LORENTZ GROUPS

In these notes I shall explain the Lorentz symmetry groups of the Minkowski spacetime. But as a warm-up exercise, let's start with the rotation symmetries of the ordinary 3D space. In terms of the (x,y,z) coordinates, a rotation is a linear coordinate transform

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R^{xx} & R^{xy} & R^{xz} \\ R^{yx} & R^{yy} & R^{yz} \\ R^{zx} & R^{zy} & R^{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

— or in index notations

$$x'^i = R^{ij}x^j \quad \langle\!\langle \text{implicit } \textstyle\sum_j \rangle\!\rangle, \tag{2}$$

— where $R = \left\|R^{ij}\right\|$ is a 3×3 real *orthogonal* matrix. That is, a matrix obeying

$$R^\top R = 1_{3\times 3} \quad \Longleftrightarrow \quad R^{ij}R^{ik} = \delta^{jk}, \tag{3}$$

which assures that the dot product of space vectors — and hence the vector's magnitudes — are invariant under rotations:

$$\mathbf{x}'\cdot\mathbf{y}' = x'^i y'^i = R^{ij}x^j \times R^{ik}y^k = \left(R^{ij}R^{ik} = \delta^{jk}\right) \times x^j y^k = x^j y^j = \mathbf{x}\cdot\mathbf{y}. \tag{4}$$

The 3×3 orthogonal matrices form a *group* called $O(3)$. That is: (1) the product of any two orthogonal matrices is itself an orthogonal matrix; (2) the unit matrix is orthogonal; (3) every orthogonal matrix is invertible and its inverse is also an orthogonal matrix.

The $O(3)$ matrices form a continuous manifold in the space of all 3×3 matrices. However, this *group manifold* is not connected but has two separate connected components disconnected from each other because the determinant of an $O(3)$ matrix takes 2 discrete values $\det(R) = \pm 1$. The matrices of $\det(R) = +1$ form a continuous subgroup of $O(3)$ called $SO(3)$ for *special* orthogonal 3×3 matrices. Physically, it's the $SO(3)$ matrices which correspond to the rotations of space. Indeed, a rotation through any angle ϕ around any axis (denoted by a unit vector $\mathbf{n}$) corresponds to an $SO(3)$ matrix

$$R^{ij}(\phi,\mathbf{n}) \;=\; \cos\phi\times\delta^{ij} \;+\; \sin\phi\times\epsilon^{ijk}n^k \;+\; (1-\cos\phi)\times n^i n^j. \tag{5}$$

Conversely, any $SO(3)$ matrix R corresponds to a rotation $R(\phi,\mathbf{n})$ through some angle ϕ around some axis $\mathbf{n}$. Moreover, two successive rotation symmetries $\mathbf{x}\to\mathbf{x}'\to\mathbf{x}''$,

$$x'^j \;=\; R_1^{jk}x^k, \qquad x''^i \;=\; R_2^{ij}x'^j \;=\; R_2^{ij}R_1^{jk}x^k \;=\; (R_2R_1)^{ik}x^k, \tag{6}$$

amount to a rotation corresponding to the matrix product $R_{1,2}=R_2R_1$. Consequently, *the rotations symmetries themselves form a group isomorphic to the $SO(3)$*.

As to the $O(3)$ matrices with negative determinants $\det(R)=-1$, they correspond to combinations of a rotation with the *space reflection* — also called the *parity* —

$$P:(x,y,z)\;\to\;(-x,-y,-z). \tag{7}$$

In particular, mirror reflections have this form: they combine the complete space reflection P with a 180° rotation in the plane of the mirror. For example, the reflection of a mirror in the (xy) plane is

$$M \;=\; R(180^\circ,z)\times P: \quad (x,y,z)\;\to\;(-x,-y,-z)\;\to\;(+x,+y,-z). \tag{8}$$

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Now consider the Lorentz symmetries of the Minkowski spacetime. By analogy with the 3D rotations, a general Lorentz symmetry is a linear transform of the coordinates (ct,x,y,z),

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} L^0_{\ 0} & L^0_{\ 1} & L^0_{\ 2} & L^0_{\ 3} \\ L^1_{\ 0} & L^1_{\ 1} & L^1_{\ 2} & L^1_{\ 3} \\ L^2_{\ 0} & L^2_{\ 1} & L^2_{\ 2} & L^2_{\ 3} \\ L^3_{\ 0} & L^3_{\ 1} & L^3_{\ 2} & L^3_{\ 3} \end{pmatrix} \times \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{9}$$

$$X'^\mu \;=\; L^\mu_{\ \nu}X^\nu \quad \langle\!\langle\,\text{implicit }\textstyle\sum_\nu\,\rangle\!\rangle$$

which preserves the Lorentzian dot products of 4-vectors:

$$X' \cdot Y' \;=\; X'^{\mu} g_{\mu\nu} Y'^{\nu} \;=\; L^{\mu}_{\;\alpha} X^{\alpha} \times g_{\mu\nu} \times L^{\nu}_{\;\beta} Y^{\beta} \;=\text{should}=\; X^{\alpha} g_{\alpha\beta} Y^{\beta} \;=\; X \cdot Y. \tag{10}$$

This equality — for any X^{α} and any Y^{β} — calls for

$$L^{\mu}_{\;\alpha} \times g_{\mu\nu} \times L^{\nu}_{\;\beta} \;=\; g_{\alpha\beta}\,, \tag{11}$$

or in matrix form

$$L^{\top} g L \;=\; g. \tag{12}$$

This is the Lorentzian generalization of the $R^{\top}R = 1$ orthogonality constraint for the rotation matrices.

The real 4×4 matrices obeying the *pseudo-orthogonality* constraint (12) form a group called $O(3,1)$, where $(3,1)$ refers to space and time dimensions of the Minkowski spacetime. Similar to rotations, two successive general Lorentz symmetries corresponding to respective $O(3,1)$ matrices L_1 and L_2 combine to another general Lorentz symmetry $L_{1,2}$ that is a matrix product $L_2 L_1$:

$$\begin{gathered} X \;\to\; X' \;\to\; X'', \qquad X'^{\mu} \;=\; L^{\mu}_{1\lambda} X^{\lambda}, \qquad X''^{\nu} \;=\; L^{\nu}_{2\mu} X'^{\mu} \\ X''^{\nu} \;=\; L^{\nu}_{2\mu} X'^{\mu} \;=\; L^{\nu}_{2\mu} L^{\mu}_{1\lambda} X^{\lambda} \;=\; (L_2 L_1)^{\nu}_{\;\lambda} X^{\lambda}. \end{gathered} \tag{13}$$

Thus, *the general Lorentz symmetries form a group isomorphic to the $O(3,1)$*. Consequently, the $O(3,1)$ group itself is often called the *general Lorentz group.*

Now consider the $O(3,1)$ group manifold, *i.e.* the manifold spanned by the $O(3,1)$ matrices in the space of all 4×4 matrices. Unlike the $O(3)$ group manifold, the $O(3,1)$ group manifold has 4 rather than 2 disconnected components because the $O(3,1)$ matrices have 2 independent discrete parameters:

- First, for any $O(3,1)$ matrix L, the time-time element $L^{0}_{\;0}$ has absolute value $|L^{0}_{\;0}| \geq 1$, so its sign acts as a discrete parameter $\operatorname{sign}(L^{0}_{\;0})$.
- Second, the sign of the determinant $\det(L) = \pm 1$ provides another discrete parameter.

- The two signs are independent, so together they parametrize 4 disconnected components of the $O(3,1)$ group manifold.
- The Lorentz symmetries with $\det(L) = +1$ are called *proper*, the symmetries with $L^0_{\ 0} > 0$ are called *orthochronous*, and the symmetries obeying both conditions are called *proper orthochronous.*

The *proper orthochronous* Lorentz symmetries form the maximal continuous subgroup of the general Lorentz group $O(3,1)$; this subgroup is called the *restricted Lorentz group* or the *continuous Lorentz group* and denoted $SO^+(3,1)$. Physically, this group comprises:

- ⋆ Lorentz *boosts* of all possible velocities $|\mathbf{v}| < c$.
- ⋆ Rotations of space (through any angle around any axis).
- ⋆ All products of space rotations and Lorentz boosts.
- ⋆ And this is it: Any $SO^+(3,1)$ symmetry is a product $L = RB$ or a pure Lorentz boost B and a pure space rotation R.

Proof: Consider the top row of an $SO^+(3,1)$ matrix L for a symmetry $X'^\mu = L^\mu_{\ \nu} X^\nu$:

$$ct' \;=\; X'^0 \;=\; \gamma ct \;+\; \gamma\vec{\beta}\cdot\mathbf{x} \tag{14}$$

where $\gamma^2 \times (1 - \vec{\beta}^2) = 1$ and $\gamma > 0$. Let B be the pure Lorentz boost by the velocity $c\vec{\beta}$, then $X''^\mu = B^\mu_{\ \nu} X^\nu$ also has

$$ct'' \;=\; \gamma ct \;+\; \gamma\vec{\beta}\cdot\mathbf{x} \;=\; ct'. \tag{15}$$

Now let $R = LB^{-1}$ so that

$$X^\nu \;=\; (B^{-1})^\nu_{\ \lambda} X''^\lambda \quad\Longrightarrow\quad X'^\mu \;=\; L^\mu_{\ \nu} X^\nu \;=\; L^\mu_{\ \nu}(B^{-1})^\nu_{\ \lambda} X''^\lambda \;=\; \bigl(LB^{-1} = R\bigr)^\mu_{\ \lambda} X''^\lambda. \tag{16}$$

In this context $ct' = ct''$ means that the top row of the R matrix is $(1,0,0,0)$ and hence the whole R matrix has form

$$R \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \tag{17}$$

where the lower right 3×3 block is an $SO(3)$ matrix. Consequently, R is a pure space rotation, hence $L = RB$ is a product of a pure boost and a pure rotation. *Quod erat demonstrandum.*

Note that the Lorentz boosts $B(\mathbf{v})$ by themselves do not form a group: Alas, a product B_1B_2 of two boosts in different directions is not a pure boost but a combination of a boost and a rotation. Moreover, a product or 3 or more boosts whose velocities relativistically add up to zero — for example

$$\mathcal{R} \;=\; B(-\mathbf{v}_{1+2}) \times B(\mathbf{v}_2) \times B(\mathbf{v}_1) \tag{18}$$

— leaves the time coordinate invariant but rotates the 3 space coordinates. Consequently, any symmetry group that includes the Lorentz boosts in all direction must also include the space rotations and hence also all products of boosts and rotations. And that's why the smallest Lorentz symmetry group is the restricted Lorentz group $SO^+(3,1)$: It does include the boosts, the rotations, and their products, but it does NOT include the time reversal

$$T : (ct, x, y, z) \;\to\; (-ct, +x, +y, +z), \tag{19}$$

the space reflection

$$P : (ct, x, y, z) \;\to\; (+ct, -x, -y, -z), \tag{20}$$

or their combination

$$PT : (ct, x, y, z) \;\to\; (-ct, -x, -y, -z). \tag{21}$$

Besides the general Lorentz group $O(3,1)$ and the restricted Lorentz group $SO^+(3,1)$, there are intermediate Lorentz groups. Of particular importance is the *orthochronous Lorentz group* $O^+(3,1)$ which requires $L^0_{\;0} > 0$ but allows both signs of $\det(L)$. Physically, this group comprises the space reflection P and all its combinations with space rotations and Lorentz boosts. In particular, it includes reflections off moving mirrors (as long as the mirror's velocity is slower than light). OOH, the orthochronous Lorentz group does not include the time reversal T or its combinations with other symmetries. That's why it's called orthochronous: it preserves the general direction of the time flow.

A good example of an $O^+(3,1)$ invariant object is the *mass shell* of a relativistic particle: The manifold spanned by the allowed energy-momenta p^μ for a relativistic particle of a given mass m:

$$\text{all } p^\mu \text{ such that } p\cdot p \;=\; p^\mu p_\mu \;=\; m^2 \text{ and } p^0 \;>\; 0, \tag{22}$$

or equivalently

$$\text{all } (p^0,\mathbf{p}) \text{ with } p^0 \;=\; +E_{\mathbf{p}} \;=\; +\sqrt{m^2+\mathbf{p}^2}. \tag{23}$$

Geometrically, this mass shell is a 3D hyperboloid in the 4D momentum space, or rather one 'sheet' of a two-'sheeted' hyperboloid. The hyperboloid as a whole (defined by the $p\cdot p = m^2$ condition) is invariant under all Lorentz symmetries — orthochronous or not, — but only the orthochronous symmetries also preserve the positive sign of the energy p^0.

The proper Lorentz group $SO(3,1)$ is another intermediate Lorentz group which requires $\det(L) = +1$ but allows for either sign of $L^0{}_0$. Physically, this group allows simultaneous space and time reversal $PT : X^\mu \to -X^\mu$ and its combinations with Lorentz boosts and space rotations, but it does not allow separate time reversal T or space reflection P. Finally, there is yet another intermediate Lorentz group which allows the time reversal T but not the space reflection P. I don't know the mathematical name of this group; all I know that it comprises the $O(3,1)$ matrices L with $\mathrm{sign}(L^0{}_0) = \mathrm{sign}(\det(L))$.

Invariant Measure on the Mass Shell

Consider the momentum space of a single quantum particle. For a non-relativistic particle, this momentum space is a flat 3D space with a uniform measure

$$d^3\mathbf{p} \;=\; dp_x\,dp_y\,dp_z \tag{24}$$

which is invariant under Galilean boosts:

$$\begin{aligned} t' &= t, \quad \mathbf{x}' \;=\; \mathbf{x} \,+\, \mathbf{v}t, \\ \mathbf{p}' &= \mathbf{p} \,+\, m\mathbf{v}, \\ E' &= E \,+\, \mathbf{v}\cdot\mathbf{p} \,+\, \tfrac{1}{2}m\mathbf{v}^2, \end{aligned} \tag{25}$$

hence

$$d^3\mathbf{p}' \;=\; d^3\mathbf{p}. \tag{26}$$

In the context of the integral

$$I \;=\int\! d^3\mathbf{p}\, F(\mathbf{p}) \tag{27}$$

of some function $F(\mathbf{p})$ over the entire momentum space, the Galilean boost amounts to

$$F'(\mathbf{p}') \;=\; F(\mathbf{p}) \;=\; F(\mathbf{p}' - m\mathbf{v}), \tag{28}$$

hence

$$\int\! d^3\mathbf{p}'\, F(\mathbf{p}' - m\mathbf{v}) \;=\int\! d^3\mathbf{p}\, F(\mathbf{p}). \tag{29}$$

Now consider a relativistic particle whose momentum space is the mass shell (23). We want the integration measure on this mass shell to be invariant under all its $O^+(3,1)$ geometric symmetries, especially under the Lorentz — rather than Galilean — boosts. Such measure is unique up to an overall coefficient; specifically

$$d^4p \times \delta(p^2 - m^2) \times \Theta(p^0 - |\mathbf{p}|). \tag{30}$$

That is, the integral of some function $F(p)$ of the relativistic momentum over the mass shell is

$$I \;=\; \int\limits_{\substack{\text{mass}\\ \text{shell}}} F(p)\, d\,\text{measure}(p) \;=\; \int\limits_{\substack{\text{whole}\\ p-\text{space}}} d^4p\, \delta(p^2 - m^2)\Theta(p^0 - |\mathbf{p}|) \times F(p). \tag{31}$$

Note that while the integral here formally runs over the whole 4D momentum space, the delta-function $\delta(p^2 - m^2)$ restricts the relevant part of the integration range to the $p^2 = m^2$ hyperboloid, and the step-function $\Theta(p^0 - |\mathbf{p}|)$ further restricts it to the positive-energy 'sheet' of the hyperboloid.

Clearly, the measure (30) is invariant under the orthochronous Lorentz symmetries $p^\mu \to p'^\mu = L^\mu_{\ \nu} p^\nu$,

$$d\text{ measure}(p') \;=\; d\text{ measure}(p). \tag{32}$$

Indeed, factor by factor in eq. (30): First, in the context of the integral (31),

$$d^4p' \;=\; |\det(L)| \times d^4p \;=\; d^4p, \tag{33}$$

so the d^4p factor is invariant under all general Lorentz symmetries $L \in O(3,1)$.[⋆] Second, for any general Lorentz symmetry

$$p'^2 \;=\; p^2 \quad\Longrightarrow\quad \delta(p'^2 - m^2) \;=\; \delta(p^2 - m^2), \tag{34}$$

so the delta-function factor is also invariant. Finally, the step-function factor $\Theta(p^0 - |\mathbf{p}|)$ restricts the integration range to $p^0 \geq |\mathbf{p}|$, which is the momentum-space analogue of the future light cone. This future light cone is invariant under all the orthochronous Lorentz symmetries but not under the time-reversal T. Consequently, the whole measure (30) on the mass shell is invariant under the orthochronous Lorentz group $O^+(3,1)$.

In the context of the relativistic momentum integral (31), the $O^+(3,1)$ invariance of the measure means that for any two functions of the particle's momentum related by an orthochronous Lorentz transform L,

$$F(p) \quad\text{and}\quad F'(p) \;=\; F(L^{-1}p) \quad\text{so that } F(p' = Lp) \;=\; F(p), \tag{35}$$

we have

$$\int\limits_{\substack{\text{mass}\\\text{shell}}} F'(p)\, d\text{ measure}(p) \;=\; \int\limits_{\substack{\text{mass}\\\text{shell}}} F(p)\, d\text{ measure}(p). \tag{36}$$

Indeed,

$$\int\limits_{\substack{\text{mass}\\\text{shell}}} F'(p)\, d\text{ measure}(p) \;=\; \int\limits_{\substack{\text{mass}\\\text{shell}}} F'(p')\, d\text{ measure}(p') \;=\; \int\limits_{\substack{\text{mass}\\\text{shell}}} F(p)\, d\text{ measure}(p) \tag{37}$$

because $F'(p') = F(p)$ and d measure$(p') = d$ measure(p).

[⋆] Out of the integral context, $d^4p' = \det(L) \times d^4p = \pm d^4p$. But the time reversal T or the space reflection P also interchanges the limits of the integrals over the appropriate components of the p'^μ, so in the context of the integral (31) we get an extra sign which precisely cancels the sign of the determinant $\det(L)$. And that's why in the context of the integral (31), $d^4p' = d^4p$ regardless of the sign of $\det(L)$.

A good coordinate system for the mass shell hyperboloid comprises the 3 space components $\mathbf{p} = (p^x, p^y, p^z)$ of the 4-momentum p^μ. Indeed, any combinations of the (p^x, p^y, p^z) components are allowed on the mass shell, and for any such combination $\mathbf{p}$ the energy component has a unique value $p^0 = +E_{\mathbf{p}} = +\sqrt{m^2 + \mathbf{p}^2}$. In terms of the (p^x, p^y, p^z) coordinates on the mass shell, the Lorentz-invariant measure (30) becomes

$$d\ \text{measure}(p) \ = \ \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}\,, \tag{38}$$

thus

$$I \ = \int\limits_{\substack{\text{mass}\\\text{shell}}} d\ \text{measure}(p) \times F(p) \ = \int\limits_{\substack{\text{3D momentum}\\\text{space}}} \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}} \times F(\mathbf{p}, p^0 = +E_{\mathbf{p}}). \tag{39}$$

Indeed,

$$I \ = \int\limits_{\substack{\text{mass}\\\text{shell}}} d\ \text{measure}(p) \times F(p) \ = \int d^4p\, \delta(p^2 - m^2)\Theta(p^0 - |\mathbf{p}|) \times F(p) \ = \int d^3\mathbf{p}\, J(\mathbf{p}) \tag{40}$$

where

$$J(\mathbf{p}) \ = \int\limits_{-\infty}^{+\infty} dp^0\, \delta(p^2 - m^2)\Theta(p^0 - |\mathbf{p}|) \times F(\mathbf{p}, p^0). \tag{41}$$

Note that

$$p^2 \ - \ m^2 \ = \ (p^0)^2 \ - \ \mathbf{p}^2 \ - \ m^2 \ = \ (p^0)^2 \ - \ E^2_{\mathbf{p}}\,, \tag{42}$$

hence

$$\begin{aligned} J(\mathbf{p}) \ &= \int\limits_{-\infty}^{+\infty} dp^0\, \delta\big((p^0)^2 - E^2_{\mathbf{p}}\big)\Theta(p^0 - |\mathbf{p}|) \times F(\mathbf{p}, p^0) \\ &= \sum_{p^0} F(\mathbf{p}, p^0) \Bigg/ \left| \frac{\partial\big((p^0)^2 - E^2_{\mathbf{p}}\big)}{\partial p^0} \right| \\ &\qquad \text{over the points } p^0 \text{ such that } (p^0)^2 - E^2_{\mathbf{p}} \ = \ 0 \text{ and } p^0 > |\mathbf{p}|. \end{aligned} \tag{43}$$

There is precisely one such point, namely $p^0 = +E_{\mathbf{p}}$; at this point

$$\frac{\partial\big((p^0)^2 - E^2_{\mathbf{p}}\big)}{\partial p^0} \ = \ 2p^0 \ = \ +2E_{\mathbf{p}}\,, \tag{44}$$

hence

$$J(\mathbf{p}) \;=\; F(\mathbf{p},+E_{\mathbf{p}}) \Big/ 2E_{\mathbf{p}} \tag{45}$$

and therefore

$$I \;=\; \int\limits_{\substack{\text{mass}\\ \text{shell}}} d\ \text{measure}(p) \times F(p) \;=\; \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}\, F(\mathbf{p}, p^0 = +E^{\mathbf{p}}), \tag{46}$$

exactly as in eq. (39).

Relativistic Normalization of States and Operators

Consider the definite-momentum quantum states $|\mathbf{p}\rangle$ of a spinless quantum particle. For a particle in a finite $L \times L \times L$ box (with periodic boundary conditions), the momentum space is discrete,

$$p^x, p^y, p^z \;=\; \frac{2\pi}{L} \times \text{an integer}, \tag{47}$$

and the definite-momentum wavefunctions

$$\psi_{\mathbf{p}}(\mathbf{x}) \;=\; L^{-3/2} \exp(i\mathbf{p}\cdot\mathbf{x}) \tag{48}$$

are normalized so that

$$\langle \mathbf{p} | \mathbf{p}' \rangle \;=\; \int\limits_{\text{box}} d^3\mathbf{x}\, \psi^*_{\mathbf{p}}(\mathbf{x}) \psi_{\mathbf{p}'}(\mathbf{x}) \;=\; L^{-3} \int\limits_{\text{box}} d^3\mathbf{x}\, \exp(i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}) \;=\; \delta_{\mathbf{p},\mathbf{p}'}\,. \tag{49}$$

Consequently, the unit operator in the 1-particle Hilbert space decomposes to the series

$$\hat{1} \;=\; \sum_{\mathbf{p}} |\mathbf{p}\rangle\, \langle \mathbf{p}| \tag{50}$$

without any extra coefficients.

In the infinite space limit $L \to \infty$, the momentum space becomes continuous, so the definite-momentum states $\mathbf{p}$ can no longer be normalized according to eq. (49). Instead, let us simply drop the $L^{-3/2}$ factor from the wavefunctions (48), thus

$$\psi_{\mathbf{p}}(\mathbf{x}) \;=\; \exp(i\mathbf{p}\cdot\mathbf{x}) \tag{51}$$

without any extra normalization factors. Consequently,

$$\langle\mathbf{p}|\mathbf{p}'\rangle \;=\; \int\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{x}\,\exp(i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}) \;=\; (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{52}$$

and hence

$$\hat{1} \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\;|\mathbf{p}\rangle\,\langle\mathbf{p}|\,. \tag{53}$$

Note: the $d^3\mathbf{p}/(2\pi)^3$ measure of this integral is precisely the inverse of the $|\mathbf{p}\rangle$ state normalization (52), and for any different normalization of the momentum states, the measure of the integral in the decomposition of the unit operator is always the inverse of the inverse of the norm $\langle\mathbf{p}|\mathbf{p}'\rangle$. For example, some Quantum Mechanics textbooks prefer to normalize the wave functions to

$$\psi_{\mathbf{p}}(\mathbf{x}) \;=\; (2\pi)^{-3/2}\exp(i\mathbf{p}\cdot\mathbf{x}) \tag{54}$$

so that

$$\begin{aligned}\langle\mathbf{p}|\mathbf{p}'\rangle \;&=\; \delta^{(3)}(\mathbf{p}-\mathbf{p}'),\\ \hat{1} \;&=\int d^3\mathbf{p}\;|\mathbf{p}\rangle\,\langle\mathbf{p}|\,.\end{aligned} \tag{55}$$

But in the Quantum Field Theory the conventional normalization is (51) and hence (52) and (53).

Or rather, this is the conventional non-relativistic normalization. Indeed, the measure of the momentum-space integral is the flat non-relativistic measure $d^3\mathbf{p}\times\text{const}$ that's invariant under the Galilean rather than the Lorentz boosts. In a relativistic QFT, we want the relativistic momentum space measure

$$\frac{d^3\mathbf{p}}{2E_\mathbf{p}}\times\text{const},\quad\text{specifically}\quad\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_\mathbf{p}}\,. \tag{56}$$

Thus, we normalize the relativistic momentum eigenstates to

$$|\mathbf{p}\rangle_{\text{rel}} \;=\; \sqrt{2E_\mathbf{p}}\,|\mathbf{p}\rangle_{\text{non.rel}} \quad\Longrightarrow\quad \psi^{\text{rel}}_\mathbf{p}(\mathbf{x}) \;=\; \sqrt{2E_\mathbf{p}}\,\exp(i\mathbf{p}\cdot\mathbf{x}), \tag{57}$$

which leads to

$$\langle\mathbf{p}|\mathbf{p}'\rangle_{\text{rel}} \;=\; 2E_\mathbf{p}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{58}$$

and hence

$$\hat{1} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_\mathbf{p}}\;|\mathbf{p}\rangle_{\text{rel}}\,\langle\mathbf{p}|_{\text{rel}}\;. \tag{59}$$

Note: the Lorentz invariance of the measure in this integral means that the relativistic normalization (58) is also Lorentz invariant. That is, for any orthochronous Lorentz transform L and for any on-shell momenta p^μ and p'^μ,

$$\langle Lp|Lp'\rangle_{\text{rel}} \;=\; \langle p|p'\rangle_{\text{rel}}\;. \tag{60}$$

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Finally, let's go to the QFT's Fock space and consider the creation and the annihilation operators $\hat{a}^\dagger_\mathbf{p}$ and $\hat{a}_\mathbf{p}$ for particles with definite momenta. Where the QFT is relativistic or not, in a finite box or in infinite space, we want to identify the 1-particle states $|\mathbf{p}\rangle$ with the Fock-space states

$$|\mathbf{p}\rangle \;=\; |1\,\text{particle},\ \text{mom}=\mathbf{p}\rangle \;=\; \hat{a}^\dagger_\mathbf{p}\,|\text{vac}\rangle\,, \tag{61}$$

and likewise

$$\langle\mathbf{p}| \;=\; \langle\text{vac}|\,\hat{a}_\mathbf{p}\,. \tag{62}$$

Consequently, we should normalize the creation and annihilation operators so that the

Schrödinger-picture bosonic commutation relations become

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}\right] = \left[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right] = 0 \tag{63}$$

while

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right] = \left\langle \mathbf{p}, \mathbf{p}' \right\rangle . \tag{64}$$

In particular, in a finite box

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right] = \delta_{\mathbf{p},\mathbf{p}'} , \tag{65}$$

while in the infinite space

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right]_{\text{non.rel}} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \tag{66}$$

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right]_{\text{rel}} = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{67}$$

To implement this relativistic normalization, we take

$$(\hat{a}_{\mathbf{p}})_{\text{rel}} = \sqrt{2E_{\mathbf{p}}} \times (\hat{a}_{\mathbf{p}})_{\text{non.rel}} , \qquad (\hat{a}^{\dagger}_{\mathbf{p}})_{\text{rel}} = \sqrt{2E_{\mathbf{p}}} \times (\hat{a}^{\dagger}_{\mathbf{p}})_{\text{non.rel}} . \tag{68}$$

In my next set of notes, we shall use these relations to expand a relativistic scalar field $\Phi(x)$ into a family of relativistic creation and annihilation operators.