OPTICAL THEOREM

The optical theorem relates the imaginary part of the elastic scattering amplitude in the forward direction to the total cross-section — elastic + inelastic — of the initial particles. In the non-relativistic normalization,

$$\operatorname{Im} f_{\text{elastic}}(\theta=0) \;=\; \frac{k_{\text{reduced}}}{4\pi}\times\sigma_{\text{total}}\,. \tag{1}$$

You have probably seen this formula in an undergraduate QM class in the context of the partial wave analysis of potential scattering; but in case you have never studied the subject, here are my notes on scattering in QM — including the partial wave analysis — from the likbez lecture I gave in November. Anyway, for a purely elastic scattering

$$f(\theta) \;=\; \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\times\left(\frac{e^{2i\delta_\ell}-1}{2i}\;=\;e^{i\delta_\ell}\sin\delta_\ell\right), \tag{2}$$

where δ_ℓ is the phase shift for the partial wave ℓ. Note that for small phase shifts $\delta_\ell \ll 1$, the real part of the scattering amplitude is much larger than its imaginary part, $\operatorname{Re} f \sim \delta$ while $\operatorname{Im} f \sim \delta^2$. For the scattering amplitude (2), the total cross-section is

$$\sigma_{\text{total}} \;=\; \int |f|^2\,d^2\Omega \;=\; \frac{4\pi}{k^2}\times\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\,, \tag{3}$$

while the imaginary part of the forward scattering amplitude is

$$\operatorname{Im} f(\theta=0) \;=\; \frac{1}{k}\times\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\,, \tag{4}$$

in manifest agreement with the optical theorem (1).

In relativistic notations, the optical theorem (1) becomes

$$\operatorname{Im}\mathcal{M}\binom{\text{elastic}}{\text{forward}} \;=\; 2E_1E_2|\mathbf{v}_1-\mathbf{v}_2|\times\sigma_{\text{total}}\,. \tag{5}$$

where the pre-factor $2E_1E_2|\mathbf{v}_1-\mathbf{v}_2|$ is invariant under Lorentz boosts along the axis of collision. In the center-of-mass frame, it amounts to $4E_{\text{cm}}^{\text{net}}|\mathbf{p}_{\text{cm}}|$.

Proof

The optical theorem follows from the unitarity of the S-matrix — or rather, the scattering operator, — $\hat{S}^\dagger\hat{S} = 1$. To see how this works, let's separate the scattering from non-scattering events, $\hat{S} = 1 + i\hat{T}$, and express the unitarity of $\hat{S}$ in terms of $\hat{T}$:

$$1 \;=\; (\hat{S}^\dagger = 1 - \hat{T}^\dagger)(\hat{S} = 1 + 1\hat{T}) \;=\; 1 \;+\; i\hat{T} \;-\; i\hat{T}^\dagger \;+\; \hat{T}^\dagger\hat{T}, \tag{6}$$

hence

$$i\hat{T}^\dagger \;-\; i\hat{T} \;=\; \hat{T}^\dagger\hat{T}. \tag{7}$$

Now let's take the diagonal matrix elements $\langle i|\cdots|i\rangle$ of both sides of this formula for some state $|i\rangle$, which we shall eventually take to be the initial state $|1+2\rangle$ of two particles about to collide. On the LHS of eq. (7)

$$\langle i|\, i\hat{T}^\dagger - i\hat{T}\,|i\rangle \;=\; i\,\langle i|\,\hat{T}\,|i\rangle^* \;-\; i\,\langle i|\,\hat{T}\,|i\rangle \;=\; 2\,\mathrm{Im}\,\langle i|\,\hat{T}\,|i\rangle \tag{8}$$

while on the RHS of eq. (7)

$$\langle i|\,\hat{T}^\dagger\hat{T}\,|i\rangle \;=\; \sum_{|f\rangle} \langle i|\,\hat{T}^\dagger\,|f\rangle\,\langle f|\,\hat{T}\,|i\rangle \;=\; \sum_{|f\rangle} \Bigl|\langle f|\,\hat{T}\,|i\rangle\Bigr|^2, \tag{9}$$

hence

$$2\,\mathrm{Im}\,\langle i|\,\hat{T}\,|i\rangle \;=\; \sum_{|f\rangle} \Bigl|\langle f|\,\hat{T}\,|i\rangle\Bigr|^2. \tag{10}$$

Physically, the sum on the RHS here is over all possible final states of the initial particles' scattering.

Next, let's factor out the energy-momentum conservation from the matrix elements of the $\hat{T}$ operator,

$$\langle f|\,\hat{T}\,|i\rangle \;=\; (2\pi)^4\delta^{(4)}(p_f - p_i) \times \langle f|\,\widehat{\mathcal{M}}\,|i\rangle\,. \tag{11}$$

Plugging this formula directly into eq. (10) gives us

$$2\,\mathrm{Im}\,\langle i|\,\widehat{\mathcal{M}}\,|i\rangle \times (2\pi)^4\delta^{(4)}(p_i - p_i) \;=\; \sum_{|f\rangle} \Bigl|\langle f|\,\widehat{\mathcal{M}}\,|i\rangle\Bigr|^2 \times \Bigl((2\pi)^4\delta^{(4)}(p_f - p_i)\Bigr)^2, \tag{12}$$

with troublesome δ-functions on both sides of the equation. To resolve this trouble, consider the state $|i'\rangle$ which is very similar to the initial state $|i\rangle$ except for a tiny difference in the net momentum $p_{i'}$. By similar I mean similar enough that we may approximate

$$\left\langle i'\right| \widehat{\mathcal{M}} \left|i\right\rangle \;=\; \langle i| \widehat{\mathcal{M}} |i\rangle \quad \text{and} \quad \left\langle i'\right| \widehat{\mathcal{M}} \left|f\right\rangle \;=\; \langle i| \widehat{\mathcal{M}} |f\rangle \;\; \forall \left|f\right\rangle. \tag{13}$$

Consequently, taking the slightly-off-diagonal matrix elements $\langle i'|\cdots|i\rangle$ of both sides of eq. (7), we get on the LHS

$$\left\langle i'\right| i\hat{T}^\dagger - i\hat{T} \left|i\right\rangle \;=\; (2\pi)^4\delta^{(4)}(p_{i'}-p_i)\times\langle i|\, i\widehat{\mathcal{M}}^\dagger - i\widehat{\mathcal{M}}\, |i\rangle \;=\; (2\pi)^4\delta^{(4)}(p_{i'}-p_i)\times 2\,\mathrm{Im}\,\langle i| \widehat{\mathcal{M}} |i\rangle\,, \tag{14}$$

and on the RHS

$$\begin{aligned}
\left\langle i'\right| \hat{T}^\dagger\hat{T} \left|i\right\rangle \;&=\; \sum_{|f\rangle} \langle f| \hat{T} \left|i'\right\rangle^* \langle f| \hat{T} |i\rangle \\
&=\; \sum_{|f\rangle} \left|\langle f| \widehat{\mathcal{M}} |i\rangle\right|^2 \times (2\pi)^4\delta^{(4)}(p_f - p_{i'}) \times (2\pi)^4\delta^{(4)}(p_f - p_i) \\
&=\; (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \times \sum_{|f\rangle} \left|\langle f| \widehat{\mathcal{M}} |i\rangle\right|^2 \times (2\pi)^4\delta^{(4)}(p_f - p_i).
\end{aligned} \tag{15}$$

Equating the last two formulae and dropping the overall factor $(2\pi)^4\delta^{(4)}(p_{i'}-p_i)$ from both, we arrive at the improved version of eq. (12), namely

$$2\,\mathrm{Im}\,\langle i| \widehat{\mathcal{M}} |i\rangle \;=\; \sum_{\langle f|} \left|\langle f| \widehat{\mathcal{M}} |i\rangle\right|^2 \times (2\pi)^4\delta^{(4)}(p_f - p_i). \tag{16}$$

Now let $|i\rangle = |1+2\rangle$ be the initial state of two particles about to collide, so that $|f\rangle$ runs over all possible final states of this collision. Thus, summing over $|f\rangle$ means summing over all possible reaction *channels* — *i.e.*, sets of final particles, — and then for each final-state particle integrating over its momentum with the relativistic measure and summing over its discrete quantum numbers like spin:

$$\begin{aligned} \langle f| &= \big\langle \text{channel} : (p', s')_1, \ldots, (p', s')_n \big| \,, \\ \sum_{\langle f|} &= \sum_{\text{channels}} \prod_{a=1}^{n} \left(\sum_{s'_a} \int \frac{d^3\mathbf{p}'_a}{(2\pi)^3 2E'_a} \right). \end{aligned} \tag{17}$$

Consequently, eq. (16) becomes

$$\begin{aligned} 2\,\text{Im}\,\langle i|\,\widehat{\mathcal{M}}\,|i\rangle \;=\; \sum_{\text{channels}} \prod_{a=1}^{n} \left(\sum_{s'_a} \int \frac{d^3\mathbf{p}'_a}{(2\pi)^3 2E'_a} \right) \Big| \big\langle \text{channel} : (p', s')_1, \ldots, (p', s')_n \big|\,\widehat{\mathcal{M}}\,|i\rangle \Big|^2 \times \\ \times\, (2\pi)^4 \delta^{(4)}(p'_1 + \cdots + p'_n - p_i^{\text{net}}). \end{aligned} \tag{18}$$

Note that the momentum integral (and spin sum) on the RHS here is precisely the phase space integral (and spin sum) for the scattering process $(1 + 2 \to 1' + \cdots + n')$. Specifically,

$$\begin{aligned} \sigma_{\text{net}}(1 + 2 \to 1' + \cdots + n') \;=\; \frac{1}{4E_1E_2|\mathbf{v}_{12}^{\text{rel}}|} \prod_{a=1}^{n} \left(\sum_{s'_a} \int \frac{d^3\mathbf{p}'_a}{(2\pi)^3 2E'_a} \right) \Big| \langle (p', s')_1, \ldots, (p', s')_n|\,\widehat{\mathcal{M}}\,|1 + 2\rangle \Big|^2 \times \\ \times\, (2\pi)^4 \delta^{(4)}(p'_1 + \cdots + p'_n - p_i^{\text{net}}), \end{aligned} \tag{19}$$

cf. my notes on the phase space. Therefore, we may rephrase eq. (18) in terms of the net cross-sections for each channel and ultimately in terms of the total cross-section for all the channel combined:

$$\begin{aligned} 2\,\text{Im}\,\langle 1 + 2|\,\widehat{\mathcal{M}}\,|1 + 2\rangle \;&=\; \sum_{\text{channels}} 4E_1E_2|\mathbf{v}_{12}^{\text{rel}}| \times \sigma_{\text{net}}(1 + 2 \to 1' + \cdots + n') \\ &=\; 4E_1E_2|\mathbf{v}_{12}^{\text{rel}}| \times \sum_{\text{channels}} \sigma_{\text{net}}(1 + 2 \to 1' + \cdots + n') \\ &=\; 4E_1E_2|\mathbf{v}_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1 + 2 \to \text{anything}). \end{aligned} \tag{20}$$

Finally, we identify the matrix element on the LHS here as the amplitude for the forward elastic scattering $|1 + 2\rangle \to |\text{exactlythe same state}\rangle$, thus

$$\text{Im}\,\mathcal{M}\binom{\text{elastic}}{\text{forward}} \;=\; 2E_1E_2|\mathbf{v}_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1 + 2 \to \text{anything}). \tag{21}$$

And this completes my proof of the optical theorem.

BTW, there is a similar version of the optical theorem for the decays of unstable particles. Indeed, let in eq. (18) $|i\rangle$ be the initial state of a single unstable particle (instead of two particles about to collide). Then the RHS of eq. (18) becomes the phase space integral (and the spin sum) for the net decay rate of the initial particle into all the available decay channels. Indeed,

$$\Gamma_{\text{net}}(1 \to 1' + \cdots + n') = \frac{1}{2M} \prod_{a=1}^{n} \left(\sum_{s'_a} \int \frac{d^3\mathbf{p}'_a}{(2\pi)^3 2E'_a} \right) \left| \langle (p', s')_1, \ldots, (p', s')_n | \widehat{\mathcal{M}} |1\rangle \right|^2 \times \\ \times (2\pi)^4 \delta^{(4)}(p'_1 + \cdots + p'_n - p_i^{\text{net}}), \tag{22}$$

— *cf.* my notes on the phase space, — hence

$$\begin{aligned} \text{Im} \langle 1| \widehat{\mathcal{M}} |1\rangle &= \frac{1}{2} \sum_{\text{channels}} 2M \times \Gamma_{\text{net}}(1 \to 1' + \cdots + n') \\ &= M \times \sum_{\text{channels}} \Gamma_{\text{net}}(1 \to 1' + \cdots + n') \\ &= M \times \Gamma_{\text{total}}(1 \to \text{anything}). \end{aligned} \tag{23}$$

We shall return to this formula in a few lectures, once we have a better understanding of the one-particle to one-particle amplitudes.

Application to the $\lambda\phi^4$ Theory.

In potential scattering, the first Born approximation yields a real scattering amplitude, but at the higher orders of perturbation theory the amplitude becomes complex, with both real and imaginary parts. Likewise, in quantum field theory the tree-level amplitudes are real, — at least for the elastic scattering in the forward direction, — but the loop corrections make for complex amplitudes. The reason for this behavior is the optical theorem (21), plus power-of-the-coupling counting in the perturbation theory. Let's see how this works for the $\lambda\phi^4$ theory.

For the elastic scattering $\mathcal{M}_{\text{tree}} = -\lambda$ (regardless of momenta) while the loop corrections are of the higher order in λ, hence

$$\frac{d\sigma^{\text{elastic}}}{d\Omega_{\text{cm}}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s} \tag{24}$$

and

$$\sigma_{\text{net}}^{\text{elastic}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s} \times \frac{4\pi}{2} = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}. \tag{25}$$

For the inelastic processes $2 \to n$ ($n \geq 4$), the tree amplitude is $O(\lambda^{n/2})$ and the loop corrections are of higher order, hence $\mathcal{M} = O(\lambda^{n/2})$ and

$$\sigma^{\text{inelastic}} = O(\lambda^n) \quad \text{for } n \geq 4. \tag{26}$$

Consequently,

$$\sigma_{\text{total}} = \sigma_{\text{net}}^{\text{elastic}} + O(\lambda^4) = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}. \tag{27}$$

By the optical theorem, this gives us the imaginary part of the forward elastic amplitude as

$$\operatorname{Im} \mathcal{M} \begin{pmatrix} \text{elastic} \\ \text{forward} \end{pmatrix} = 2E_1 E_2 |\mathbf{v}_{12}^{\text{rel}}| \times \frac{\lambda^2 + O(\lambda^3)}{32\pi s}, \tag{28}$$

where in the center of mass frame

$$2E_1 E_2 |\mathbf{v}_{12}^{\text{rel}}| = 2E^2 \times 2v = s \times v \tag{29}$$

(where v is the speed of each particle relative to the CM), hence

$$\operatorname{Im} \mathcal{M} \begin{pmatrix} \text{elastic} \\ \text{forward} \end{pmatrix} = \frac{\lambda^2 v}{32\pi} + O(\lambda^3). \tag{30}$$

In terms of the loop counting for the amplitude on the LHS here, this means

$$\operatorname{Im} \mathcal{M}_{\text{tree}} \begin{pmatrix} \text{elastic} \\ \text{forward} \end{pmatrix} = 0, \tag{31}$$

$$\text{while} \quad \operatorname{Im} \mathcal{M}_{1\,\text{loop}} \begin{pmatrix} \text{elastic} \\ \text{forward} \end{pmatrix} = \frac{\lambda^2 v}{32\pi} > 0. \tag{32}$$

Note: the optical theorem not only explains why the tree amplitude is real while the one-loop amplitude is complex, it also gives us specific predictions for the imaginary part of the one-loop amplitude in terms of the tree-level total cross-section. In the same way, it would give us a specific prediction for the imaginary part of the two-loop amplitude in terms of the one-loop total cross-section, *etc.*, *etc.* But let's not get too far into the higher-order calculations.

Instead, let's verify the tree-level and 1-loop level predictions (31) and (32). At the tree level, we indeed have

$$\mathcal{M}^{\text{elastic}}_{\text{tree}} \;=\; -\lambda, \quad \operatorname{Im}\mathcal{M}^{\text{elastic}}_{\text{tree}} \;=\; 0 \tag{33}$$

for any scattering angle θ and not just $\theta = 0$. Likewise, we shall see im a moment that

$$\forall\theta: \quad \operatorname{Im}\mathcal{M}^{\text{elastic}}_{1\,\text{loop}} \;=\; \frac{\lambda^2 v}{32\pi} \tag{34}$$

where

$$v \;=\; \frac{|\mathbf{p}|}{E} \;=\; \sqrt{1-\frac{m^2}{E^2}} \;=\; \sqrt{1-\frac{4m^2}{s}}\,. \tag{35}$$

Indeed, we have seen earlier in class that

$$\mathcal{M}^{\text{elastic}}_{1\,\text{loop}} \;=\; \frac{\lambda^2}{32\pi^2}\Big(J(4) \;-\; J(t/m^2) \;-\; J(u/m^2) \;-\; J(s/m^2)\Big), \tag{36}$$

where

$$J(t/m^2) \;=\; \int\limits_0^1 dx\,\log\frac{m^2 - tx(1-x)}{m^2} \tag{37}$$

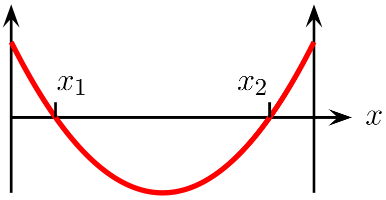
and likewise for $J(u/m^2)$ and $J(s/m^2)$. For elastic scattering $t < 0$, hence

$$J(t) \;=\; \int\limits_0^1 dx\,\log\frac{m^2 + \text{positive}}{m^2}, \tag{38}$$

which is real and positive. Likewise, $u < 0$ and hence $J(u/m^2)$ is real and positive. But $s > +4m^2$, which makes $J(s/m^2)$ complex rather than real. Indeed,

$$J(s/m^2) \;=\; \int\limits_0^1 dx\,\log\frac{m^2 - sx(1-x)}{m^2}\,, \tag{39}$$

and for $s > +4m^2$ the argument of the logarithm here becomes negative for some x:

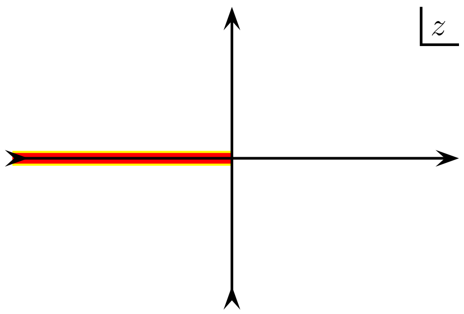


$$x_{1,2} \;=\; \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{m^2}{s}} \;=\; \frac{1 \mp v}{2}\,. \tag{40}$$

For a negative or complex argument the logarithm becomes complex,

$$\log(z) \;=\; \log(|z|) \;+\; i\arg(z), \tag{41}$$

which has a branch cut in the complex z plane along the negative half of the real axis,



thus

$$\log(\text{negative} \pm i\epsilon) \;=\; \text{real} \;\pm\; i\pi. \tag{42}$$

Consequently,

$$\operatorname{Im}\log\frac{m^2 - sx(1-x)}{m^2} \;=\; \begin{cases} \pm\pi & \text{for } x_1 < x < x_2, \\ 0 & \text{otherwise,} \end{cases} \tag{43}$$

and therefore

$$\operatorname{Im} J(s/m^2) \;=\; \int_0^1 dx\; \operatorname{Im}\log\frac{m^2 - sx(1-x)}{m^2} \;=\; \pm\pi \times (x_2 - x_1) \;=\; \pm\pi v \tag{44}$$

(where v is the particles' speed in the CM frame). The sign of this imaginary part depends on the side of the logarithm's branch cut we end up on for negative $m^2 - sx(1-x)$, which in turn depends on the analytic continuation from a real s to $s+i\epsilon$ or to $s-i\epsilon$:

$$\begin{aligned} \text{for } s &\to s+i\epsilon, & m^2-sx(1-x) &= \text{real} - i\epsilon & \Longrightarrow \quad \log &= \text{real} - i\pi, \\ \text{while for } s &\to s-i\epsilon, & m^2-sx(1-x) &= \text{real} + i\epsilon & \Longrightarrow \quad \log &= \text{real} + i\pi, \end{aligned} \tag{45}$$

and hence

$$\operatorname{Im} J\left((s\pm i\epsilon)/m^2\right) = \mp\pi v. \tag{46}$$

In the context of the one-loop elastic amplitude (36), this means

$$\operatorname{Im}\mathcal{M}^{\text{elastic}}_{1\,\text{loop}}(s\pm i\epsilon,t) = -\frac{\lambda^2}{32\pi^2}\operatorname{Im}J((s\pm i\epsilon)/m^2) = \pm\frac{\lambda^2 v}{32\pi}. \tag{47}$$

As we see, this formula is in perfect agreement with the optical theorem's prediction (32), *provided we interpret s as $s+i\epsilon$*.

The reason for the choice of $s\to s+i\epsilon$ rather than $s\to s-i\epsilon$ stems from the origin of the $J(s/m^2)$ in the s-channel amplitude

$$\mathcal{F}(s) = \frac{\lambda^2}{2}\int_0^1 dx\int_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{[k^2-\Delta(x,s)+i\epsilon]^2}, \tag{48}$$

cf. eq. (20) of my introductions to the one-loop amplitudes for a similar t-channel amplitude. After the Wick rotation to the Euclidean momentum space, eq. (48) becomes

$$\mathcal{F}(s) = \frac{\lambda^2}{2}\int_0^1 dx\int_{\text{reg}}\frac{d^4k_E}{(2\pi)^4}\,\frac{1}{[k_E^2+\Delta(x,s)-i\epsilon]^2}. \tag{49}$$

For a similar t-channel amplitude, we may disregard the $-i\epsilon$ term in the denominator because $\Delta(x,t)>0$ for all x so there are no poles in the Euclidean momentum space. But in the s channel $\Delta(x,s)$ turns negative for some x, so we should retain the $-i\epsilon$ term. Hence, in all the following formulae for the s channel we should replace $\Delta(x,s)$ with

$$\Delta(x,s) \;-\; i\epsilon \;=\; m^2 \;-\; x(1-x)\times s \;-\; i\epsilon \;=\; m^2 \;-\; x(1-x)\times(s+i\epsilon), \qquad (50)$$

which is equivalent to replacing $s \to s + i\epsilon$. And that's how we end up with $J((s+i\epsilon)/m^2$ in the s-channel amplitude and therefore

$$\mathcal{M}^{\text{elastic}}_{1\,\text{loop}} \;=\; \frac{\lambda^2}{32\pi^2}\Big(J(4) \;-\; J(t/m^2) \;-\; J(u/m^2) \;-\; J((s+i\epsilon)/m^2)\Big). \qquad (51)$$

In my next set of notes about the field correlation functions I shall give you a more general reason for always interpreting s as $s+i\epsilon$ rather than $s-i\epsilon$.

Cutkosky Cutting Rules

Going back to eq. (47), we see that the 1-loop elastic scattering amplitude is a discontinuous function of s; specifically, it has a discontinuity at real $s \geq 4m^2$, where the real part of the amplitude is continuous but the imaginary part changes sign. Thus,

$$\operatorname{disc}\mathcal{M}(s,t) \;\stackrel{\text{def}}{=}\; \mathcal{M}(s+i\epsilon,t) \;-\; \mathcal{M}(s-i\epsilon,t) = \;2i\times\operatorname{Im}\mathcal{M}(s+i\epsilon,t), \qquad (52)$$

hence by the Optical theorem we should have

$$\operatorname{disc}\mathcal{M}^{\text{elastic}}(s;t=0) \;=\; 4iE_1E_2v_{\text{rel}}\times\sigma^{\text{total}}(s). \qquad (53)$$

Similar Optical Theorems apply to the scattering amplitudes in other field theories such as QED or QCD. Moreover, they can be derived in the Feynman diagram language from the *Cutkosky's cutting rules* discovered by Richard E. Cutkosky in 1960. Here are these cutting rules:

1. Consider all Feynman diagrams contributing to the amplitude in question which can be separated into 2 parts by cutting some propagators such that:
   - All the incoming particles are on one side of the cut and all the outgoing particles on the other side, and

- All the cut propagators can be simultaneously put on shell for some values of the loop momenta.

If a diagram cannot be cut to obey these conditions, ignore it. But if it can be cut in several ways, consider all such cuttings separately.

2. For each cutting, replace each cut propagator with the delta-function putting it on-shell

$$\frac{i}{q^2 - m^2 + i\epsilon} \longrightarrow 2\pi\delta(q^2 - m^2). \tag{54}$$

But keep all other factors — vertices, un-cut propagators, combinatorial factors, numerators of non-scalar propagators, *etc.*, — the same as in the regular Feynman rules. Then evaluate the loop momentum integrals.

3. Total up the contributions from all the all possible cutting of all cuttable diagrams.

4. The result is the discontinuity of the original amplitude between $s + i\epsilon$ and $s - i\epsilon$.

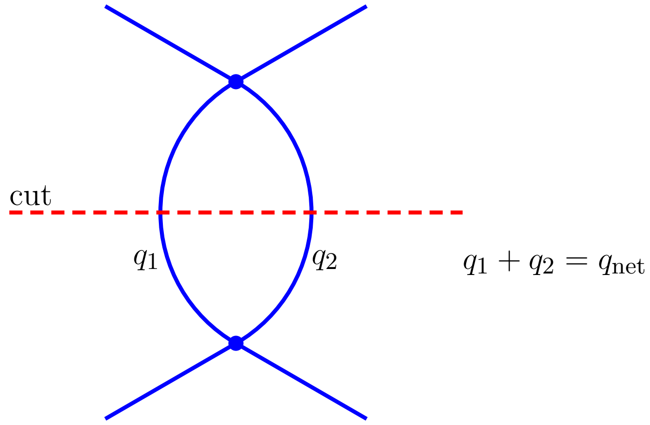
To see how this works, let's go back to the $\lambda\Phi^4$ theory and the 1-loop elastic scattering amplitude

$$\mathcal{M}^{\text{elastic}}_{1\,\text{loop}}(s,t) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u). \tag{55}$$

Breaking any of the 3 one-loop diagrams into 2 parts means cutting both propagators, and then putting both cut propagators on shell, $q_1^2 = m^2$ and $q_2^2 = m^2$, requires

$$q_{\text{net}}^2 = (q_1 + q_2)^2 \geq +4m^2. \tag{56}$$

Clearly, this is possible in the s-channel but not in the t-channel or in the u-channel, so the Cutkosky's rules tell us that $\mathcal{F}(t)$ and $\mathcal{F}(u)$ should be real and continuous — and indeed we have seen that they are — while $\mathcal{F}(s)$ should have a discontinuity between $s + i\epsilon$ and $s - i\epsilon$. Let's find this discontinuity by direct evaluation of the s-channel diagram and then compare it to what obtains from the Cutkosky's rules.



(57)

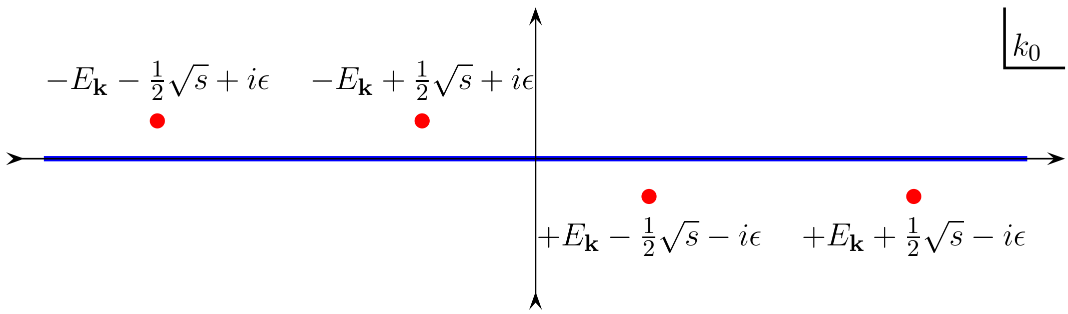
Let's work in the center-of-mass frame where $q^\mu_{\rm net} = (\sqrt{s}, \mathbf{0})$, so we may set

$$q_1^\mu \;=\; (\tfrac12\sqrt{s} + k_0\,,\, +\mathbf{k}), \quad q_2^\mu \;=\; (\tfrac12\sqrt{s} - k_0\,,\, -\mathbf{k}) \tag{58}$$

in terms of the independent loop momentum k^μ, thus

$$\begin{aligned} \mathcal{F}(s) \;&=\; \frac{i\lambda^2}{2} \int\!\frac{d^4k}{(2\pi)^4}\, \frac{i}{q_1^2 - m^2 + i\epsilon}\, \frac{i}{q_2^2 - m^2 + i\epsilon} \\ &\quad \langle\!\langle \text{ integrating over } k_0 \text{ before integrating over } \mathbf{k} \,\rangle\!\rangle \\ &=\; \frac{-i\lambda^2}{2} \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \int\!\frac{dk_0}{2\pi}\, \frac{1}{(k_0 + \frac12\sqrt{s})^2 - E_{\mathbf{k}}^2 + i\epsilon}\, \frac{1}{(k_0 - \frac12\sqrt{s})^2 - E_{\mathbf{k}}^2 + i\epsilon} \end{aligned} \tag{59}$$

where $E_{\mathbf{k}}^2 = m^2 + \mathbf{k}^2$. From the k_0 point of view, the integrand here has 4 poles



(60)

It is also regular at $k_0 = \infty$, which allows us to move the end points of the integration contour away from ∞ and close the contour either above or below the real axis. Let's close the contour below the real axis, so the poles enclosed within it are the poles at $k_0 = \pm\frac{1}{2}\sqrt{s} + E_{\mathbf{k}} - i\epsilon$. The residues at these poles are

$$\text{at } k_0 \;=\; +\tfrac{1}{2}\sqrt{s} \,+\, E_{\mathbf{k}} \,-\, i\epsilon : \quad \text{residue} \;=\; \frac{1}{(2\mathrm{E}_{\mathbf{k}})(2\mathrm{E}_{\mathbf{k}} + \sqrt{\mathrm{s}})(+\sqrt{\mathrm{s}})}, \tag{61}$$

$$\text{at } k_0 \;=\; -\tfrac{1}{2}\sqrt{s} \,+\, E_{\mathbf{k}} \,-\, i\epsilon : \quad \text{residue} \;=\; \frac{1}{(2\mathrm{E}_{\mathbf{k}})(2\mathrm{E}_{\mathbf{k}} - \sqrt{\mathrm{s}})(-\sqrt{\mathrm{s}})}, \tag{62}$$

hence

$$\int \frac{dk_0}{2\pi}\Bigl(\cdots\Bigr) \;=\; \frac{-i}{2E_{\mathbf{k}}\sqrt{s}} \times \left(\frac{1}{2E_{\mathbf{k}} + \sqrt{s}} \,-\, \frac{1}{2E_{\mathbf{k}} - \sqrt{s}}\right). \tag{63}$$

When we further integrate this expression over $d^3\mathbf{k}$, the first 2 factors stay real and finite and do not have any discontinuities between $s + i\epsilon$ and $s - i\epsilon$. The first term inside $(\cdots)$ also stays real and finite and does not have any discontinuities, but the second terms blows up when $E_{\mathbf{k}} = \frac{1}{2}\sqrt{s}$, so it is this term which generates the discontinuity. Thus,

$$\operatorname{disc}\mathcal{F}(s) \;=\; \frac{\lambda^2}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{k}}\sqrt{s}} \times \operatorname{disc}\left(\frac{1}{2E_{\mathbf{k}} - \sqrt{s}}\right) \tag{64}$$

where

$$\begin{aligned} \operatorname{disc}\left(\frac{1}{2E_{\mathbf{k}} - \sqrt{s}}\right) \;&=\; \frac{1}{2E_{\mathbf{k}} - \sqrt{s} - i\epsilon} \,-\, \frac{1}{2E_{\mathbf{k}} - \sqrt{s} + i\epsilon} \\ &=\; \frac{2i\epsilon}{(2E_{\mathbf{k}} - \sqrt{s})^2 + \epsilon^2} \\ &\xrightarrow[\epsilon\to 0]{}\; 2\pi i\,\delta(2E_{\mathbf{k}} - \sqrt{s}), \end{aligned} \tag{65}$$

hence

$$\operatorname{disc}\mathcal{F}(s) \;=\; \frac{i\lambda^2}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{k}}\sqrt{s}} \times 2\pi\,\delta(2E_{\mathbf{k}} - \sqrt{s}). \tag{66}$$

Now let's compare this discontinuity with what comes out from the Cutkosky's cutting rules applied to the amplitude (59):

$$\mathcal{D}(s) \;=\; \frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\,2\pi\delta(q_1^2 - m^2) \times 2\pi\delta(q_2^2 - m^2) \tag{67}$$

$\langle\!\langle$ integrating over k^0 before integrating over $\mathbf{k}$ $\rangle\!\rangle$

$\langle\!\langle$ then replacing $\int dk^0$ with $\int dq_1^0 \int dq_2^0\, \delta(q_1^0 + q_2^0 - \sqrt{s})$ $\rangle\!\rangle$

$$= \frac{i\lambda^2}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int dq_1^0 \int dq_2^0\, \delta(q_1^0 + q_2^0 - \sqrt{s}) \times 2\pi\delta((q_1^0)^2 - \mathbf{k}^2 - m^2)\delta((q_2^0)^2 - \mathbf{k}^2 - m^2) \tag{67}$$

where

$$\int dq_1^0\, \delta((q_1^0)^2 - E_\mathbf{k}^2) \times f(q_1^0) \;=\; \frac{1}{2E_\mathbf{k}} \times f(q_1^0 = E_\mathbf{k}) \tag{68}$$

and likewise for the $\int dq_2^0$. Consequently,

$$\mathcal{D}(s) \;=\; \frac{i\lambda^2}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(2E_\mathbf{k})^2} \times 2\pi\delta(2E_\mathbf{k} - \sqrt{s}), \tag{69}$$

and comparing this formula to eq. (66), we immediately see that

$$\mathcal{D}(s) \;=\; \operatorname{disc} \mathcal{F}(s) \;=\; \operatorname{disc} \mathcal{M}(s;t). \tag{70}$$

Thus, *Cutkosky's cutting rules indeed yield the correct discontinuity of the scattering amplitude.*

Note that nothing in the above argument depends on the two cut propagators having similar masses, so it can be generalized to a theory of several fields with different masses, and ultimately to a theory of fields with different spins. And indeed, the Cutkosky's cutting rules work for all kinds of quantum field theories at all loop levels, and not just at one loop.

Finally, let me explain the relation of the Cutkosky's rules to the optical theorem. Once we cut a diagram à la Cutkosky, its bottom half — below the cut — become the the amplitude $\mathcal{M}(1{+}2 \to \text{cut})$ of turning the initial state $|1+2\rangle$ into the intermediate state $|\text{cut}\rangle$ comprised of particles corresponding to the cut on-shell propagators. For the one-loop example in question, $|\text{cut}\rangle = |q_1, q_2\rangle$ and

$$\mathcal{M}_{\text{below}} \;=\; \langle q_1, q_2|\, \mathcal{M}^{\text{tree}}\, |k_1, k_2\rangle \;=\; -\lambda. \tag{71}$$

Likewise, the top half of a cut diagram — above the cut — yields the amplitude of turning the $|\text{cut}\rangle$ state into the final state $|k'_1, k'_2\rangle$: In our case

$$\mathcal{M}_{\text{above}} = \langle k'_1, k'_2 | \mathcal{M}^{\text{tree}} | q_1, q_2 \rangle = -\lambda, \tag{72}$$

while more generally

$$\mathcal{M}_{\text{above}} = \langle \text{out} | \widehat{\mathcal{M}} | \text{cut} \rangle = \langle \text{cut} | \widehat{\mathcal{M}} | \text{out} \rangle^* , \tag{73}$$

$$\mathcal{M}_{\text{below}} = \langle \text{cut} | \widehat{\mathcal{M}} | \text{in} \rangle . \tag{74}$$

Finally, the remaining integral over the on-shell cut momenta q_1^μ and q_2^μ (together with the combinatorial factor $\frac{1}{2}$) is precisely the phase-space integral

$$\frac{1}{2} \int \frac{d^3\mathbf{q}_1}{(2\pi)^3 \times 2E_1} \int \frac{d^3\mathbf{q}_2}{(2\pi)^3 \times 2E_2} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - q_{\text{net}}) \tag{75}$$

over the intermediate on-shell states $|q_1, q_2\rangle$. Similarly, for other processes the remaining integral over the cut propagators' momenta also becomes the phase space integral over all possible intermediate $|\text{cut}\rangle$ states. Finally, when there are several diagrams which can be cut à la Cutkosky — and/or several ways to cut them — different cuts corresponds to different types of the intermediate $|\text{cut}\rangle$ states, *i.e.* different numbers or species of the intermediate particles. In other words, they correspond to different *channels* of processes $|\text{in}\rangle \to |\text{cut}\rangle$ or $|\text{out}\rangle \to |\text{cut}\rangle$. Altogether,Cutkosky's cutting rules yield the *generalized optical theorem:*

$$\text{disc}\, \langle \text{out} | \widehat{\mathcal{M}} | \text{in} \rangle = i \sum_{\langle \text{cut}|}^{\text{channels}} \int d\binom{\text{phase}}{\text{space}}(\text{cut})\ \langle \text{cut} | \widehat{\mathcal{M}} | \text{out} \rangle^* \times \langle \text{cut} | \widehat{\mathcal{M}} | \text{in} \rangle . \tag{76}$$

For the elastic scattering in the forward direction, the outgoing 2-particle state is the same as the incoming state, $|\text{out}\rangle = |\text{in}\rangle$, so the generalized optical theorem (76) becomes the ordinary optical theorem

$$\begin{aligned} \text{disc}\, \mathcal{M}^{\text{elastic}}_{\text{forward}}(s) &= i \sum_{\langle \text{cut}|}^{\text{channels}} \int d\binom{\text{phase}}{\text{space}}(\text{cut}) \left| \langle \text{cut} | \widehat{\mathcal{M}} | \text{in} \rangle \right|^2 \\ &= i \sum_{\langle \text{cut}|}^{\text{channels}} 4E_1 E_2 v_{12}^{\text{rel}} \times \sigma(\text{in} \to \text{cut}) \\ &= 4iE_1 E_2 v_{12}^{\text{rel}} \times \sigma_{\text{total}}(\text{in} \to \text{anything}). \end{aligned} \tag{77}$$