PSEUDO GOLDSTONE BOSONS

In my previous notes I explained spontaneous breakdown of *exact* symmetries of quantum field theories. Now let's consider spontaneous breakdown of *approximate* symmetries. For example, consider the theory of a complex scalar field $\Phi(x)$ with Lagrangian

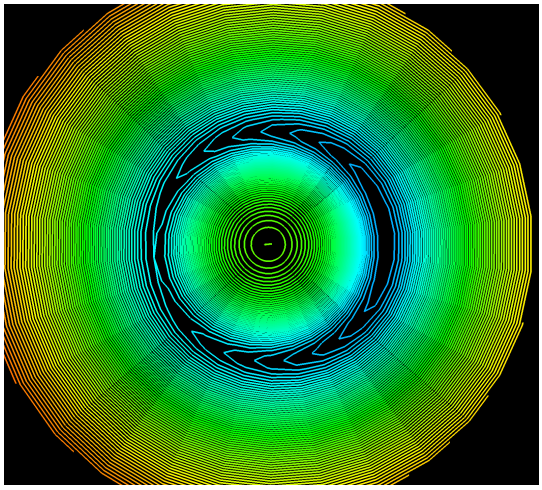
$$\mathcal{L} \;=\; \partial_\mu \Phi^* \partial^\mu \Phi \;-\; \frac{\lambda}{2}\left(\Phi^*\Phi - \frac{v^2}{2}\right)^2 \;+\; \beta\lambda v^3 \times \mathrm{Re}\,\Phi \tag{1}$$

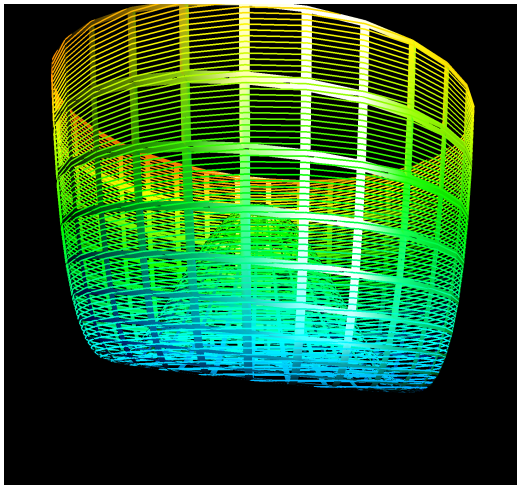
for a small $\beta \ll 1$. In the $\beta \to 0$ limit, the theory has a global $U(1)$ phase symmetry spontaneously broken by

$$\langle \Phi \rangle \;=\; \frac{v}{\sqrt{2}} \times e^{i\Theta}. \tag{2}$$

Thus, there is a continuous family of exactly degenerate vacua with different phases Θ, and there is an exactly massless Goldstone boson corresponding to $\delta\Phi(x)$ in the $\delta\Theta$ direction.

For $\beta \neq 0$ this $U(1)$ symmetry is *explicitly* broken, although for small β it remains an approximate symmetry of the theory. Consequently, the would-be vacuum states with VEVs (2) are approximately degenerate, but not exactly so. Instead, the circular ring of the potential's minima becomes tilted, as shown on these contour plots:





Or in terms of the phase Θ,

$$V(\Theta) \;=\; \text{const} \;-\; \frac{\beta\lambda v^4}{\sqrt{2}} \times \cos\Theta, \tag{3}$$

so there is a unique overall minimum at $\Theta = 0$. In terms of the fluctuations $\delta\Phi(x)$ in the Θ

direction, they are no longer potential-less and hence massless but cost

$$\delta V \;\approx\; +\frac{\beta\lambda v^4}{2\sqrt{2}} \times \delta\Theta^2, \tag{4}$$

or in terms of canonically normalized $\phi = v \times \Theta$,

$$\delta V \;\approx\; +\frac{\beta\lambda 2^4}{2\sqrt{2}} \times \phi^2. \tag{5}$$

Consequently, instead of the exactly massless Goldstone field, we have a light but massive *pseudo-Goldstone* field ϕ of mass^2

$$m_\phi^2 \;=\; \beta \times \frac{\lambda v^2}{\sqrt{2}}\,. \tag{6}$$

This is an example of a general rule: For each spontaneously broken generator of an *approximate* symmetry there is a light but massive *pseudo-Goldstone boson*. Similar to the massless Goldstone bosons, the pseudo-Goldstone bosons are created from the vacuum by the currents operators of the broken symmetries, and they have the same quantum numbers (WRT the unbroken symmetries) as the generators of the broken symmetries. However, the scattering amplitudes involving the pseudo-Goldstone bosons do not quite vanish in the $p^\mu_{\rm PGB} \to 0$ limit but merely become relatively small. We shall see an example of this behavior in the future homework#11 .

An important real-life example of a spontaneously broken approximate symmetry is the chiral symmetry of QCD. Since the two lightest quark flavors u and d are much lighter that the energy scale $\Lambda \sim 500$ MeV where QCD becomes non-perturbatively strong, we may approximate $m_d \approx m_u \approx 0$. In this approximation, QCD has the $SU(2)_L \times SU(2)_R$ chiral symmetry acting on the two lightest quark flavors as

$$\big(\psi'_L(x)\big)^{c,i} \;=\; (U_L)^i{}_j \psi_L^{c,j}(x), \quad \big(\psi'_R(x)\big)^{c,i} \;=\; (U_R)^i{}_j \psi_R^{c,j}(x), \quad i,j = u,d, \quad U_L, U_R \in SU(2). \tag{7}$$

This symmetry is spontaneously broken down to $SU(2)_V$ isospin by the non-perturbative

expectation value of the quark-antiquark bilinear condensate

$$\left\langle \overline{\Psi}\Psi \right\rangle \;\approx (285\text{ MeV})^3 \;\neq\; 0. \tag{8}^{\star}$$

To see how this condensate breaks the chiral symmetry, note that

$$\overline{\Psi}\Psi \;=\; \psi_L^\dagger \psi_R \;+\; \psi_R^\dagger \psi_L \tag{9}$$

and let me spell out the color and the flavor indices of these bilinears:

$$\mathcal{M}^i{}_j \;=\; (\psi_R^\dagger)_{c,j}(\psi_L)^{c,i} \quad\text{and}\quad (\mathcal{M}^\dagger)^j{}_i \;=\; (\psi_L^\dagger)_{c,i}(\psi_R)^{c,j}. \tag{10}$$

The $\mathcal{M}$ here is a 2×2 matrix which transforms under the chiral symmetry (7) as

$$\begin{aligned}
\mathcal{M}'^i{}_j \;&=\; (\psi'^\dagger_R)_{c,j}(\psi'_L)^{c,i} \\
&=\; \big((U_R^\dagger)^\ell{}_j(\psi_R^\dagger)_{c,\ell})\big)\big((U_L)^i{}_k(\Psi_L)^{c,k}\big) \\
&=\; (U_L)^i{}_k \times (\psi_R^\dagger)_{c,\ell})(\Psi_L)^{c,k} \times (U_R^\dagger)^\ell{}_j \\
&=\; (U_L)^i{}_k \times \mathcal{M}^k{}_\ell (U_R^\dagger)^\ell{}_j,
\end{aligned} \tag{11}$$

or in the matrix form

$$\mathcal{M}' \;=\; U_L \times \mathcal{M} \times U_R^\dagger. \tag{12}$$

Consequently, the quark-antiquark bilinear condensate

$$\langle \mathcal{M} \rangle \;=\; C \times \mathbf{1}_{2\times 2} \tag{13}$$

for a constant $C \neq 0$ spontaneously breaks the $SU(2)_L \times SU(2)_R$ chiral symmetry down to the $SU(2)_V$ in which $U_L = U_R$.

$\star$ The numerical value is per-flavor for the u and d flavors.

The 6 generators of the $SU(2)_L \times SU(2)_R$ comprise Q_V^a and Q_A^a for $a = 1, 2, 3$ corresponding to the vector and the axial isospin currents

$$J_V^{\mu a} \;=\; \overline{\Psi}\frac{\tau^a}{2}\gamma^\mu\Psi \quad \text{and} \quad J_A^{\mu a} \;=\; \overline{\Psi}\frac{\tau^a}{2}\gamma^\mu\gamma^5\Psi. \tag{14}$$

The Q_V^a generator remain unbroken by the bilinear condensate (13) while the Q_A^a become spontaneously broken. Since the $J_A^{\mu a}$ currents are axial vectors, the broken Q_A^a generators are pseudoscalars, so the corresponding Goldstone bosons should be pseudoscalars rather than true scalars. Also, the three Q_a^a form an isospin triplet (WRT the unbroken $SU(2)_V$), so the 3 Goldstone bosons should also form an isotriplet.

In real life, the chiral symmetry of QCD is only approximate, so instead of massless Goldstone bosons we get light bot massive pseudo-Goldstone bosons. But just as in the chiral limit, these pseudo-Goldstone bosons are pseudoscalars and form an isospin triplet: the pi-mesons π^a, or rather

$$\pi^0 \;=\; \pi^3 \quad \text{and} \quad \pi^\pm \;=\; \frac{\pi^1 \pm i\pi^2}{\sqrt{2}}\,. \tag{15}$$

Since they are pseudo-Goldstone rather that true Goldstone bosons, the pi-mesons are not massless but rather light: $m_\pi \approx 135$ MeV, much lighter than any other mesons.

We cannot calculate the pi-meson mass directly from QCD, it remains an unsolved problem. Instead, let's presume that the quark-antiquark condensate in QCD somehow has form (13), and then use the an effective theory of that condensate and its fluctuations to derive the pion's mass.

Similar to the tilted ring of the would-be vacua in the Φ theory from the example a few pages above, let's consider the would-be vacua related by the chiral symmetry to the vacuum (13). The chiral symmetries act on the condensate $\mathcal{M}$ as

$$\mathcal{M}' \;=\; U_L \times \mathcal{M} \times U_R^\dagger, \tag{12}$$

so the states related to the vacuum with $\langle M\rangle = C \times 1$ have

$$\langle M\rangle \;=\; C \times W \quad \text{for} \quad W \in SU(2). \tag{16}$$

where the $SU(2)$ matrix W is the analogue of the phase Θ from the earlier example. Thus, W parametrizes *approximately* degenerate would-be vacuum states, and to work with the

fluctuations along the W directions we need to promote W to a slowly-varying matrix-valued field $W(x)$.

Note: The $W(x)$ field would map the Minkowski spacetime to a *non-linear field space*, namely the $SU(2)$ group manifolds. Such non-linear field theories are called *non-linear sigma models*, and I hope to give an extra lecture on this subject sometimes in the Spring semester.

Meanwhile, let me use a different approach called the *linear sigma model*. Since the $SU(2)$ group manifold is a sphere S^3 in 4 dimensions, we may add a radial direction to that sphere and make a linear theory of 4 fields $\phi^{1,2,3,4}$:

$$W(x) \;=\; \frac{1}{v}\Big(\phi^4(x)\times 1_{2\times 2} \,+\, \phi^1(x)\times i\tau^1 \,+\, \phi^2(x)\times i\tau^2 \,+\, \phi^3(x)\times i\tau^3\Big) \tag{17}$$

(where $\tau^1, \tau^2\tau^3$ are isospin Pauli matrices) is an $SU(2)$ matrix when $(\phi^1)^2+\cdots+(\phi^4)^2 = v^2$. So let's give the ϕ^i fields an $SO(4)$ invariant potential with a radius $= v$ sphere of minima,

$$\mathcal{L}_{\text{chiral}} \;=\; \sum_i \tfrac{1}{2}(\partial_\mu\phi^i)^2 \;-\; \frac{\lambda}{8}\left(\sum_i(\phi^i)^2 \,-\, v^2\right)^2, \tag{18}$$

then we may identify the bilinear condensate and its fluctuations as

$$\mathcal{M}(x) \;=\; \frac{C}{v}\Big(\phi^4(x)\times 1_{2\times 2} \,+\, \phi^1(x)\times i\tau^1 \,+\, \phi^2(x)\times i\tau^2 \,+\, \phi^3(x)\times i\tau^3\Big). \tag{19}$$

In terms of the ϕ^i fields, the chiral symmetries (12) act as $SO(4)$ rotations of the 4D field space. Indeed, $SU(2)_L\times SU(2)_R \cong \text{Spin}(4)$. Also, the $SU(2)_V$ unbroken symmetry subgroup acts as $\text{Spin}(3) \subset \text{Spin}(4)$ which leaves the ϕ^4 invariant while rotating the other three $\phi^{1,2,3}$ into each other. Thus, the $SU(2)_L \times SU(2)_R \to SU(2)_V$ spontaneous symmetry breaking can be modeled by the $SO(4) \to SO(3)$ spontaneous symmetry breaking of the linear sigma model made from the $\phi^{1,2,3,4}$ fields.

Now consider the explicit chiral symmetry breaking by the quark mass terms

$$\mathcal{L}_{\text{mass}} \;=\; m_u\overline{\Psi}_u\Psi^u \;\; m_d\overline{\Psi}_d\Psi^d. \tag{20}$$

In terms of the quark-antiquark bilinears $\mathcal{M}$ and $\mathcal{M}^\dagger$, these terms amount to

$$\mathcal{L}_{\text{mass}} \;=\; 2m_u\,\text{Re}\big(\mathcal{M}^u_{\;u}\big) \;+\; 2m_d\,\text{Re}\big(\mathcal{M}^d_{\;d}\big). \tag{21}$$

In the linear sigma model where the $\mathcal{M}$ matrix obtains as (19), we have

$$\mathrm{Re}\,\mathcal{M}^u_{\ u} \;=\; \mathrm{Re}\,\mathcal{M}^d_{\ d} \;=\; \frac{C}{v}\times\phi^4, \tag{22}$$

hence

$$\begin{aligned}\mathcal{L} \;&=\; \mathcal{L}_{\rm chiral} \;+\; \mathcal{L}_{\rm mass}\\ &=\; \frac{1}{2}\sum_i(\partial_\mu\phi^i)^2 \;-\; \frac{\lambda}{8}\left(\sum_i(\phi^i)^2 \;-\; v^2\right)^2 \;+\; \frac{2(m_u+m_d)C}{v}\times\phi^4.\end{aligned} \tag{23}$$

You are going to work out this linear sigma model in detail in problem 3 of your homework#11, where I had renamed f to v and $2(m_u+m_d)C/\lambda v^3$ to β. The $SO(4)$-breaking term in the potential can be treated as a small perturbation provided

$$\frac{\beta}{f} \;=\; \frac{2C(m_u+m_d)}{\lambda v^4} \;\ll\; 1. \tag{24}$$

Working through the problem, you shall see that the model has only one vacuum state where $\langle\phi^4\rangle \approx v$ while $\langle\phi^{1,2,3}\rangle = 0$, and that the fluctuation fields are all massive, but the 3 pion fields $\pi^{1,2,3} = \phi^{1,2,3}$ are light while the $\sigma = \phi^4 - \langle\phi^4\rangle$ is heavy,

$$M_\pi^2 \;\approx\; \lambda\beta f \;=\; \frac{2C(m_u+m_d)}{v^2} \quad \text{while } M_\sigma^2 \;\approx\; \lambda f^2 \;=\; \lambda v^2, \tag{25}$$

hence

$$\frac{M_\pi^2}{M_\sigma^2} \;\approx\; \frac{2C(m_u+m_d)}{\lambda v^4} \;\ll\; 1. \tag{26}$$

Identifying the σ particle with a rather broad resonance with central mass $M \approx 500$ MeV, we see that

$$\frac{M_\pi^2}{M_\sigma^2} \;\approx\; 0.073, \tag{27}$$

which is indeed rather small. Also, identifying v as the pion decay constant $f_\pi \approx 93$ MeV, we see that the linear sigma model needs a rather strong coupling $\lambda \approx 30$. Finally, from eq. (25) for the pion mass, we obtain the value of the $\bar{q}q$ bilinear condensate in QCD,

$$\langle\overline{\Psi}\Psi\rangle \;=\; 2C \;=\; \frac{M_\pi^2 f_\pi^2}{m_u+m_d} \;=\; (285\ \text{MeV})^3 \tag{28}$$

for $m_u = 2.2$ MeV and $m_d = 4.6$ MeV.