

INTRODUCTION TO SPONTANEOUS SYMMETRY BREAKING

Consider a physical system of some kind whose dynamics has a symmetry group G . That is, the system's Lagrangian and Hamiltonian are invariant under G while the equations of motion are covariant WRT G . But the system does not have a G -invariant ground state; instead, there is a degenerate family of ground states related to each other by the symmetry actions. Symmetries like that are called *spontaneously broken*.

Classically, there are plenty of examples of spontaneously broken symmetries. For example, consider a roulette wheel

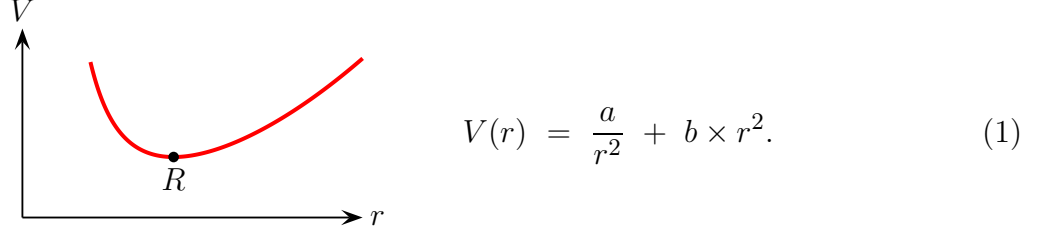


The motion of the ball on this wheel has exact $\mathbf{Z}_{38}$ rotational symmetry. The ground states of the ball are the places where it can rest once it runs out of both kinetic and potential energy. But instead of a single rotationally invariant ground state, the ball has 38 exactly degenerate states — one for each sector of the roulette — related to each other by the $\mathbf{Z}_{38}$ symmetry.

For another example, consider a pencil standing on its point on a flat table. The motion of this pencil has exact $SO(2)$ rotational symmetry in the horizontal plane, but the only $SO(2)$ invariant state of the pencil — standing vertically up — is not a ground state but a local maximum of the potential energy. Instead, the ground states of the pencil correspond to it laying down on the table; there is a continuous degenerate family of such states with different orientations of the pencil, and they are all related to each other by the $SO(2)$ rotations.

On the other hand, in Quantum Mechanics of a single particle (or several particles), spontaneous symmetry breaking (SSB) is not allowed. For example, consider the $SO(3)$

rotational symmetry of a particle moving in a central potential $V(r)$. Suppose the potential has a local maximum at $r = 0$, global minimum at some finite $r = R$, and grows up at $r \rightarrow \infty$, for example



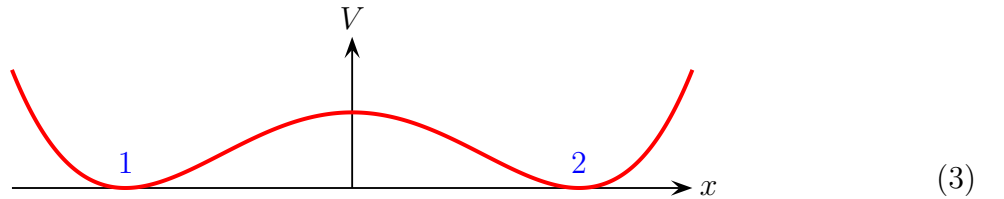
Classically, the ground states are minima of this potential — which form a degenerate sphere of radius R . At every point of this sphere, the particle has a different direction $\mathbf{n}$, which indicates spontaneous breakdown of the rotational $SO(3)$ symmetry of the system.

But in the quantum theory, the particle's ground state is smeared all over that sphere in an s-wave state

$$\begin{aligned} \psi(r, \theta, \phi) &= f(r) \times Y_{0,0}(\theta, \phi) \\ \text{where } Y_{0,0}(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} = \text{const} \end{aligned} \quad (2)$$

while $f(r)$ is a radial wave-function that has a peak at $r \approx R$. The wave-function (2) is invariant under all $SO(3)$ rotations of space, so the rotation symmetry is *not* spontaneously broken.

For a discrete symmetry example, consider the $\mathbf{Z}_2$ symmetry $x \rightarrow -x$ or a 1D particle in a double-well potential



Semiclassically, this particle has two degenerate ground states related by the $x \rightarrow -x$ symmetry:

$$\begin{aligned} |1\rangle &\text{ particle at bottom of well\#1,} \\ |2\rangle &\text{ particle at bottom of well\#2,} \end{aligned} \quad (4)$$

although in both cases the particle is not quite resting at the bottom of a well but is in the

zero-point state motion around the bottom. Anyway, having two degenerate ground states related by the symmetry indicates the spontaneous breaking of that symmetry.

But in the fully quantum theory, the particle may tunnel between the wells, which leads to a tiny but non-zero matrix element $\langle 1 | \hat{H} | 2 \rangle$. Consequently, the would-be ground states (4) are not exactly eigenstates of the Hamiltonian, and the true eigenstates are the linear combinations

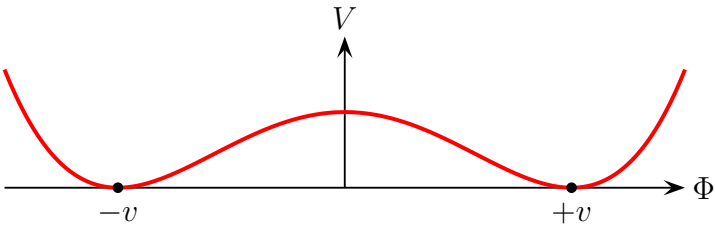
$$|+\rangle = \sqrt{\frac{1}{2}}(|1\rangle + |2\rangle) \quad \text{and} \quad |-\rangle = \sqrt{\frac{1}{2}}(|1\rangle - |2\rangle). \quad (5)$$

Moreover, these true eigenstates have slightly different energies,

$$E_- - E_+ \text{ is tiny but positive,} \quad (6)$$

so there is only one true ground state of the theory: the $|+\rangle$, which happens to be invariant under the $x \rightarrow -x$ symmetry. Thus, there is no spontaneous symmetry breaking.

However, in the quantum field theory in $d > 0$ space dimensions, the tunneling is suppressed, which allows for the spontaneous symmetry breaking. For example, consider a real scalar field in $d + 1$ dimensions (d space one time) with the double-well potential

$$V(\phi) = \frac{\lambda}{8}(\Phi^2 - v^2)^2, \quad (7)$$


This potential — and hence the whole QFT — is invariant under the discrete $\mathbf{Z}_2$ symmetry $\Phi(x) \rightarrow -\Phi(x)$, but semiclassically this symmetry is spontaneously broken: there are two degenerate vacuum states related by the symmetry,

$$\begin{aligned} |\text{vac}\#1\rangle & \quad \text{with} \quad \langle \text{vac}\#1 | \hat{\Phi}(x) | \text{vac}\#1 \rangle = -v, \\ |\text{vac}\#2\rangle & \quad \text{with} \quad \langle \text{vac}\#2 | \hat{\Phi}(x) | \text{vac}\#2 \rangle = +v. \end{aligned} \quad (8)$$

To understand the tunneling (or rather lack of tunnelling) between these two states, let's

expand the $\Phi(\mathbf{x}, t)$ field into the momentum modes,

$$\Phi(\mathbf{x}, t) = \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \Phi_{\mathbf{p}}(t) \quad (9)^{\star}$$

Since in both vacuum states (8) the field $\Phi(\mathbf{x})$ (or rather its expectation value) is uniform throughout the space, the $\mathbf{p} = 0$ mode Φ_0 is the only mode responsible for the tunneling between the two vacua. The effective action for just this mode is

$$S_{\text{eff}} = L^d \times \int dt \left(\frac{1}{2} \dot{\Phi}_0^2 - V(\Phi_0) \right), \quad (10)$$

hence the canonical conjugate variable to the Φ_0 is $\Pi_0 = L^d \times \dot{\Phi}_0$, and the effective Hamiltonian for the quantum theory of the $\mathbf{p} = 0$ mode is

$$\hat{H}_{\text{eff}} = \frac{1}{2L^d} \hat{\Pi}_0^2 + L^d \times V(\hat{\Phi}_0). \quad (11)$$

Consequently, the WKB approximation to the wavefunction of a tunneling state is

$$\Psi(\phi_0) \sim \exp \left(\pm \int d\phi_0 \sqrt{2L^{2d} \times V(\phi_0)} \right), \quad (12)$$

hence the tunneling amplitude between the two vacuum states (8)

$$\langle \text{vac}\#2 | \hat{H} | \text{vac}\#1 \rangle \sim \exp \left(-L^d \times \int_{-v}^{+v} d\phi_0 \sqrt{2V(\phi_0)} \right). \quad (13)$$

For the quartic double-well potential (7), the integral here evaluates to

$$\int_{-v}^{+v} d\phi_0 \sqrt{2V(\phi_0)} = \frac{2}{3} \lambda^{1/2} v^3, \quad (14)$$

but its specific value is not important. What's important is that *in the infinite volume limit* $L^d \rightarrow \infty$ *the tunneling amplitude* (13) *dies off exponentially*,

$$\left(\begin{array}{c} \text{tunneling} \\ \text{amplitude} \end{array} \right) \sim \exp(-\text{const} \times L^d) \xrightarrow{L^d \rightarrow \infty} 0. \quad (15)$$

Thus, in the quantum field theory in infinite space there is no tunneling between the two

$\star$ For the moment, I take the space to be a box of large but finite volume L^d , hence discrete momenta $\mathbf{p}$. Nevertheless, let me normalize the modes as in eq. (9), without the $L^{-d/2}$ factors.

would-be ground states (8), and we do not need to take the symmetric linear combination $|+\rangle$ of those two states to get the true ground state of the theory.

The above argument leaves open a question: Just because we do not have to take a linear combinations of the two degenerate ground states (8), why don't we take such a linear combination just for the fun of it? Indeed, from the purely QM point of view, any linear combination of tow exactly degenerate ground states is itself a Hamiltonian eigenstate of exactly same lowest energy. And the answer to this question is: **in QFT, the physical vacuum is more than just the lowest-energy eigenstate of the Hamiltonian, it should also obey the *cluster decomposition rule*.** Among other things, this rule forbids infinite-range connected correlation functions of quantum fields or other kinds of local operators. Thus, the two-point connected correlation function

$$\begin{aligned} G_2^{\text{conn}}(x-y) &\stackrel{\text{def}}{=} \langle \text{vac} | (\hat{\Phi}(x) - \langle \Phi \rangle) (\hat{\Phi}(y) - \langle \phi \rangle) | \text{vac} \rangle \\ &= \langle \text{vac} | \hat{\Phi}(x) \hat{\Phi}(y) | \text{vac} \rangle - \langle \text{vac} | \hat{\Phi} | \text{vac} \rangle^2 \\ &\text{must} \longrightarrow 0 \text{ when } |x-y| \rightarrow \infty. \end{aligned} \quad (16)$$

For each of the two vacuum states (8), such correlation function obtains from the two-point connected Feynman diagrams of the shifted theory (*i.e.*, theory expanded in terms of $\Phi'(x) = \Phi(x) - \langle \Phi \rangle$), and it indeed dies off for $|x-y| \rightarrow \infty$. But now consider a linear combination

$$|\text{mix}\rangle = \alpha |\text{vac}\#1\rangle + \beta |\text{vac}\#2\rangle \quad \text{for } |\alpha|^2 + |\beta|^2 = 1 \quad (17)$$

of the two vacuum states (8). In such a linear combination

$$\begin{aligned} \langle \text{mix} | \hat{\Phi}(x) \hat{\Phi}(y) | \text{mix} \rangle &= |\alpha|^2 \times \langle \text{vac}\#1 | \hat{\Phi}(x) \hat{\Phi}(y) | \text{vac}\#1 \rangle \\ &\quad + |\beta|^2 \times \langle \text{vac}\#2 | \hat{\Phi}(x) \hat{\Phi}(y) | \text{vac}\#2 \rangle \\ &\quad + 2\Re(\alpha^* \beta \times \langle \text{vac}\#1 | \hat{\Phi}(x) \hat{\Phi}(y) | \text{vac}\#2 \rangle) \\ &\xrightarrow{(x-y) \rightarrow \infty} |\alpha|^2 \times v^2 + |\beta|^2 \times v^2 + 2\Re(\alpha^* \beta \times 0) \\ &= 1 \times v^2, \end{aligned} \quad (18)$$

while

$$\begin{aligned}
\langle \text{mix} | \hat{\Phi} | \text{mix} \rangle &= |\alpha|^2 \times \langle \text{vac\#1} | \hat{\Phi} | \text{vac\#1} \rangle + |\beta|^2 \times \langle \text{vac\#2} | \hat{\Phi} | \text{vac\#2} \rangle \\
&\quad + 2\Re(\alpha^* \beta \times \langle \text{vac\#1} | \hat{\Phi} | \text{vac\#2} \rangle) \\
&= |\alpha|^2 \times (+v) + |\beta|^2 \times (-v) + 2\Re(\alpha^* \beta \times 0) \\
&= v \times (|\beta|^2 - |\alpha|^2).
\end{aligned} \tag{19}$$

Consequently, in the mixed would-be-vacuum state

$$G_2^{\text{conn}}(x-y) \xrightarrow{(x-y) \rightarrow \infty} = v^2 - v^2 \times (|\beta|^2 - |\alpha|^2)^2 = v^2 \times 4|\alpha|^2|\beta|^2 \neq 0, \tag{20}$$

so the mixed states do not obey the cluster decomposition rules. Thus, the only valid vacuum states of the QFT are the two states

$$\begin{aligned}
|\text{vac\#1}\rangle \quad \text{with} \quad \langle \text{vac\#1} | \hat{\Phi}(x) | \text{vac\#1} \rangle &= -v, \\
|\text{vac\#2}\rangle \quad \text{with} \quad \langle \text{vac\#2} | \hat{\Phi}(x) | \text{vac\#2} \rangle &= +v,
\end{aligned} \tag{8}$$

but not any of their linear combinations.

The bottom line is, the QFT in question has two exactly degenerate vacuum states (8), neither of which is symmetric WRT $\Phi \rightarrow -\Phi$; instead, the symmetry relates the two vacua to each other. Thus, the $\Phi \rightarrow -\Phi$ symmetry of the theory is spontaneously broken. And this is an example of the general rule: In $D = 1 + 0$ spacetime dimensions (*i.e.*, in quantum mechanics) any spontaneous symmetry breaking is forbidden, but in all higher dimensions $D \geq 1 + 1$, a *discrete* symmetry may be spontaneously broken.

Or rather, this is the general at zero temperature. At finite temperatures — in condensed matter, or in the early Universe — a spontaneous breaking of a discrete symmetry requires $D \geq 1 + 2$ spacetime dimensions. In $D = 1 + 1$ — *i.e.*, in 1 space dimension — the thermal equilibrium state at any $T > 0$, however low, is not any of the ground states but a chain of finite-length domains separated by domain walls. In each domain, the QFT fluctuates around a ground state, but different domains fluctuate around different ground states, so on average over the infinite space the discrete symmetry remains unbroken. However, at low temperatures the average domain lengths grow exponentially with $1/T$, so locally within a long domain the symmetry appears to be spontaneously broken.

So here is the bottom line for the spontaneous breaking of discrete symmetries:

- In $D = 1 + 0$ dimensions (QM) there is no spontaneous symmetry breaking.
- In $D = 1 + 1$ dimensions, SSB is possible at zero temperature but not at finite temperatures.
- In higher dimensions $d \geq 1 + 2$, SSB is possible at both $T = 0$ and $T > 0$.

As to spontaneous breaking of continuous symmetries, see [my next set of notes](#).