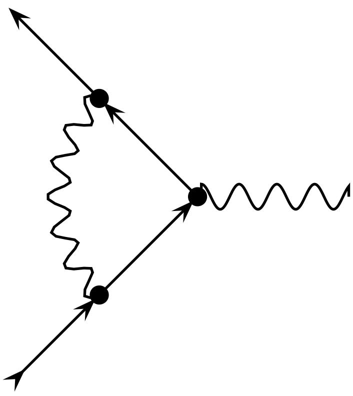
Gauge Dependence

In QED, the on-shell physical amplitudes do not depend on the gauge-fixing condition, but that's unfortunately not true for the off-shell amplitudes. Even the UV divergences — and hence the counterterms which cancel them — depend on the gauge-fixing conditions. In particular, in the Lorenz-invariant gauges where the photon propagator is

$$= \frac{-i}{k^2 + i0}\left[g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right], \tag{1}$$

the off-shell amplitudes and the counterterms depend on the ξ parameter. In these notes, we shall focus on the ξ dependence of the δ_1 and the δ_2 counterterms.

Let's start with the one-loop δ_1 counterterm which cancels the UV divergence of the vertex correction



(2)

Evaluating this diagram for the general ξ gauge, we get

$$\begin{aligned}
ie\Gamma^\mu_{1\,\text{loop}}(p', p) &= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, ie\gamma_\nu \times \frac{i}{\not{p}' + \not{k} - m + i0} \times ie\gamma^\mu \times \frac{i}{\not{p} + \not{k} - m + i0} \times ie\gamma_\lambda \times \\
&\qquad \times \frac{-i}{k^2 + i0}\left[g^{\lambda\nu} + (\xi - 1)\frac{k^\lambda k^\nu}{k^2 + i0}\right] \\
&= e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 + i0} \times \gamma^\nu \frac{1}{\not{p}' + \not{k} - m + i0}\gamma^\mu \frac{1}{\not{p} + \not{k} - m + i0}\gamma_\nu \\
&\quad + (\xi - 1)e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k^2 + i0)^2} \times \not{k}\, \frac{1}{\not{p}' + \not{k} - m + i0}\gamma^\mu \frac{1}{\not{p} + \not{k} - m + i0}\, \not{k} \\
&= ie\Gamma^\mu_F(p', p) + (\xi - 1) \times ie\Delta\Gamma^\mu(p', p)
\end{aligned} \tag{3}$$

where Γ^μ_F stands for the $\Gamma^\mu_{1\,\mathrm{loop}}$ which obtains in the Feynman gauge $\xi = 0$ — see my notes on the dressed QED vertex for detail, — while

$$\Delta\Gamma^\mu(p',p) \;=\; -ie^2\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\times \not{k}\,\frac{1}{\not{p}'+\not{k}\;-m+i0}\,\gamma^\mu\,\frac{1}{\not{p}+\not{k}\;-m+i0}\,\not{k} \tag{4}$$

is the gauge-dependent correction. Fortunately, this correction drastically simplifies for the on-shell electrons when $\Delta\Gamma^\mu$ appears in the context of $\bar{u}(p')\Delta\Gamma^\mu u(p)$. Indeed, in this context

$$\frac{1}{\not{p}+\not{k}\;-m+i0}\,\not{k} \;=\; 1\;-\;\frac{1}{\not{p}+\not{k}\;-m+i0}\,(\not{p}-m)\;\cong\;1$$

because $(\not{p}-m)u(p)=0$, and likewise

$$\not{k}\,\frac{1}{\not{p}'+\not{k}\;-m+i0}\;=\;1\;-\;(\not{p}'-m)\,\frac{1}{\not{p}'+\not{k}\;-m+i0}\;\cong\;1.$$

Consequently, eq. (4) simplifies to

$$\Delta\Gamma^\mu \;=\; e^2\gamma^\mu\times\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \tag{5}$$

which does not depend on any momenta, p, p', or q, but only on the UV and the IR regulators.

Finally, since the complete dressed vertex involves not only the loop diagram (2) but also the δ_1 counterterm, we see that the gauge-dependent correction (5) can be completely canceled by the gauge-dependent correction to the δ_1, namely

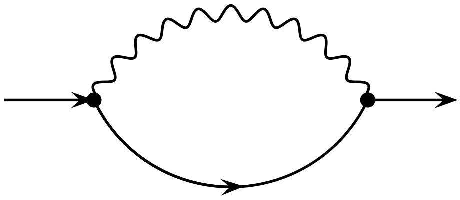
$$\delta_1(\xi) \;=\; \delta_1^{\rm Feynman\,gauge}\;-\;(\xi-1)\times e^2\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{6}$$

Now consider the δ_2 counterterm, which together with the δ_m counterterm cancel the UV divergence of the electron's self energy

$$\Sigma^e_{\rm net}(\not{p}) \;=\; \Sigma^e_{\rm loops}(\not{p}) \;+\; \delta_m \;-\; \delta_2\,\not{p}. \tag{7}$$

At the one loop level,

$$-i\Sigma_{1\,\rm loop}(\not{p}) \;=\; \tag{8}$$



which evaluates to

$$\begin{aligned}
-i\Sigma^{1\,\rm loop}(\not{p}) \;&=\; \int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\, ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu\times\frac{-i}{k^2+i0}\left[g^{\lambda\nu}\;+\;(\xi-1)\frac{k^\lambda k^\nu}{k^2+i0}\right]\\
&=\; -e^2\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i0}\times\gamma^\nu\,\frac{1}{\not{k}+\not{p}-m_e+i0}\,\gamma_\nu\\
&\qquad+\;(\xi-1)\times(-e^2)\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\times\not{k}\,\frac{1}{\not{k}+\not{p}-m_e+i0}\;\not{k}\\
&=\; -i\Sigma_F(\not{p})\;-\;i(\xi-1)\times\Delta\Sigma(\not{p}),
\end{aligned} \tag{9}$$

where $\Sigma_F(\not{p})$ is the $\Sigma_{1\,\rm loop}$ which obtains in the Feynman gauge — and which you should calculate in your homework set#18, — while

$$\Delta\Sigma(\not{p}) \;=\; -ie^2\int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\times\not{k}\,\frac{1}{\not{k}+\not{p}-m_e+i0}\;\not{k} \tag{10}$$

is the gauge-dependent correction. To compensate for this correction, we should also correct the δ_2 and δ_m counterterms to assure that $\Sigma_{\rm net}$ and $d\Sigma_{\rm net}/d\not{p}$ both vanish at $\not{p}=m$, hence

$$\delta_2 \;=\; \delta_2^{\rm Feynman\,gauge}\;+\;(\xi-1)\Delta\delta_2\,,\quad \delta_m \;=\; \delta_m^{\rm Feynman\,gauge}\;+\;(\xi-1)\Delta\delta_m\,, \tag{11}$$

for

$$\Delta\delta_2 \;=\; \frac{d\Delta\Sigma}{d\not p}\bigg|_{\not p=m} \quad\text{and}\quad \Delta\delta m \;-\; m\Delta\delta_2 \;=\; -\Delta\Sigma(\not p = m). \tag{12}$$

Taking the derivative of $\Delta\Sigma$ from eq. (10), we get

$$\frac{d\Delta\Sigma}{d\not p} \;=\; e^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\times\not k\,\frac{-1}{(\not k+\not p-m_e+i0)^2}\,\not k, \tag{13}$$

where

$$\not k\,\frac{1}{(\not k+\not p-m_e+i0)^2}\,\not k \;=\; \left(1\,-\,(\not p-m)\,\frac{1}{\not p+\not k-m+i0}\right)\times\left(1\,-\,\frac{1}{\not p+\not k-m+i0}\,(\not p-m)\right)$$
$$\to\; 1 \text{ for } \not p = m. \tag{14}$$

Consequently,

$$\Delta\delta_2 \;=\; \frac{d\Delta\Sigma}{d\not p}\bigg|_{\not p=m} \;=\; -e^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{15}$$

As to the $\Delta\delta_m$ corrections to the mass counterterm, I leave its calculation to a future homework.

Finally, comparing eqs. (6) and (15), we see that the gauge-dependent corrections to the δ_1 and δ_2 counterterms are exactly the same,

$$\Delta\delta_1 \;=\; \Delta\delta_2 \;=\; -e^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{16}$$

Therefore, once we verify the Ward identity $\delta_1 = \delta_2$ in the Feynman gauge — which you hopefully do in your current homework#18, — it follows that

$$\delta_1(\xi) \;=\; \delta_2(\xi) \quad \text{in any gauge}. \tag{17}$$