TESTING SEIBERG DUALITY

Seiberg duality is the IR duality between two SQCD-like theories A and B with similar numbers of flavors $N_f(A) = N_f(B)$ but different numbers of colors, $N_c(A) \neq N_c(B)$. Instead, $N_c(A) + N_c(B) = N_f$. In these notes, I shall use short-hand notations

$$N_a \stackrel{\rm def}{=} N_c(A), \quad N_b \stackrel{\rm def}{=} N_c(B), \quad N_f \stackrel{\rm def}{=} N_f(A) \;=\; N_f(B). \tag{1}$$

Specifically, the A theory is a pure SQCD, while the B theory also has N_f^2 gauge-singlet chiral superfields with Yukawa couplings to the B-quarks,

$$W_B \;=\; \lambda \sum_{c,f,f'} \Phi^{f'f} \tilde{q}_{f',c} q^c_f \,. \tag{2}$$

In these notes I take both A and B theories being in the conformal window of SQCD,

$$\tfrac{1}{3} N_f \;<\; N_a, N_b \;<\; \tfrac{2}{3} N_f \,, \tag{3}$$

so the deep-IR limits of both theories are strongly (or moderately-strongly) coupled superconformal theories. Our knowledge of such IR-strong theories is limited, so we cannot rigorously prove the complete similarity of the deep-IR regimes of the A and B theories. Instead, I am going to verify that every IR feature of each theory we can find out indeed agrees between the A and B theories. And that's how we are going to test the Seiberg duality in these notes.

In particular, we are going to verify that:

1. Both theories have the same unbroken flavor symmetry

$$G_F \;=\; SU(N_f)_L \times SU(N_f)_D \times U(1)_B \times U(1)_R \tag{4}$$

where $U(1)_B$ is the vector-like baryon number symmetry while $U(1)_R$ is the anomaly-free combination of the pure-R symmetry and the axial $U(1)$.

2. Since the flavor symmetry (4) is chiral, we should check the 't Hooft's anomaly matching conditions for the two theories. Or at least make sure that these conditions agree with the same set of massless composite fermions, whoever they might be.
3. The two theories have similar chiral rings. In particular, they have similar generators with similar flavor quantum numbers and similar scaling dimensions Δ.
4. The two theories have similar moduli spaces.
   Note that SQCD may have non-zero VEVs of meson or baryon scalars. Such VEVs would spontaneously break the conformal symmetry of the IR theory, but they would not break the supersymmetry. Instead, the IR theory would have supersymmetric vacua, usually in continuous families parametrized by some moduli.
   To test the Seiberg duality, we should verify the A and B theory have similar moduli spaces.
5. The tree level superpotentials of the two theories may be deformed away from the conformal point by adding quark masses or O'Raifeartaigh terms for the gauge singlets. We should check that there is a 1-to-1 correspondence between such deformations of the A theory and of the B theory.

1. Flavor symmetry

Let's start with the flavor symmetries of A and B theories. The A theory is pure massless SQCD, so it's classical flavor symmetry is

$$G_F^{\text{classical}} \;=\; SU(N_f)_L \times SU(N_f)_D \times U(1)_V \times U(1)_A \times U(1)_{R0} \tag{5}$$

where the vector-like $U(1)_V$ is the baryon number symmetry, henceforth renamed $U(1)_B$, while $U(1)_{R0}$ is the pure-R symmetry. In the quantum A theory, the axial symmetry and the pure-R symmetry are anomalous, but they have an anomaly-free combination $U(1)_R$, thus the net symmetry of the quantum A theory is

$$G_F \;=\; SU(N_f)_L \times SU(N_f)_D \times U(1)_B \times U(1)_R\,. \tag{4}$$

Moreover, in SQCD with $N_f > N_c$ the spontaneous chiral symmetry breaking is not necessary, so the IR theory has the same flavor symmetry (4) as the UV theory. Or rather, the theory has a moduli space of vacua, some of which may spontaneously break the chiral symmetry but other vacua do not. In particular, the conformal point of the moduli space — where $\langle\mathcal{M}\rangle = 0$, $\langle\mathcal{B}\rangle = 0$, $\langle\widetilde{\mathcal{B}}\rangle = 0$ — has completely unbroken flavor symmetry (4).

For future reference, let me tabulate the color and the flavor quantum numbers of all the A theory's UV fields:

fields \ QN	$SU(N_a)$	$SU(N_f)_L$	$SU(N_f)_D$	B	$R(\text{boson})$	$R(\text{fermion})$
A^μ, λ^α	Adjoint	$\mathbf{1}$	$\mathbf{1}$	0	0	$+1$
Q, Ψ_Q	$\mathbf{N_c^A}$	$\mathbf{N_f}$	$\mathbf{1}$	$+\frac{1}{N_a}$	$1-\frac{N_a}{N_f}$	$-\frac{N_a}{N_f}$
$\widetilde{Q}, \widetilde{\Psi}_Q$	$\overline{\mathbf{N_a}}$	$\mathbf{1}$	$\mathbf{N_f}$	$-\frac{1}{N_a}$	$1-\frac{N_a}{N_f}$	$-\frac{N_a}{N_f}$

(6)

Note#1: The $U(1)_B$ charge is normalized as the baryon number, that's why the quarks and the squarks have $B = +1/N_a$ while the antiquarks and antisquarks have $B = -1/N_a$.

Note#2: The R charge of the quarks and antiquarks follow from canceling the color-color-R anomaly, thus

$$\text{Index(adj)} \times R(\lambda) \ + \ 2N_f \times \text{Index(fund)} \times R(\Psi_Q, \widetilde{\Psi}_Q) \ = \ 0, \tag{7}$$

where $\text{Index(adj)} = N_c(A) = N_a$ and $\text{Index(fund)} = \frac{1}{2}$. Also, for any mixture of R-symmetry with the axial symmetry $R(\lambda) = +1$, hence $R(\Psi_Q, \widetilde{\Psi}_Q) = -N_a/N_f$ as in the table (6). As to the squarks and antisquarks, their R charge follows as $R(Q, \widetilde{Q}) = R(\Psi_Q, \widetilde{\Psi}_Q) + 1$.

Now consider the B theory. Its SQCD core has exactly the same anomaly-free flavor symmetry

$$G_F \ = \ SU(N_f)_L \times SU(N_f)_D \times U(1)_B \times U(1)_R \tag{4}$$

as the A theory. Adding the gauge-singlet $\Phi^{f'f}$ fields with Yukawa couplings to the quarks and antiquarks does not introduce additional symmetries because we cannot transform the

Φ's — and only the Φ′ — in any way without affecting the superpotential

$$W_B \;=\; \lambda \sum_{c,f,f'} \Phi^{f'f} \tilde{q}_{f',c} q^c_f\,. \tag{2}$$

At the same time, this superpotential is invariant under all the flavor symmetries (4) provided the quarks, the antiquarks, and the singlets transform in correlated ways. Thus, the net flavor symmetry of the B theory is the same (4) as of the A theory, *quod erat demonstrandum.*

Again, for future reference, let me tabulate the color and the flavor quantum numbers of all the B theory's UV fields:

fields \ QN	$SU(N_b)$	$SU(N_f)_L$	$SU(N_f)_D$	B	R(boson)	R(fermion)
A^μ, λ^α	Adjoint	$\mathbf{1}$	$\mathbf{1}$	0	0	$+1$
q, ψ^α_q	$\mathbf{N_b}$	$\overline{\mathbf{N_f}}$	$\mathbf{1}$	$+\frac{1}{N_b}$	$1-\frac{N_b}{N_f}$	$-\frac{N_b}{N_f}$
$\tilde{q}, \tilde{\psi}^\alpha_q$	$\overline{\mathbf{N^B_c}}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	$-\frac{1}{N_b}$	$1-\frac{N_b}{N_f}$	$-\frac{N_b}{N_f}$
Φ, ψ^α_ϕ	$\mathbf{1}$	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$+2\frac{N_b}{N_f}$	$2\frac{N_b}{N_f}-1$

(8)

Note#3: It's a matter of convention whether we assign the quarks to a fundamental of an antifundamental multiplet of the $SU(N_f)_L$ flavor group, and likewise for the antiquarks belonging to an $\mathbf{N_f}$ or an $\overline{\mathbf{N_f}}$ multiplet of the $SU(N_f)_D$. For the duality purposes, we want the A-quarks Q and the B-quarks q to have opposite $SU(N_f)_L$ quantum number, and ditto for the A-antiquarks $\widetilde{Q}$ and the B-antiquarks $\tilde{q}$. To avoid confusion, I chose both Q and $\widetilde{Q}$ to be in the fundamental flavor multiplets while q and $\tilde{q}$ are therefore in the antifundamental multiplets.

Note#4: The quantum numbers of the gauge singlets Φ follow from invariance of the superpotential (2) under all the flavor symmetries. Or rather, W_B should be invariant WRT $SU(N_f)_L \times SU(N_f)_D \times U(1)_B$ and have R-charge $R(W) = +2$.

2. Anomaly matching conditions

If the flavor symmetry (4) is **not** spontaneously broken, the IR regime of a theory must have massless composite fermions whose spectrum obeys 't Hooft's anomaly matching conditions:

$$\forall F_1, F_2, F_3 \in G_F : \quad \mathrm{tr}_{\mathrm{composite}}\bigl(F_1\{F_2, F_3\}\bigr) \;=\; \mathrm{tr}_{\mathrm{elementary}}\bigl(F_1\{F_2, F_3\}\bigr). \tag{9}$$

Note: both traces are taken over massless LH Weyl fermions — elementary or composite — while the RH Weyl fermions are treated as conjugates of the LH Weyl fermions.

Now, suppose two theories A and B are IR-dual to each other. Then they have the same unbroken flavor symmetry and the same spectrum of massless composite fermions, whatever it might be, hence

$$\mathrm{tr}_{\mathrm{A,\,composite}}\bigl(F_1\{F_2, F_3\}\bigr) \;=\; \mathrm{tr}_{\mathrm{B,\,composite}}\bigl(F_1\{F_2, F_3\}\bigr). \tag{10}$$

OOH, the UV regimes of the two theories have different spectra of elementary fermionic fields, the anomaly matching conditions (9) for both theories with eq. (10) gives us

$$\mathrm{tr}_{\mathrm{A,\,elementary}}\bigl(F_1\{F_2, F_3\}\bigr) \;=\; \mathrm{tr}_{\mathrm{A\,or\,B,\,composite}}\bigl(F_1\{F_2, F_3\}\bigr) \;=\; \mathrm{tr}_{\mathrm{B,\,elementary}}\bigl(F_1\{F_2, F_3\}\bigr). \tag{11}$$

Thus, the two theories A and B can be IR dual to each other only if their respective spectra of elementary fermions have matching flavor anomalies,

$$\forall F_1, F_2, F_3 \in G_F : \quad \mathrm{tr}_{\mathrm{A,\,elementary}}\bigl(F_1\{F_2, F_3\}\bigr) \;=\; \mathrm{tr}_{\mathrm{B,\,elementary}}\bigl(F_1\{F_2, F_3\}\bigr). \tag{12}$$

So let us check these conditions for the Seiberg's A theory and B theory. Since the flavor symmetry (4) has two non-abelian $SU(N_f)$ factors and two abelian factors $U(1)_B$ and $U(1)_R$, the mathematically allowed anomalies are limited to the following combinations of generators $F_{1,2,3}$:

1. All 3 generators belong to the same $SU(N_f)$ subgroup:

$$\begin{aligned} \mathrm{tr}\big(L^a\{L^b, L_c\}\big) &= d^{abc} \times I_3^{\mathrm{net}} \text{ WRT } SU(N_f)_L\,, && (13) \\ \mathrm{tr}\big(D^a\{D^b, D_c\}\big) &= d^{abc} \times I_3^{\mathrm{net}} \text{ WRT } SU(N_f)_D\,, && (14) \end{aligned}$$

where I_3 is the cubic index of a multiplet (WRT the appropriate $SU(N_f)$ factor) and I_3^{net} is the net cubic index of all the LH fermions of the theory. For a theory where all fermions belong to fundamental, antifundamental, or adjoint multiplets

$$I_3^{\mathrm{net}} = \#(\mathbf{N_f}) - \#(\overline{\mathbf{N_f}}). \qquad (15)$$

Therefore, to check the anomaly match between the A and B theories, we should simply count the fundamental and the antifundamental multiplets of the two theories' fermions. Thus, for the $SU(N_f)_L$ factor, we have:

A theory has N_a fundamentals in Ψ_Q and no antifundamentals, hence $I_3^{\mathrm{net}} = +N_a$.

B theory has N_b antifundamentals in ψ_q and N_f fundamentals in ψ_ϕ, hence

$$I_3^{\mathrm{net}} = -N_b + N_f = +N_a\,. \qquad (16)$$

Match.

For the $SU(N_f)_D$ factor we have a similar counting:

A theory has N_a fundamentals in $\widetilde{\Psi}_Q$ and no antifundamentals, hence $I_3^{\mathrm{net}} = +N_a$.

B theory has N_b antifundamentals in $\tilde{\psi}_q$ and N_f fundamentals in ψ_ϕ, hence

$$I_3^{\mathrm{net}} = -N_b + N_f = +N_a\,. \qquad (17)$$

Match.

2. Two generators belong to the same $SU(N_f)$ factor while the third generator is abelian. In this case,

$$\begin{aligned}
\operatorname{tr}\bigl(B\{L^a,L^b\}\bigr) &= 2\delta^{ab}\times\mathcal{A}^{BLL}, && (18)\\
\mathcal{A}^{BLL} &= \sum_{\text{LHWF}} B\times\text{Index}[SU(n_f)_L]\times\text{Multiplicity}[\text{color},SU(N_d)_D], && (19)\\
\operatorname{tr}\bigl(B\{D^a,D^b\}\bigr) &= 2\delta^{ab}\times\mathcal{A}^{BDD}, && (20)\\
\mathcal{A}^{BDD} &= \sum_{\text{LHWF}} B\times\text{Index}[SU(n_f)_D]\times\text{Multiplicity}[\text{color},SU(N_d)_L], && (21)\\
\operatorname{tr}\bigl(R\{L^a,L^b\}\bigr) &= 2\delta^{ab}\times\mathcal{A}^{RLL}, && (22)\\
\mathcal{A}^{RLL} &= \sum_{\text{LHWF}} R\times\text{Index}[SU(n_f)_L]\times\text{Multiplicity}[\text{color},SU(N_d)_D], && (23)\\
\operatorname{tr}\bigl(R\{D^a,D^b\}\bigr) &= 2\delta^{ab}\times\mathcal{A}^{RDD}, && (24)\\
\mathcal{A}^{RDD} &= \sum_{\text{LHWF}} R\times\text{Index}[SU(n_f)_D]\times\text{Multiplicity}[\text{color},SU(N_d)_L], && (25)
\end{aligned}$$

so to make sure all these anomalies match between the A and B theories, we need to check that

$$\begin{aligned}
\mathcal{A}^{BLL}[A] &= \mathcal{A}^{BLL}[B], \quad \mathcal{A}^{BDD}[A] = \mathcal{A}^{BDD}[B],\\
\mathcal{A}^{RLL}[A] &= \mathcal{A}^{RLL}[B], \quad \mathcal{A}^{RDD}[A] = \mathcal{A}^{RDD}[B].
\end{aligned} \tag{26}$$

Let's start with the $\mathcal{A}^{BLL}$ anomaly coefficients.

A theory: the only fermions with non-trivial $SU(N_f)_L$ quantum numbers are the quarks Ψ_Q, so other fermions do not contribute to the anomaly in question. The quark multiplet has $B = +1/N_a$, $SU(N_f)_L$ Index $= \frac{1}{2}$, and color $\times$ $SU(N_f)_D$ multiplicity $= N_a \times 1$, hence

$$\mathcal{A}^{BLL}[A] \;=\; \frac{+1}{N_a}\times\frac{1}{2}\times N_a \;=\; \frac{1}{2}\,. \tag{27}$$

B theory: this time, both the quarks ψ_q and the ψ_ϕ gauge singlets have non-trivial $SU(N_f)_L$ quantum numbers, but the ψ_ϕ has zero baryon charge, so only the quarks

ψ_q contribute to the anomaly in question. Specifically, the ψ_q have $B = +1/N_b$, $SU(N_f)_L$ Index $= \frac{1}{2}$, and color $\times$ $SU(N_f)_D$ multiplicity $= N_b \times 1$, hence

$$\mathcal{A}^{BLL}[B] \;=\; \frac{+1}{N_b} \times \frac{1}{2} \times N_b \;=\; \frac{1}{2}\,. \tag{28}$$

Match.

Next, the $\mathcal{A}^{BDD}$ anomalies of the A and B theories **match** in exactly similar way, so let me skip the formulae.

Now, the less obvious $\mathcal{A}^{RLL}$ anomalies of the two theories.

A theory: again, only the quarks contribute to this anomaly because all other fermions are $SU(N_f)_L$ singlets. The R-charge of the quarks is $-N_a/N_f$, hence

$$\mathcal{A}^{RLL}[A] \;=\; \frac{-N_a}{N_f} \times \frac{1}{2} \times N_a \;=\; -\frac{N_a^2}{2N_f}\,. \tag{29}$$

B theory: this time, both the quarks ψ_q and the color-singlets ψ_ϕ contribute to the $\mathcal{A}^{RLL}$ anomaly. Both types of fermion belong to multiplets of Index $= \frac{1}{2}$; the quarks ψ_q have $R = -N_b/N_f$ and color $\times$ $SU(N_f)_D$ multiplicity $= N_b \times 1$, while the ψ_ϕ have $R = 2(N_b/N_f) - 1$ and multiplicity $= 1 \times N_f$. Consequently,

$$\begin{aligned}
\mathcal{A}^{RLL}(\psi_q) \;&=\; \frac{-N_b}{N_f} \times \frac{1}{2} \times N_b \;=\; -\frac{N_b^2}{2N_f}\,, && (30)\\
\mathcal{A}^{RLL}(\psi_\phi) \;&=\; \frac{2N_b - N_f}{N_f} \times \frac{1}{2} \times N_f \;=\; \frac{2N_b - N_f}{2}\,, && (31)\\
\mathcal{A}^{RLL}[B](\text{net}) \;&=\; \mathcal{A}^{RLL}(\psi_q) \;+\; \mathcal{A}^{RLL}(\psi_\phi) \\
&=\; -\frac{1}{2N_f}\Big(N_f(2N_b - N_f) \;-\; N_b^2\Big). && (32)
\end{aligned}$$

$*$ But

$$N_f(2N_b - N_f) \;-\; N_b^2 \;=\; -(N_f - N_b)^2 \;=\; -N_a^2, \tag{33}$$

so the $\mathcal{A}^{RLL}$ anomalies of the A and B theories **match**.

Next, the $\mathcal{A}^{RDD}$ anomalies are similar to the $\mathcal{A}^{RLL}$ anomalies, so let me be brief. In the A theory, only the antiquarks $\widetilde{\Psi}_Q$ contribute to this anomaly, while in the B theory this anomaly comes from both the antiquarks $\tilde{\psi}_q$ and the color-singlet fields ψ_ϕ. The relevant indices, multiplicities, and R charges of these fields are similar to what we had in the $\mathcal{A}^{RLL}$ case, so we get

$$\mathcal{A}^{RDD}[A] \;=\; \mathcal{A}^{RDD}(\widetilde{\Psi}_Q) \;=\; \frac{-N_a}{N_f} \times \frac{1}{2} \times N_a \;=\; -\frac{N_a^2}{2N_f}\,, \tag{34}$$

while

$$\begin{aligned}
\mathcal{A}^{RDD}(\tilde{\psi}_q) \;&=\; \frac{-N_b}{N_f} \times \frac{1}{2} \times N_b \;=\; -\frac{N_b^2}{2N_f}\,, && (35)\\
\mathcal{A}^{RDD}(\psi_\phi) \;&=\; \frac{2N_b - N_f}{N_f} \times \frac{1}{2} \times N_f \;=\; \frac{2N_b - N_f}{2}\,, && (36)\\
\mathcal{A}^{RDD}[B](\text{net}) \;&=\; \mathcal{A}^{RDD}(\psi_q) \;+\; \mathcal{A}^{RD}(\psi_\phi)\\
&=\; -\frac{1}{2N_f}\Big(N_f(2N_b - N_f) \;-\; N_b^2\Big)\\
&=\; -\frac{N_a^2}{2N_f}\,.(37)
\end{aligned}$$

Match.

3. Finally, the purely abelian anomalies $\mathrm{tr}(R^3)$ and $\mathrm{tr}(RB^2)$, plus the trace anomaly $\mathrm{tr}(R)$. In principle, one should also check the $\mathrm{tr}(B^3)$, $\mathrm{tr}(BR^2$, and $\mathrm{tr}(B)$ anomalies, but by the charge-conjugation symmetry of both A and B theories, the last 3 traces automatically vanish. Thus, all we need to calculate are the 3 abelian anomaly coefficients

$$\begin{aligned}
\mathcal{A}^{BBR} &= \sum_{\text{LHWF}} B^2 \times R \times \text{mult},\\
\mathcal{A}^{RRR} &= \sum_{\text{LHWF}} R^3 \times \text{mult},\\
\mathcal{A}^{R} &= \sum_{\text{LHWF}} R \times \text{mult},
\end{aligned} \tag{38}$$

where 'mult' denotes net multiplicity WRT to both color and flavor symmetries. So let's calculate these three coefficients for both A and B theories and make sure they match.

A theory fermions:

$$\begin{aligned}
\Psi_Q \quad \text{of } B &= \frac{+1}{N_a}, \quad R = -\frac{N_a}{N_f}, \quad \text{mult} = N_a N_f\,, \\
\widetilde{\Psi}_Q \quad \text{of } B &= \frac{-1}{N_a}, \quad R = -\frac{N_a}{N_f}, \quad \text{mult} = N_a N_f\,, \\
\lambda \quad \text{of } B &= 0, \qquad R = +1, \qquad \text{mult} = N_a^2 - 1.
\end{aligned} \tag{39}$$

Consequently,

$$\begin{aligned}
\mathcal{A}^{BBR}[A] &= \Bigg(\frac{1}{N_a}\Bigg(^{2} \times \frac{-N_a}{N_f} \times N_f N_a \times 2 \;+\; 0 \\
&= -2, \qquad (40) \\
\mathcal{A}^{RRR}[A] &= \left(\frac{-N_a}{N_f}\right)^3 \times \times N_f N_a \times 2 \;+\; (+1)^3 \times (N_a^2 - 1) \\
&= -2\frac{N_a^4}{N_f^2} \;+\; N_a^2 \;-\; 1, \qquad (41) \\
\mathcal{A}^{R}[A] &= \left(\frac{-N_a}{N_f}\right) \times \times N_f N_a \times 2 \;+\; (+1) \times (N_a^2 - 1) \\
&= -2N_a^2 \;+\; N_a^2 \;-\; 1 \;=\; -N_a^2 \;-\; 1. \qquad (42)
\end{aligned}$$

B theory fermions:

$$\begin{aligned}
\psi_q \quad \text{of } B &= \frac{+1}{N_b}, \quad R = -\frac{N_b}{n_f}, \qquad \text{mult} = N_b N_f\,, \\
\tilde{\psi}_q \quad \text{of } B &= \frac{-1}{N_b}, \quad R = -\frac{N_b}{n_f}, \qquad \text{mult} = N_b N_f\,, \\
\psi_\phi \quad \text{of } B &= 0, \qquad R = \frac{2N_b}{N_f} - 1, \quad \text{mult} = N_f^2\,, \\
\lambda \quad \text{of } B &= 0, \qquad R = +1, \qquad \text{mult} = N_b^2 - 1.
\end{aligned} \tag{43}$$

Consequently,

$$\begin{aligned}
\mathcal{A}^{BBR}[B] &= \left(\frac{1}{N_b}\right)^2 \times \frac{-N_b}{N_f} \times N_b N_f \times 2 \;+\; 0 \;+\; 0 \\
&= -2,
\end{aligned} \tag{44}$$

$$\begin{aligned}
\mathcal{A}^{RRR}[B] &= \left(\frac{-N_b}{N_f}\right)^3 \times N_b N_f \times 2 \;+\; \left(\frac{2N_b - N_f}{N_f}\right)^3 \times N_f^2 \;+\; (+1)^3 \times (N_b^2 - 1) \\
&= -2\frac{N_b^4}{N_f^2} \;+\; \frac{(2N_b - N_f)^3}{N_f} \;+\; N_b^2 \;-\; 1, \qquad (45) \\
\mathcal{A}^{R}[B] &= \left(\frac{-N_b}{N_f}\right) \times N_b N_f \times 2 \;+\; \left(\frac{2N_b - N_f}{N_f}\right) \times N_f^2 \;+\; (+1) \times (N_b^2 - 1) \\
&= -2N_b^2 \;+\; N_f(2N_b - N_f) \;+\; N_b^2 \;-\; 1. \qquad (46)
\end{aligned}$$

By inspection, $\mathcal{A}^{BBR}[A] = \mathcal{A}^{BBR}[B]$, **match**, while the other two anomalies take a bit of algebra:

$$\mathcal{A}^{R}[B] \;=\; -N_b^2 + 2N_f N_b - N_f^2 - 1 \;=\; -(N_f - N_b)^2 - 1 \;=\; -N_a^2 - 1 \;=\; \mathcal{A}^{R}[A], \quad (47)$$

match, and

$$\begin{aligned}
\mathcal{A}^{RRR}[B] &= -2\frac{N_b^4}{N_f^2} \;+\; \frac{(2N_b - N_f)^3}{N_f} \;+\; N_b^2 \;-\; 1 \\
&= -2\frac{N_b^4}{N_f^2} \;+\; 8\frac{N_b^3}{N_f} \;-\; 12N_b^2 \;+\; 6N_b N_f \;-N_f^2 +\; N_b^2 \;-\; 1 \\
&= \frac{-2}{N_f^2}\Big(N_b^4 \;-\; 4N_b^3 N_f \;+\; 6N_b^2 \;-\; 4N_b N_f^3 \;+\; N_f^4\Big) \qquad (48) \\
&\quad -\; 2N_b N_f \;+\; N_f^2 \;+\; N_b^2 \;-\; 1 \\
&= -2\frac{(N_f - N_b)^4}{N_f^2} \;+\; (N_f - N_b)^2 \;-\; 1 \;=\; -2\frac{N_a^4}{N_f^2} \;+\; N_a^2 \;-\; 1 \\
&= \mathcal{A}^{RRR}[A],
\end{aligned}$$

match.

Altogether, all the flavor anomalies of the A theory and the B theory match each other.

3. Chiral rings

As I explained earlier in class, the chiral ring of the A theory is generated by the mesons $\mathcal{M}^{f'f}$, baryons $\mathcal{B}^{f_1,\ldots,f_n}$ (totally antisymmetric in their $n = N_a$ flavor indices, and antibaryons $\widetilde{\mathcal{B}}^{f'_1,\ldots,f'_n}$ (also totally antisymmetric in their flavor indices). In principle there is also the gaugino condensate S, but the the theory with massless quarks $S \stackrel{\text{c.r.}}{=} 0$. Here are the flavor quantum numbers of the chiral ring generators:

fields \ QN	$SU(N_f)_L$	$SU(N_f)_D$	B	R
$\mathcal{M}$	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$2 - \frac{2N_a}{N_f}$
$\mathcal{B}$	$\binom{\text{antisymmetric tensor}}{N_a \text{ upper indices}}$	$\mathbf{1}$	$+1$	$\frac{N_a(N_f-N_a)}{N_f}$
$\widetilde{\mathcal{B}}$	$\mathbf{1}$	$\binom{\text{antisymmetric tensor}}{N_a \text{ upper indices}}$	-1	$\frac{N_a(N_f-N_a)}{N_f}$

(49)

As to the B theory, it also has chiral mesons $M_{f'f}$, baryons $b_{f_1,\ldots f_{n'}}$ (antisymmetric in $n' = N_b$ flavor indices) and antibaryons $\tilde{b}_{f'_a,\ldots,f'_{n'}}$, albeit with lower rather then upper indices. But the mesons vanish in the on-shell chiral ring by the $\Phi^{f'f}$ equations of motion, $M_{f'f} \stackrel{\text{c.r.}}{=} 0$. Instead, the chiral ring is generated by the baryon, antibaryon, and the Φ operators. Here are their flavor quantum numbers:

fields \ QN	$SU(N_f)_L$	$SU(N_f)_D$	B	R
Φ	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$\frac{2N_b}{N_f}$
b	$\binom{\text{antisymmetric tensor}}{N_b \text{ lower indices}}$	$\mathbf{1}$	$+1$	$\frac{N_b(N_f-N_b)}{N_f}$
$\tilde{b}$	$\mathbf{1}$	$\binom{\text{antisymmetric tensor}}{N_b \text{ lower indices}}$	-1	$\frac{N_b(N_f-N_b)}{N_f}$

(50)

Since $N_b = N_f - N_c$, — and hence an $SU(N_f)$ antisymmetric tensor with N_b lower indices is equivalent to an antisymmetric tensor with N_a upper indices, — the B-theory baryons b

have the same flavor quantum numbers as the A-baryons $\mathcal{B}$, the B-theory antibaryons $\tilde{b}$ have the same flavor QN as the A-theory baryons $\widetilde{\mathcal{B}}$, and the B-theory Φ fields have the same flavor QN as the A-theory mesons $\mathcal{M}$.

Finally, in the superconformal deep IR limits of the two theories, the scaling dimensions of primary chiral operators follow from their R-charges,

$$\Delta \;=\; coeff32R. \tag{51}$$

Consequently,

$$\Delta(\Phi) \;=\; \Delta(\mathcal{M}) \;=\; 3\frac{N_b}{N_f} \;=\; 3 \;-\; 3\frac{N_a}{N_f} \tag{52}$$

while

$$\Delta(b) \;=\; \Delta(\tilde{b}) \;=\; \Delta(B) \;=\; \Delta(\widetilde{\mathcal{B}}) \;=\; \frac{3N_aN_b}{2N_f}\,. \tag{53}$$

Altogether, the two theories' chiral rings have similar generators with similar flavor quantum numbers and similar scaling dimensions in the IR SCFT. This is in good agreement with the A and B theories being IR-dual to each other.

4. Moduli Spaces and Chiral Ring Equations

The generators of a chiral ring are usually related by some polynomial constraints and/or on-shell equations of motion. Thus, to completely specify a choral ring of a theory, we should not only specify the generators but also all the off-shell and on-shell relations between them. In the previous section, we saw that the chiral rings of the A and B theories have similar generators with similar flavor quantum numbers, and in this section we shall see that the two chiral rings also have similar relations between their generators.

We shall also look at the moduli spaces of the A and B theories. Indeed, SQCDs with $N_f \geq N_c$ have continuous families of supersymmetric vacua parametrized by VEVs of various mesons, baryons, and antibaryons. Although any such non-zero VEV would spontaneously break the conformal symmetry of the IR theory, it would not break SUSY, so the moduli space should have a Kähler geometry. Moreover, the complex structure of the moduli space

— *i.e.*, the holomorphic relations between meson, baryon, and antibaryon VEVs — follows from the chiral ring equations for the corresponding generators. For example, we saw that in SQCD with $N_f = N_c$, the chiral ring generators $\mathcal{M}^{f'f}$, $\mathcal{B}$, and $\widetilde{\mathcal{B}}$ obey the constraint equation

$$\det(\mathcal{M}) \;-\; \widetilde{\mathcal{B}}\times\mathcal{B} \;\stackrel{\text{c.r.}}{=}\; \Lambda^{2N}. \tag{54}$$

Consequently, the moduli space of the theory is parametrized by the VEVs $\left\langle\mathcal{M}^{f'f}\right\rangle$, $\langle\mathcal{B}\rangle$, $\langle\widetilde{\mathcal{B}}\rangle$ subject to a holomorphic relation

$$\det\bigl(\left\langle\mathcal{M}^{f'f}\right\rangle\bigr) \;-\; \langle\widetilde{\mathcal{B}}\rangle\times\langle\mathcal{B}\rangle \;=\; \Lambda^{2N}. \tag{55}$$

Now consider the Seiberg's A and B theories. For small VEVs of various operators, the moduli space geometry of either theory should depend only on the theory's IR physics rather than on its UV features. So if the A and B theory are truly IR-dual to each other, they should have similar moduli spaces geometries, at least in the vicinity of the superconformal point where all VEVs $= 0$. Similar geometries means similar complex structures and similar Kähler functions, but alas the non-perturbative Kähler functions are out of our reach, so we cannot check their similarities for the A-theory's and B-theory's moduli spaces. On the other hand, the complex structure of the moduli space follows from the chiral ring equations, so once we see that the A and B theories have similar chiral ring equations, we shall know that their respective moduli spaces indeed have similar complex structures.

With all these preliminaries, let's look at the chiral ring equations of the two theories, starting with the pure-SQCD A theory. *Classically,* the mesons, the baryons, and the antibaryons of the A theory obey several constraints:

$$\text{rank}(\mathcal{M}\text{ matrix}) \;\le\; n \;\stackrel{\text{def}}{=}\; N_a\,, \tag{CA1}$$

$$\mathcal{M}^{f'[f}\times\mathcal{B}^{f_1,\ldots,f_n]} \;=\; 0, \tag{CA2}$$

$$\widetilde{\mathcal{B}}^{[f'_1,\ldots,f'_n}\times\mathcal{M}^{f']f} \;=\; 0, \tag{CA3}$$

$$\det\Bigl(\bigl[n\times n\text{ submatrix}(\mathcal{M})\bigr]^{f'_1,\ldots,f'_n;f_1,\ldots,f_n}\Bigr) \;=\; \widetilde{\mathcal{B}}^{f'_1,\ldots,f'_n}\times\mathcal{B}^{f_1,\ldots,f_n}. \tag{CA4}$$

Indeed, in matrix notations $\mathcal{M} = \widetilde{Q} \times Q$ where $\widetilde{Q}$ and Q are rectangular matrices with shorter side $N_a < N_f$, hence the rank constraint (CA1). Next,

$$\mathcal{M}^{f'[f} \times \mathcal{B}^{f_1,\ldots,f_n]} \;=\; \widetilde{Q}^{f'} \times Q^{[f} \times Q^{f_1} \cdots Q^{f_n]} \quad \langle\!\langle\,\text{color indices suppressed}\,\rangle\!\rangle. \tag{56}$$

Since the quarks are scalar bosons, antisymmetrizing their flavor indices requires also antisymmetrizing their colors, which is impossible for $n+1$ squark fields and only n colors, hence the constraint (CA2). The (CA3) constraint obtains in exactly the same way, so let me skip the details. Finally, the (CA4) constraint is $SU(N_f)_L \times SU(N_f)_D$ invariant, so we may check it in any particular flavor basis we like. So let's work in the basis where non-zero matrix elements of the rank $\le n$ matrix $\mathcal{M}$ are in the $n \times n$ upper left block of the matrix,

$$\mathcal{M} \;=\; \begin{pmatrix} \widehat{M} & 0 \\ 0 & 0 \end{pmatrix} \tag{57}$$

Consequently, in the same basis

$$Q \;=\; (\,\widehat{Q} \quad 0\,), \quad \widetilde{Q} \;=\; \begin{pmatrix} \widehat{\widetilde{Q}} \\ 0 \end{pmatrix}, \quad \text{where} \quad \widehat{M} \;=\; \widehat{\widetilde{Q}} \times \widehat{Q}, \tag{58}$$

and therefore

$$\mathcal{B}^{1,\ldots,n} \;=\; \det \widehat{Q}, \quad \widetilde{\mathcal{B}}^{1,\ldots,n} \;=\; \det \widehat{\widetilde{Q}}, \quad \text{all other} \quad \mathcal{B}^{f_1,\ldots f_n} \;=\; 0, \quad \widetilde{\mathcal{B}}^{f'_1,\ldots f'_n} \;=\; 0. \tag{59}$$

At the same time

$$\det(\widehat{\mathcal{M}}) \;=\; \det(\widehat{\widetilde{Q}}) \times \det(\widehat{Q}) \;=\; \widetilde{\mathcal{B}}^{1,\ldots,n} \times \mathcal{B}^{1,\ldots,n} \tag{60}$$

while

$$\det\bigl(\text{ any other } n \times n \text{ submatrix of } \mathcal{M}\bigr) \;=\; 0, \tag{61}$$

in perfect agreement with the constraint (CA4).

The constraints (CA1) through (CA4) are classical, but what happens to them in the quantum theory? In principle, they could suffer from quantum corrections — just like they do for $N_f = N_c$ — but in SQCD with $N_f > N_c$ massless flavors there are no such corrections. In particular, in the conformal window of SQCD any such corrections would prevent the theory from having a superconformal vacuum with $\langle\mathcal{M}\rangle = 0$, $\langle\mathcal{B}\rangle = 0$, and $\langle\widetilde{\mathcal{B}}\rangle = 0$, so we know the quantum theory has the same constraints (CA1–4) as the classical theory.

Also, in the massless SQCD the only on-shell chiral ring equation is $S \overset{\text{c.r.}}{=} 0$, so as far as the $\mathcal{M}, \mathcal{B}, \widetilde{\mathcal{B}}$ chiral ring generators are concerned, the only relevant chiral ring equations are the constraints (CA1–4).

Now consider the B theory. It's also an SQCD but with extra singlets $\Phi^{f'f}$, thanks to which the dual mesons $M_{f'f}$ has no VEVs, or in chiral ring terms obey $M_{f'f} \overset{\text{c.r.}}{=} 0$. Instead the chiral ring is generated by the dual baryons b, dual antibaryons $\tilde{b}$, and the Φ fields themselves. In terms of these generators, the chiral ring equations (CA1–4) of the A theory become

$$\begin{aligned}
\operatorname{rank}(\Phi\text{ matrix}) &\leq N_a = N_f - N_b\,, && \text{(CB1)}\\
\Phi^{f'f} \times b_{f,\ldots} &= 0, && \text{(CB2)}\\
\tilde{b}_{f',\ldots} \times \Phi^{f'f} &= 0, && \text{(CB3)}\\
\big[\operatorname{minor}_{N_b}(\Phi)\big]_{f'_1,\ldots,f'_{N_b};f_1,\ldots,f_{N_b}} &= \tilde{b}_{f'_1,\ldots,f'_{N_b}} \times b_{f_1,\ldots,f_{N_b}}\,. && \text{(CB4}*\text{)}
\end{aligned}$$

so our next task is to check that the $b, \tilde{b}, \Phi$ generators indeed obey these equations.

Let's start with the (CB1) constraint. From the dual quarks point of view $\langle\Phi\rangle$ is a mass matrix, for $\langle\Phi\rangle \neq 0$ the B theory has $\operatorname{rank}(\langle\Phi\rangle)$ massive quark flavors and $N_f - \operatorname{rank}(\langle\Phi\rangle)$ massless flavors. But we have learned earlier in class that SQCD with a number N_{f0} of massless flavors in the range

$$0 < N_{f0} < N_c \tag{62}$$

— plus any number of massive flavors — has no supersymmetric vacuum states; instead, the massless squark VEVs run away to infinity. Consequently, in a SUSY vacuum of the B theory the number of massless flavors must lie outside the range (62), thus either

$$N_{f0} \;=\; N_f \;-\; \mathrm{rank}(\langle\Phi\rangle) \;\geq\; N_b \quad\Longrightarrow\quad \mathrm{rank}(\langle\Phi\rangle) \;\leq\; N_f \;-\; N_b \;=\; N_a \tag{63}$$

— exactly as in eq. (CB1), — or else

$$\mathrm{rank}(\langle\Phi\rangle) \;=\; N_f \tag{64}$$

and hence no massless flavors at all. But an SQCD without any massless flavors has a non-zero gaugino condensate

$$\langle S\rangle \;=\; \left[\Lambda_B^{3N_b-N_f}\times\det(\lambda\Phi)\right]^{1/N_b} \;\times\; \sqrt[N_b]{1}, \tag{65}$$

which generates the non-perturbative superpotential for the Φ fields,

$$W_{\text{n.p.}}(\Phi) \;=\; \binom{\text{numeric}}{\text{constant}}\times\left[\Lambda_B^{3N_b-N_f}\times\det(\lambda\Phi)\right]^{1/N_b}. \tag{66}$$

Unfortunately, this superpotential has no stationary points for any Φ matrices of rank $= N_f$:

$$\forall\Phi\ \text{with}\ \det(\Phi)\neq 0: \quad \frac{\partial W_{\text{n.p.}}}{\partial\Phi^{f'f}} \;\neq\; 0, \tag{67}$$

which means we cannot have $\langle\Phi\rangle$ of rank $= N_f$ in any supersymmetric vacuum of the B theory. Since earlier we have similarly ruled out $\langle\Phi\rangle$ matrices of ranks between N_a and N_f, the only allowed option is to have $\mathrm{rank}(\langle\Phi\rangle)\leq N_a$, exactly as in the constraint (CB1).

Next, consider the baryons and antibaryons of the B theory. If any of the N_b dual squarks q_f comprising a baryon $b_{f,\ldots}$ happens to be massive, that squark cannot have a classical VEV hence also $\left\langle b_{f,\ldots}\right\rangle = 0$. Likewise, if any of the N_b antisquarks $\tilde{q}_{f'}$ comprising an antibaryon $\tilde{b}_{f',\ldots}$ is massive then that antibaryon also has a zero VEV. In terms of the $\langle\Phi\rangle$ matrix, these classical constraints amount to (CB2) and (CB3).

In the quantum theory, we have the same constraints as chiral ring equations. To see how that works, consider an infinitesimal non-linear variation of the anti-quark superfields

$$\delta\tilde{q}_{c,f'} = \epsilon C_{f'}^{f_2,\ldots,f_{N_b}} \epsilon_{c,c_2,\ldots,c_{N_b}} q_{c2}^{f_2} \cdots q_{c_{N_b}}^{f_{N_b}} \tag{68}$$

for some fixed tensor $C_{f'}^{f_2,\ldots,f_{N_b}}$ in flavor indices. At the same times, we do not vary the quarks at all, $\delta q_f^c = 0$. Under this variation, the tree-level superpotential of the B theory varies by

$$\begin{aligned}\delta W_{\text{tree}} &= \lambda \Phi^{f'f} q_f^c \times \epsilon C_{f'}^{f_2,\ldots,f_{N_b}} \epsilon_{c,c_2,\ldots,c_{N_b}} q_{c2}^{f_2} \cdots q_{c_{N_b}}^{f_{N_b}} \\ &= \epsilon\lambda C_{f'}^{f_2,\ldots,f_{N_b}} \times \Phi^{f'f} b_{f,f_2,\ldots,f_{N_b}} .\end{aligned} \tag{69}$$

At the same,

$$\delta K = Z\,\mathrm{tr}(\delta\tilde{q} \times e^{-2V}\overline{q}) = \epsilon \times \text{a combination of } N_b - 1 \text{ squarks and one } e^{-2V}\overline{q}, \tag{70}$$

Viewing the RHS here as some kind of a current, we see that it involves chiral and antichiral SF of different species, so it has no Konishi anomaly. Consequently,

$$\delta W_{\text{tree}} \stackrel{\text{c.r.}}{=} 0 \tag{71}$$

and therefore

$$\Phi^{f'f} b_{f,f_2,\ldots,f_{N_b}} \stackrel{\text{c.r.}}{=} 0 \tag{CB2}$$

for any combinations of the un-contracted flavor indices.

Likewise, a non-linear variation of the B-theory's quarks

$$\delta q_f^c = \epsilon C_f^{f'_2,\ldots,f'_{N_b}} \epsilon^{c,c_2,\ldots,c_{N_b}} \tilde{q}_{c2,f'_2} \cdots \tilde{q}_{c_{N_b},f'_{N_b}} \tag{72}$$

leads to

$$\delta W_{\text{tree}} = \epsilon\lambda C_f^{f'_2,\ldots,f'_{N_b}} \times \Phi^{f'f} \tilde{b}_{f',f'_2,\ldots,f'_{N_b}} \stackrel{\text{c.r.}}{=} \text{anomaly} = 0 \tag{73}$$

and hence

$$\Phi^{f'f} \times \tilde{b}_{f',f'_2,\ldots,f'_{N_b}} \stackrel{\text{c.r.}}{=} 0. \tag{CB3}$$

Finally, the constraint (CB4$*$). The way I wrote it, I identified the $\Phi^{f'f}$ of the B theory with the $\mathcal{M}^{f'f}$ mesons of the A theory. But such literal identification $\Phi^{f'f} = \mathcal{M}^{f'f}$ works only for the renormalized fields of the same non-canonical dimension $\Delta = 3N_b/N_f$. In terms of the un-renormalized fields which keep their canonical dimensions — 1 for the elementary $\Phi^{f'f}$ but 2 for the mesons $\mathcal{M}^{f'f}$ made from 2 elementary scalars — the duality between A and B theory works as

$$\mathcal{M}^{f'f} \;=\; \mu\times\Phi^{f'f} \tag{74}$$

for some constant μ of dimension mass^{+1}. I shall let you calculate this constant in your next homework set#9. Meanwhile, let me simply rewrite the constraint (CB4∗) as

$$\tilde{b}_{f'_1,\ldots,f'_{N_b}}\times b_{f_1,\ldots,f_{N_b}} \;=\; \text{const}\times\bigl[\text{minor}_{N_b}(\Phi)\bigr]_{f'_1,\ldots,f'_{N_b};f_1,\ldots,f_{N_b}}. \tag{CB4}$$

To check the constraint (CB4), we first note that it's invariant WRT $SU(N_f)_L\times SU(N_f)_D$, so once we verify it in any particular flavor basis it's then automatically valid in any other flavor bases. So let's wok in the basis where the $\langle\Phi\rangle$ matrix is diagonal,

$$\langle\Phi\rangle \;=\; \text{diag}\bigl(\Phi^{11},\Phi^{22},\ldots,\Phi^{nn},0,\ldots,0\bigr) \qquad \langle\!\langle\,\text{note rank}(\langle\Phi\rangle)\le n=N_a\,\rangle\!\rangle. \tag{75}$$

In this basis, eq. (CB4) becomes

$$\tilde{b}_{n+1,\ldots,N_f}\times b_{n+1,\ldots,N_f} \;=\; \text{const}\times\Phi^{11}\Phi^{22}\cdots\Phi^{nn}, \tag{76}$$

$$\text{all other }\tilde{b}_{f'_1,\ldots,f'_{N_b}}\times b_{f_1,\ldots,f_{N_b}} \;=\; 0. \tag{77}$$

To verify these equations, let first assume rank$(\langle\Phi\rangle)=N_a$ so there are precisely N_a massive and N_b massless flavors of dual quarks and antiquarks. All baryons comprising a massive quark flavor — and all antibaryons comprising a massive antiquark flavor — have zero VEVs, hence eq. (77). As to the massless flavors — whose number is the same as the number of B-theory's colors — the mesons and the baryons we can make from them obey

$$\det(M_{\text{massless}}) \;-\; \tilde{b}_{\text{massless}}\times b_{\text{massless}} \;\stackrel{\text{c.r.}}{=}\; \Lambda_{\text{eff}}^{2N_b}\,. \tag{78}$$

In this formula,

$$\det(M_{\text{massless}}) \;\stackrel{\text{c.r.}}{=}\; 0 \tag{79}$$

because in the B theory all $M_{f'f} \overset{\text{c.r.}}{=} 0$,

$$\tilde{b}_{\text{massless}} \times b_{\text{massless}} \;=\; \tilde{b}_{n+1,\ldots,N_f} \times b_{n+1,\ldots,N_f}\,, \tag{80}$$

and Λ_{eff} is the Λ of the effective low-energy theory from which all massive quarks have been integrated out,

$$\Lambda_{\text{eff}}^{2N_b} \;=\; \Lambda_B^{3N_b-N_f} \times (\lambda\Phi^{11})(\lambda\Phi^{22})\cdots(\lambda\Phi^{nn}). \tag{81}$$

Plugging all these formula into eq. (78), we immediately obtain

$$\tilde{b}_{n+1,\ldots,N_f} \times b_{n+1,\ldots,N_f} \;=\; \text{const} \times \Phi^{11}\Phi^{22}\cdots\Phi^{nn}, \tag{82}$$

exactly as in eq. (76).

Finally, suppose $\text{rank}(\langle\Phi\rangle) < N_a$, so the B theory has less then N_a massive flavors and more than N_b massless flavors. Again, the baryons and the antibaryons comprising any massive flavors have vanishing VEVs. As to the baryons and the antibaryons made only from the massless flavors, their products are related to the minors of the B-theory's meson matrix $M_{f'f}$ without any quantum corrections because $N_{f,\text{massless}} > N_b$. But in the B theory all $M_{f'f} \overset{\text{c.r.}}{=} 0$, hence $\tilde{b}\times b \overset{\text{c.r.}}{=} 0$. Altogether, all $\tilde{b}\times b \overset{\text{c.r.}}{=} 0$ for all flavors, massive and massless. At the same time, all degree-N_b minors of matrix of rank $< N_f - N_b$ vanish, so we have zeros on both sides of eq. (CB4). *Quod erat demonstrandum.*

5. Deformations

Consider slight deformations of the Seiberg's A and B theories. In particular, we may give small quark masses to the A theory, thus

$$W_A^{\text{tree}} \;=\; -\sum_{f,f'} M_{ff'}\widetilde{Q}_c f' Q^{c,f}. \tag{83}$$

In the B theory, explicit quark masses would be equivalent to constant shifts of the $\Phi^{f'f}$ fields, so there is no point of adding them to the theory. But instead, we may give small

O'Raifeartaigh terms to the gauge-singlet $\Phi^{f'f}$, thus

$$W_B^{\rm tree} \;=\; \lambda \sum_{c,f,f'} \Phi^{f'f} q_f^c \tilde{q}_{c,f'} \;-\; \mu \sum_{f,f'} M_{ff'} \Phi^{f'f}. \tag{84}$$

In either case, the deformation operator is relevant and explicitly breaks the conformal symmetry of the IR theory. However, as long as the deformation is much smaller than the energy scale E_C at which the UV theory flows to its IR fixed point and becomes approximately conformal, the effects of the deformation are governed by the approximately superconformal IR theory rather than by the UV theory. Consequently, if the Seiberg's A theory and B theory are truly IR-dual to each other, then mass deformation of the A theory and the O'Raifeartaigh deformation of the B theory should have exactly similar effects.

Verifying that this is indeed the case is an interesting exercise, but I am going to leave it out from these notes and make it a part of your next homework set#9.

6. Double Duality

To complete these notes, let me show that the Seiberg dual of the Seiberg dual theory B is equivalent to the original theory A. Indeed, at the core of the B theory is SQCD with N_f flavors and N_c^B colors, so its Seiberg dual — which I shall temporarily call the C theory — includes SQCD with the same N_f flavors but $N_c^C = N_f - N_c^B = N_c^A$ colors, same as the original theory A. Also, since the B-theory's quarks q_f^c and antiquarks $\tilde{q}_{c,f'}$ have lower flavor indices, the C-theory's quarks and antiquarks should have upper flavor indices, same as $Q^{c,f}$ and $\widetilde{Q}_c^{f'}$ of the A theory. In addition, the Seiberg duality endows the C theory with N_f^2 gauge-singlet fields $\phi_{f'f}$ with the same flavor quantum numbers as the B-theory's mesons $M_{f'f}$ and with Yukawa couplings to the C-theories quarks and antiquarks,

$$W_C \;\supset\; \lambda' \sum_{c,f,f'} \phi_{f'f} \times \widetilde{Q}_c^{f'} Q^{c,f}. \tag{85}$$

But besides its SQCD core, the B theory has its own gauge-singlet fields $\Phi^{f'f}$, and the second Seiberg duality $B \to C$ does not take them away. Furthermore, in the B-theory's Yukawa couplings

$$W_B \;=\; \lambda \sum_{c,f,f'} \Phi^{f'f} \times q^c_f \tilde{q}_{c,f'} \;=\; \lambda \sum_{f,f'} \Phi^{f'f} \times M_{f'f}\,, \tag{86}$$

the B-theory's mesons $M_{f'f}$ are replaced in the C theory with the $\phi_{f'f}$ fields, or rather with the $\mu'\phi_{f'f}$ for some constant μ' of dimension mass^{+1}, thus

$$W^{\text{tree}}_C \;=\; \lambda' \sum_{c,f,f'} \phi_{f'f} \times \widetilde{Q}^{f'}_c Q^{c,f} \;+\; \lambda\mu' \sum_{f,f'} \Phi^{f'f} \times \phi_{f'f}. \tag{87}$$

But the red term in this superpotential is nothing but the mass term for all the $\Phi^{f'f}$ and $\phi_{f'f}$ gauge singlets, and this mass eliminates all those gauge-single fields from the deep-IR limit of the theory. Instead, all we have in the deep-IR limit of the C theory is its SQCD core with N_f flavors and $N^C_c = N^A_c$ colors — exactly as in the A theory. Thus, performing the Seiberg duality twice we indeed come back to the original A theory.