A'' \ = \ U_2 A' \ = \ A'. \tag{51}$$

In explicit matrix terms, this means

$$U_2 \ = \ \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & 0 \\ * & \cdots & * & 0 \\ * & \cdots & * & 1 \end{pmatrix}. \tag{52}$$

Note however that for an $SL(N,\mathbf{C})$ but non-unitary matrix U_2, having $U^i_N = 0$ for $i < N$ does *not* require $U^N_i = 0$ for $i < N$. Also, the U_2^{-1} matrices acting on the row vector B' also have form (52) — with restricted right column but unrestricted bottom row — so any row vector $B' = B \times U_1^{-1}$ can be rotated till it points in the $N^{\rm th}$ direction without disturbing the column vector A', thus

$$B''(y,\theta) \ = \ B'(y,\theta) \times U_2^{-1}(y,\theta) \ = \ \begin{pmatrix} 0 & \cdots & 0 & B''_N(y,\theta) \end{pmatrix}. \tag{53}$$

3. At this point, both the column vector A'' and the row vector B'' point in the $N^{\rm th}$ direction for all (y,θ), and both of these conditions are preserved by the U_3 matrices belonging to the $SL(N-1,\mathbf{C})\times\hat{\mathbf{C}}^\star$ subgroup of the $SL(N,\mathbf{C})$. The $U_3 \in SL(N-1,\mathbf{C})$ have no further effect on the A'' and B'' superfields, but $U_3 \in \hat{\mathbf{C}}$ change the values of the A''^N and B''_N:

$$A'''^N(y,\theta) \ = \ A''^N(y,\theta) \times C(y,\theta), \qquad B'''_N(y,\theta) \ = \ B''_N(y,\theta) \times C^{-1}(y,\theta). \tag{54}$$

Consequently, for any non-zero values of the $A''^N(y,\theta)$ and $B''_N(y,\theta)$, we may always

$\star$ Here $\hat{\mathbf{C}}$ denotes the complexification of the $U(1)$, the group of multiplication by non-zero complex numbers.

make them equal to each other,

$$A'''^N(y,\theta) \;\equiv\; B'''_N(y,\theta). \tag{55}$$

Altogether, this 3-stage supersymmetric gauge transform can bring any non-zero multiplets A^i and B_i of chiral superfield to the form

$$A^i(y,\theta) \;=\; \Phi(y,\theta)\times\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}, \qquad \begin{array}{c} B_i(y,\theta) \;=\; \Phi(y,\theta)\times\begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix}, \\ \text{for the same } \Phi(y,\theta). \end{array} \tag{47}$$

The bottom line here is that in the Higgs regime of SQCD we may impose the supersymmetric unitary gauge where the quark superfields $A^i(y,\theta)$ and $B_i(y,\theta)$ are restricted to have form (47), or equivalently, are required to have

$$\begin{aligned} A^i(y,\theta) &\equiv 0 \quad \text{for } i<N,\\ B_i(y,\theta) &\equiv 0 \quad \text{for } i<N,\\ A^N(y,\theta) &\equiv B_N(y,\theta). \end{aligned} \tag{56}$$

These conditions eliminate $2N-1$ out of $2N$ chiral superfields of the theory and leave us with the only chiral superfield $\Phi(y,\theta)$. There is no scalar potential for this Φ, so it acts as the *modulus superfield* of the classical SQCD vacua. However, similar to SQED, Φ is a double-valued modulus — $\Phi(y,\theta)$ is gauge-equivalent to $-\Phi(y,\theta)$ — so it is often more convenient to use the single-valued modulus

$$\mathcal{M} \;=\; \Phi^2. \tag{57}$$

Or in gauge-invariant terms,

$$\mathcal{M} \;=\; A^i B_i\,. \tag{58}$$

The unitary gauge conditions (56) do not fix all the supersymmetrized gauge symmetries of the theory. Instead, they limit them to the $U(y,\theta)\in SL(N-1,\mathbf{C})$ subgroup of $SL(N,\mathbf{C})$, which corresponds to the ordinary gauge symmetry group restricted to the $SU(N-1)$ subgroup of $SU(N)$. Physically, this means that this $SU(N-1)$ subgroup remains un-Higgsed and the corresponding vector superfields V^a remain massless, while the other $2N-1$ vector superfields of the Higgsed-down symmetries become massive. In the the process, these massive vectors eat $2N-1$ chiral superfields, but in the SUSY unitary gauge those eaten chiral superfields are simply eliminated by the gauge conditions while the extra components of the massive vector superfields are already there but now we can no longer eliminate them by gauge transforms. Furthermore, the Lagrangian in the unitary gauge contains gauge-symmetry-breaking supersymmetric mass terms

$$\mathcal{L}_{\text{net}} \supset \mathcal{L}_{MV} = \sum_{a,b} \bigl(M_V^2\bigr)^{a,b} \times \int d^4\theta\, V^a V^b \tag{59}$$

for the vector superfields outside the unbroken $SU(N-1)$ subgroup of the $SU(N)$.

To see how this works, consider the kinetic Lagrangian terms for the quark superfields:

$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= \int d^4\theta \Bigl(\overline{A}\exp(+2\mathcal{V})A \,+\, B\exp(-2\mathcal{V})\overline{B}\Bigr) \\
\text{in the unitary gauge } &\longrightarrow \int d^4\theta\, \overline{\Phi}\Phi \times \Bigl(e^{+2\mathcal{V}} + e^{-2\mathcal{V}}\Bigr)^{\text{matrix element}}_{N,N} \\
&= \int d^4\theta\, \overline{\Phi}\Phi \,+\, |\langle\Phi\rangle|^2 \times \int d^4\theta \Bigl(4\mathcal{V}^2\Bigr)^{\text{matrix element}}_{N,N} \\
&\qquad +\ \text{interaction terms}
\end{aligned} \tag{60}$$

where the second term amounts to

$$\begin{aligned}
\mathcal{L}_{MV} &= |\langle\Phi\rangle|^2 \times \int d^4\theta \Bigl(4\mathcal{V}^2\Bigr)^{\text{matrix element}}_{N,N} \\
&= |\langle\Phi\rangle|^2 \times \int d^4\theta \left(4\left(g\sum\nolimits_a V^a t^a\right)^2\right)^{\text{matrix element}}_{N,N} \\
&= g^2\, |\langle\Phi\rangle|^2 \times \sum_{a,b} \int d^4\theta\, V^a V^b \times \bigl(2\{t^a,t^b\}\bigr)^{\text{matrix element}}_{N,N}.
\end{aligned} \tag{61}$$

In other words, it's the sum of manifestly supersymmetric mass terms for the canonically normalized vector superfields $V^a(x,\theta,\bar{\theta})$,

$$\mathcal{L}_{MV} = \sum_{a,b} \big(M_V^2\big)^{a,b} \times \int d^4\theta\, V^a V^b \tag{59}$$

for the mass^2 matrix

$$\big(M_V^2\big)^{a,b} = g^2 \,|\langle\Phi\rangle|^2 \times \big(2\{t^a,t^b\}\big)_{N,N}^{\text{matrix element}}. \tag{62}$$

For $N=2$, $2t^a$ are Pauli matrices, hence $2\{t^a,t^b\} = \delta^{a,b} \times \mathbf{2}_{2\times 2}$ and therefore

$$\big(M_V^2\big)^{a,b} = g^2 \,|\langle\Phi\rangle|^2 \times \delta^{a,b} \tag{63}$$

— all 3 vector superfields of the $SU(2)$ getting the same mass $M_V = g|\langle\Phi\rangle|$. For $N>2$, the $2t^a$ are Gell-Mann matrices which do not anticommute with each other, but nevertheless have

$$\big(2\{t^a,t^b\}\big)_{N,N}^{\text{matrix element}} = \delta^{a,b} \times \begin{cases} 0 & \text{for } a \le (N-1)^2 - 1, \\ 1 & \text{for } (N-1)^2 \le a \le N^2 - 2, \\ \frac{2(N-1)}{N} & \text{for } a = N^2 - 1. \end{cases} \tag{64}$$

Consequently, the vector mass^2 matrix (62) is diagonal but has 3 different eigenvalues, which correspond to the $SU(N-1)$ quantum numbers of the vector superfields. Indeed, the V^a form an adjoint multiplet of the original $SU(N)$ gauge group, but under the un-broken $SU(N-1)$ subgroup they break into several irreducible multiplets, namely

$$\text{adjoint} + \text{fundamental} + \text{antifundamental} + \text{singlet}. \tag{65}$$

- The adjoint multiplet comprises t^a with $a \le (N-1)^2 - 1$, so all the corresponding vector superfields are massless.
- The fundamental and antifundamental $(\mathbf{N-1}) + \overline{(\mathbf{N-1})}$ multiplets — which are antiparticles of each other — are linear combinations of t^a with $(N-1)^2 \le a \le N^2 - 2$. Consequently, all the corresponding superfields V^a have the same mass$^2 = g^2\,|\langle\Phi\rangle|^2$.
- Finally, the singlet is t^a for $a = N^2 - 1$, so the corresponding V^a has a different mass$^2 = g^2\,|\langle\Phi\rangle|^2 \times \frac{2(N-1)}{N}$.

SQCD with Several Flavors

Thus far we have focused on SQCD with a single quark flavor, but generalization to several flavors is rather straightforward. Using matrix notations for the colors but not the flavors, we have $2N_fN_c$ chiral superfields in N_f column vectors A_f and N_f row vectors B_f, and a traceless hermitian matrix

$$\mathcal{V} \;=\; g\sum_a V^a(x,\theta,\bar{\theta})\times t^a \tag{66}$$

of vector superfields. The Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm SQCD} \;&=\int\!d^4\theta\,\sum_f\Bigl(\overline{A}_f\exp(+2\mathcal{V})A_f \;+\; B_f\exp(-2\mathcal{V})\overline{B}_f\Bigr)\\ &\quad+\int\!d^2\theta\left(\frac{i\tau}{8\pi}\,{\rm tr}\bigl(\mathcal{W}^\alpha\mathcal{W}_\alpha\bigr)\;+\;\sum_f m_fB_fA_f\right)\;+\;{\rm H.\,c.},\end{aligned} \tag{67}$$

and the supersymmetrized gauge transforms act similarly to the one-flavor case:

$$\begin{aligned}A'_f \;&=\; \exp(+i\Lambda)\times A_f\,,\\ B'_f \;&=\; B_f\times\exp(-i\Lambda),\\ \overline{A}'_f \;&=\; \overline{A}_f\times\exp(-i\overline{\Lambda}),\\ \overline{B}'_f \;&=\; \exp(+i\overline{\Lambda})\times\overline{B}_f\,,\\ \exp(+2\mathcal{V}') \;&=\; \exp(+i\overline{\Lambda})\times\exp(+2\mathcal{V})\times\exp(-i\Lambda),\end{aligned} \tag{68}$$

all for

$$\Lambda \;=\; \sum_a\Lambda^a(y,\theta)\times t^a,\qquad \overline{\Lambda} \;=\; \sum_a\overline{\Lambda}(\bar{y},\bar{\theta})\times t^a. \tag{69}$$

Later in class we shall study in great detail the vacuum states of the SQCD theories with different flavor to color ratios N_f/N_c, and also different numbers of massive vs. massless flavors. Thanks to supersymmetry, we would be able to derive a bunch of exact — *i.e.*, non-perturbative — properties of those vacuum states. But in these notes we shall focus on the classical vacua of the theory and their moduli spaces. In particular, we assume $N_f < N_c$ massless quark flavors, and ignore the massive flavors altogether since classically they cannon have non-zero VEVs.

Let's start with just 2 massless flavors and $N_c = N \geq 3$ colors. At a generic point in the moduli space, the squark VEVs Higgs the $SU(N)$ gauge theory down to $SU(N-2)$ (or down to nothing for $N = 3$). To see how this works, let's turn on squark VEVs one flavor at a time. Or equivalently, let's impose the supersymmetrized unitary gauge on one flavor at a time. Thus, for any $A_1 \neq 0$ and any $B_1 \neq 0$ we may gauge transform these column and row vectors till they both point in the N^{th} direction, specifically

$$A_1(y,\theta) \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \Phi_1(y,\theta) \end{pmatrix}, \qquad B_1(y,\theta) \;=\; \begin{pmatrix} 0 & \cdots & 0 & 0 & \Phi_1(y,\theta) \end{pmatrix} \tag{70}$$

for some modulus superfield $\Phi_1(y,\theta)$. The gauge transform which brings the first flavor fields to this form uses up the $SU(N)/SU(N-1)$ SUSY gauge symmetries, but the $SU(N-1)$ subgroup remains un-fixed.

Now consider the second flavor quark fields A_2 and B_2. From the $SU(N-1)$ point of view, the A_2 quarks' colors comprise a fundamental $(\mathbf{N-1})$ multiplet plus a singlet $A_{2,N}$, or in matrix notations a column vector $\tilde{A}_2$ of length $N-1$, plus a separate A_2^N superfield. Likewise, the B_2 antiquark fields form a row vector $\tilde{B}_2$ of length $N-1$ plus a separate $B_{2,N}$ superfield. Consequently, we may use the (supersymmetrized) $SU(N-1)$ gauge transform to make the $\tilde{A}_2$ and $\tilde{B}_2$ vectors to point in the $(N-1)^{\text{st}}$ direction. In terms of the length-N vectors A_2 and B_2, this means

$$A_2(y,\theta) \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi_2(y,\theta) \\ A_2^N(y,\theta) \end{pmatrix}, \qquad B_2(y,\theta) \;=\; \begin{pmatrix} 0 & \cdots & 0 & \Phi_2(y,\theta) & B_{2,N}(y,\theta) \end{pmatrix}. \tag{71}$$

Altogether, the SUSY unitary gauge reduces $4N_c$ chiral superfields down to just 4 moduli superfields, namely Φ_1, Φ_2, A_2^N, and $B_{2,N}$. The remaining $4N_c - 4$ chiral superfields are fixed by the gauge conditions, or in gauge0-independent terms, they become eaten by the

Higgs mechanism which reduces the $SU(N_c)$ gauge group down to its $SU(N_c-2)$ subgroup. In the process,

$$\Big(N_c^2-1\Big) \;-\; \Big((N_c-2)^2-1\Big) \;=\; 4N_c \;-\; 4 \tag{72}$$

vector supermultiplets become massive, and that's why they eat up $4N_c-4$ chiral superfields.

Note that while the specific set of 4 moduli superfields Φ_1, Φ_2, A_2^N, and $B_{2,N}$ is based on sequential flavor-by-flavor imposition of the unitary gauge, the net dimension of the moduli space is gauge invariant. So there is a better set of 4 moduli space coordinates in terms of gauge-invariant bilinears of quark and antiquark fields,

$$\mathcal{M}_{f,f'} \;=\; B_{f,i}A^i_{f'}, \qquad f,f'=1,2. \tag{73}$$

Besides gauge invariance, these coordinates are useful throughout the entire moduli space, covering not only the general points where the $SU(N_c)$ gauge group is Higgsed all the way down to a $SU(N_c-2)$ subgroup but also at special locations where a larger $SU(N_c-1)$ subgroup survives the Higgsing. Furthermore, the 2×2 matrix $\mathcal{M}$ made from the 4 moduli (73) provides an easy way to find the un-Higgsed gauge group: *If* $\det(\mathcal{M})\neq 0$ *then only an* $SU(N_c-2)$ *subgroup remains un-Higgsed, but if* $\det(\mathcal{M})=0$ *(but* $\mathcal{M}\neq 0$*) then the un-Higgsed subgroup is an* $SU(N_c-1)$*.* Indeed, if $\det(\mathcal{M})=0$ then a suitable $SU(2)\times SU(2)$ flavor symmetry can bring it to a form

$$\mathcal{M} \;=\; \begin{pmatrix} \mathcal{M}_{1,1} & 0\\ 0 & 0\end{pmatrix}, \tag{74}$$

which corresponds to $A_2=0$ and $B_2=0$, while $A_1\neq 0$ and $B_1\neq 0$ Higgs the $SU(N_c)$ down to $SU(N_c-1)$. Conversely, if the un-Higgsed subgroup is an $SU(N_c-1)$ then some linear combination of A_1 and A_2 happens to vanish, and likewise for the B_1 and B_2, hence $\det(\mathcal{M})=0$.

Now let's generalize the above analysis of the Higgs mechanism and of the moduli spaces to SQCD theories with $N_f>2$ massless flavors. The N_f^2 gauge invariant quark-antiquark bilinears

$$\mathcal{M}_{f,f'} \;=\; B_{f,i} A^i_{f'}, \qquad f, f' = 1, \ldots, N_f \tag{75}$$

are independent from each other for $N_f \leq N_c$, and for $N_f \leq N_c - 1$ these are all the independent $SL(N_c, \mathbf{C}$ invariant combinations one can make from the $2N_f N_c$ chiral superfields A^i_f and $B_{f,i}$. Also, we shall see momentarily that any combination of the N_f^2 VEVs $\langle \mathcal{M}_{f,f'} \rangle$ (or rather, VEVs of the bilinears' scalar components) are allowed along the flat directions of the classical scalar potential. Consequently, the $\mathcal{M}_{f,f'}$ are the moduli superfields of SQCD with $N_f < N_c$.

Now consider the Higgs mechanism at generic points of the moduli space. In general each flavor with non-zero squark VEV Higgses down once color of the gauge theory, so a generic set of squark VEVs for $N_f < N_c$ colors should Higgs the $SU(N_c)$ gauge theory down to its $SU(N_c - N_f)$ subgroup (or down to nothing for $N_f = N_c - 1$). In the process

$$N_V \;=\; \left(N_c^2 - 1\right) \;-\; \left((N_c - N_f)^2 - 1\right) \;=\; 2N_c N_f \;-\; N_f^2 \tag{76}$$

vector superfields V^a become massive, while the same number of chiral superfields are eaten by the Higgs mechanism. Since the theory has only $2N_c N_f$ chiral superfields to begin with, only N_f^2 chiral superfields remain un-eaten. Thus, *the only un-eaten combinations of the chiral quark and antiquark superfields are the moduli superfields* (75)*.*

Furthermore, the $N_f \times N_f$ matrix $\mathcal{M}$ — or rather its VEV $\langle \mathcal{M} \rangle$ — provides a simple indicator of the un-Higgsed gauge group at any point of the moduli space: Let

$$R \;=\; \text{matrix rank}(\langle \mathcal{M} \rangle), \tag{77}$$

i.e. the biggest size of a square sub-matrix of $\langle \mathcal{M} \rangle$ with a non-zero determinant, then the un-Higgsed subgroup of the $SU(N_c)$ is a $SU(N_c - R)$. In particular, at general points of the moduli space $\det(\langle \mathcal{M} \rangle) \neq 0$, hence rank $R = N_f$ and the un-Higgsed subgroup is a $SU(N_c - N_f)$. But at special point of the moduli space $R < N_f$, hence a larger un-Higgsed subgroup.

To see how all this works, let's go to the Wess–Zumino gauge and consider the flat directions of the scalar potential for the squarks:

$$V_{\rm scalar} \;=\; \frac{g^2}{2}\sum_a \bigl(\mathcal{D}^a\bigr)^2 \tag{78}$$
$$\text{for } \mathcal{D}^a \;=\; \sum_f \Bigl(A_f^\dagger t^a A_f \;-\; B_f t^a B_f^\dagger\Bigr). \tag{79}$$

Instead of explicit flavors, let's use $N_f \times N_c$ matrix notations for the squark fields, or rather $N_c \times N_f$ matrices A and $B^\dagger$ and $N_f \times N_c$ matrices B and $A^\dagger$. In these notations, eq. (79) becomes

$$\mathcal{D}^a \;=\; \mathrm{tr}\bigl(A\times A^\dagger\times t^a\bigr) \;-\; \mathrm{tr}\bigl(B^\dagger\times B\times t^a\bigr) \;=\; \mathrm{tr}\bigl((AA^\dagger - B^\dagger B)\times t^a\bigr). \tag{80}$$

Along a flat direction of the scalar potential (78), all such $\mathcal{D}^a$ must vanish, which requires

$$(AA^\dagger - B^\dagger B) \;=\; \text{number}\times \mathbf{1}_{N_c\times N_c}\,. \tag{81}$$

Further more, the matrices $AA^\dagger$ and $B^\dagger B$ are hermitian non-negative matrices with at most $N_f < N_c$ non-zero eigenvalues, so they cannot have non-zero difference cannot be proportional to a unit matrix. Instead, they must be simply equal to each other,

$$A\times A^\dagger \;=\; B^\dagger\times B. \tag{82}$$

Furthermore, this hermitian matrix can always be diagonalized by an ordinary gauge symmetry,

$$\begin{gathered} A \;\to\; U\times A,\quad B\;\to B\times U^\dagger,\quad U\in SU(N_c),\\ \bigl(AA^\dagger \;=\; B^\dagger B\bigr) \;\to\; U\times\bigl(AA^\dagger \;=\; B^\dagger B\bigr)\times U^\dagger \;=\; \mathrm{diag.\,matrix}\Bigl(0,\ldots,0;E_1^2,\ldots,E_{N_f}^2\Bigr) \end{gathered} \tag{83}$$

for some real eigenvalues $E_1,\ldots,E_{N_f}$.

Next, a useful theorem: Any complex matrix M can be decomposed into a product of two unitary matrices and a real diagonal matrix, $M = U_1 \times D \times U_2$. Even the rectangular complex matrices like A and B can be decomposed as

$$A \;=\; U_A \times D_A \times W_A\,, \qquad B \;=\; W_B \times D_B \times U_B\,, \tag{84}$$

where U_A and U_B are $N_c \times N_c$ unitary matrices, W_A and W_B are $N_f \times N_f$ unitary matrices, while D_A and D_B are rectangular matrices with a single non-zero diagonal. In block form,

$$D_A \;=\; \begin{pmatrix} \text{zero block} \\ (N_c - N_f) \times N_f \\ \hline \text{real diagonal} \\ \text{block } N_f \times N_f \end{pmatrix}, \qquad D_B \;=\; \left(\begin{array}{c|c} \text{zero block} & \text{real diagonal} \\ N_f \times (N_c - N_f) & \text{block } N_f \times N_f \end{array}\right). \tag{85}$$

Consequently,

$$A \times A^\dagger \;=\; U_A \times \mathrm{diag.\,matrix}\Big(0, \ldots, 0; D^2_{A,1}, \ldots, D^2_{A,N_f}\Big) \times U_A^\dagger, \tag{86}$$

$$B^\dagger \times B \;=\; U_B^\dagger \times \mathrm{diag.\,matrix}\Big(0, \ldots, 0; D^2_{B,1}, \ldots, D^2_{B,N_f}\Big) \times U_B\,, \tag{87}$$

So eqs. (83) for the squark VEVs translate to

$$D^2_{A,f} \;=\; D^2_{B,f} \;=\; E^2_f \quad \text{for } f = 1, \ldots, N_f \tag{88}$$

while

$$U_A \;=\; U^\dagger, \qquad U_B \;=\; U. \tag{89}$$

Therefore, the moduli matrix $\mathcal{M} = B \times A$ becomes

$$\mathcal{M} \;=\; W_B^\dagger \times \mathrm{diag.\,matrix}\Big(E^2_1, \ldots, E^2_{N_f}\Big) \times W_A\,. \tag{90}$$

From this formula it immediately follows that

$$R \;=\; \mathrm{rank}(\mathcal{M}) \;=\; \#\text{non-zero eigenvalues } E^2_f \tag{91}$$

of matrices $AA^\dagger$ or $B^\dagger B$, and that's why the Higgs mechanism due to VEVs $\langle A\rangle$ and $\langle B\rangle$ breaks the $SU(N_c)$ gauge symmetry down to $SU(N_c - R)$.

Next, let's show that any complex $N_f \times N_f$ matrix $\mathcal{M}$ can be realized by suitable squark VEVs A and B along the flat directions (83) of the scalar potential. Indeed, any matrix $\mathcal{M}$ can be decomposed into a product (90) of two unitary matrices $W_B^\dagger$ and W_A and a diagonal matrix with real non-negative eigenvalues $E_1^2, \ldots, E_{N_f}^2$. Given such eigenvalues, we construct the 'diagonal' rectangular matrices D_A and D_B as in eq. (85) with diagonal elements $D_f^A = D_f^B = E_f$. And then we construct the A and B matrices as

$$A \;=\; U \times D_A \times W_A \,, \qquad B \;=\; W_B^\dagger \times D_B \times U^\dagger \tag{92}$$

where W_A and $W_B^\dagger$ are $N_f \times N_f$ unitary matrices from the decomposition of the $\mathcal{M}$ matrix, while U is an arbitrary $SU(N_c)$ matrix. As we saw a few lines above, any such values of the squark fields A_f^i and $B_{f,i}$ obey the zero-potential condition (82) and also $B \times A = \mathcal{M}$. *Quod erat demonstrandum.*

Finally, let's calculate the classical Kähler function $K(\mathcal{M}, \overline{\mathcal{M}})$ of the moduli space. To do that, we simply start with the free classical Kähler function for the squark fields,

$$K \;=\; \operatorname{tr}\bigl(A^\dagger \times A\bigr) \;+\; \operatorname{tr}\bigl(B \times B^\dagger\bigr), \tag{93}$$

and re-express it in terms of the moduli scalars comprising the $\mathcal{M}$ matrix and its conjugate $\mathcal{M}^\dagger$. In light of eqs. (92),

$$\begin{aligned} \operatorname{tr}\bigl(A^\dagger \times A\bigr) \;&=\; \sum_{f=1}^{N_f} \bigl(D_f^A\bigr)^2, \\ \operatorname{tr}\bigl(B \times B^\dagger\bigr) \;&=\; \sum_{f=1}^{N_f} \bigl(D_f^B\bigr)^2, \end{aligned} \tag{94}$$

and hence

$$K \;=\; 2\sum_{f=1}^{N_f} E_f^2 \,. \tag{95}$$

At the same time, the hermitian matrix

$$\mathcal{M} \times \mathcal{M}^\dagger \;=\; W_B^\dagger \times \text{diag. matrix}\Bigl(E_1^4, \ldots, E_{N_f}^4\Bigr) \times W_B \tag{96}$$

has non-negative eigenvalues E_f^4, so we may take a square root of this matrix

$$\sqrt{\mathcal{M}\times\mathcal{M}^\dagger} \;=\; W_B^\dagger \times \mathrm{diag.\,matrix}\Big(E_1^2,\ldots,E_{N_f}^2\Big) \times W_B\,. \tag{97}$$

This square root has eigenvalues E_f^2, so taking its trace yields

$$\mathrm{tr}\left(\sqrt{\mathcal{M}\times\mathcal{M}^\dagger}\right) \;=\; \sum_{f=1}^{N_f} E_f^2\,. \tag{98}$$

Comparing this formula to eq. (95), we immediately see that

$$K\big(\mathcal{M},\overline{\mathcal{M}}\big) \;=\; 2\,\mathrm{tr}\left(\sqrt{\mathcal{M}\times\overline{\mathcal{M}}}\right). \tag{99}$$

The Riemannian metric for the moduli space in the $\mathcal{M}_{f,f'}$ and $\overline{\mathcal{M}}_{f,f'}$ follows from this Kähler function by taking its second derivatives WRT to $\mathcal{M}_{f,f'}$ and $\overline{\mathcal{M}}_{f'',f'''}$, but in the interests of brevity let me skip this piece of calculus.

Let me conclude these notes by emphasizing that the analysis presented here was purely classical and is subject to the quantum corrections. In particular, the Kähler function (99) for the moduli space is subject to the perturbative corrections at all loop orders, as well as non-perturbative corrections. But my analysis of the Higgs mechanism and the chiral superfields it eats up, of the scalar potential's flat directions, and of moduli space and its complex structure — all that is valid to all orders of the perturbation theory. Only the non-perturbative effects related to instantons and / or gaugino condensation spoil the flatness of the scalar potential along the classical moduli space. We shall explore such effects in some detail later in this class.