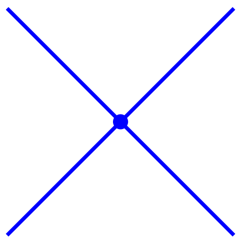
Amputating External Leg Bubbles

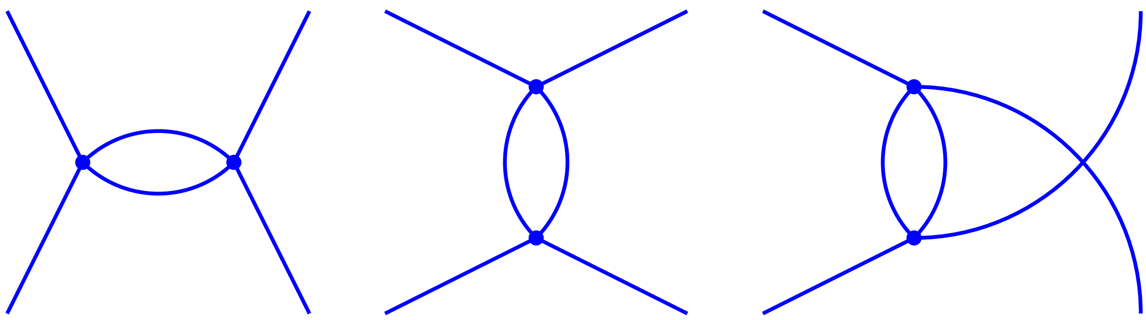
In the Fall semester of the QFT class, we have evaluated a bunch of Feynman diagrams, but only at the tree level. This Spring semester we shall focus on the loop diagrams, and our first task is to find what kind of diagrams do or do not contribute to the scattering amplitudes. Back in October, we saw that the scattering amplitudes do not include any loop diagrams with vacuum bubbles, or any other kinds of disconnected diagrams. And in these notes, we shall see that the diagrams with *external leg bubbles* also do not contribute to the scattering amplitudes.

But before we formally defined the external leg bubbles, let's see a few example diagrams where such bubbles cause trouble. For simplicity, consider the QFT of a single scalar field $\hat{\Phi}(x)$ with a $\lambda\Phi^4$ interaction, and let's focus on the elastic scattering amplitude $\mathcal{M}(1+2\to 1'+2')$. At the tree level, there is only one diagram contributing to this process,



$$\mathcal{M}_{\rm tree} \;=\; -\lambda\ \forall\text{particle momenta.} \tag{1}$$

But at the one-loop level, there are seven connected diagrams with 4 external legs, namely 3 diagrams of one topology



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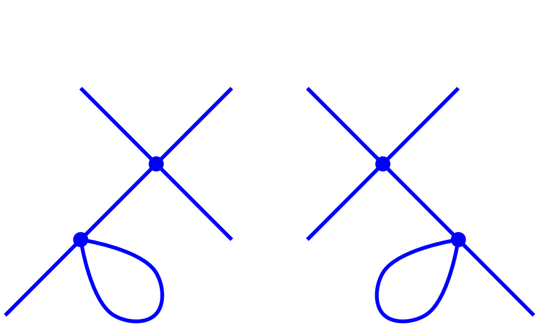
and 4 diagrams of a different topology



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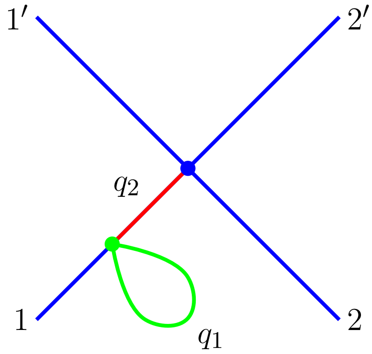
In my next set of notes we shall learn how to evaluate the 3 diagrams (2). But for the moment, let's focus on the other 4 diagrams (3) and see what's wrong with them.

Since the 4 diagrams (3) are related to each other by external leg permutations and/or crossing symmetries, it's enough to consider any one of these diagrams, say the first:



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By the momentum conservation in the green vertex, regardless of the loop momentum q_1, the red propagator carries momentum $q_2 = p_1$, the same as the first incoming particle. But that particle's momentum is on-shell, $p_1^2 = m^2$, hence $q_2^2 = m^2$, so the red propagator amounts to

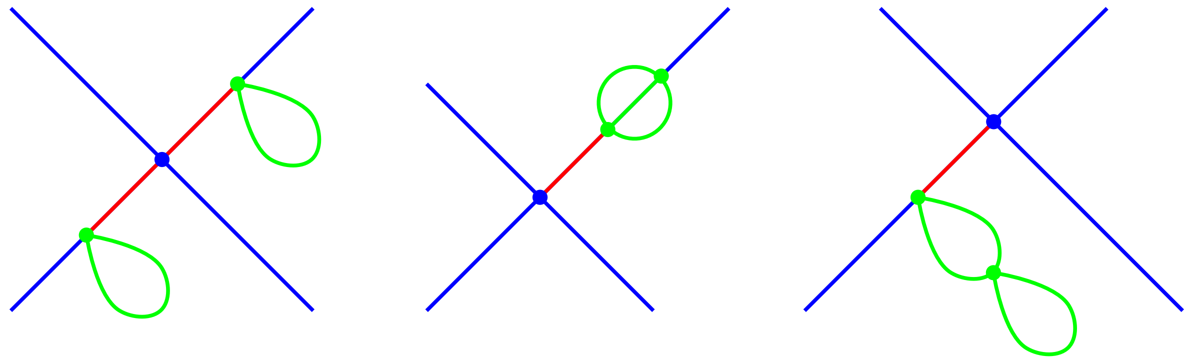
$$\text{red propagator} \;=\; \frac{i}{q_2^2 - m^2 + i0} \;=\; \frac{1}{0} \quad \text{regardless of the loop momentum } q_1. \tag{5}$$

Consequently, the diagram's amplitude

$$\mathcal{M} \;=\; \frac{1}{0} \times \int \frac{d^4 q_1}{(2\pi)^4}\,(\text{other factors}) \tag{6}$$

blows up even before we integrate over the loop momentum q_1.

Similar trouble afflicts many diagrams at the higher loop orders; here are a few two-loop examples:



(7)

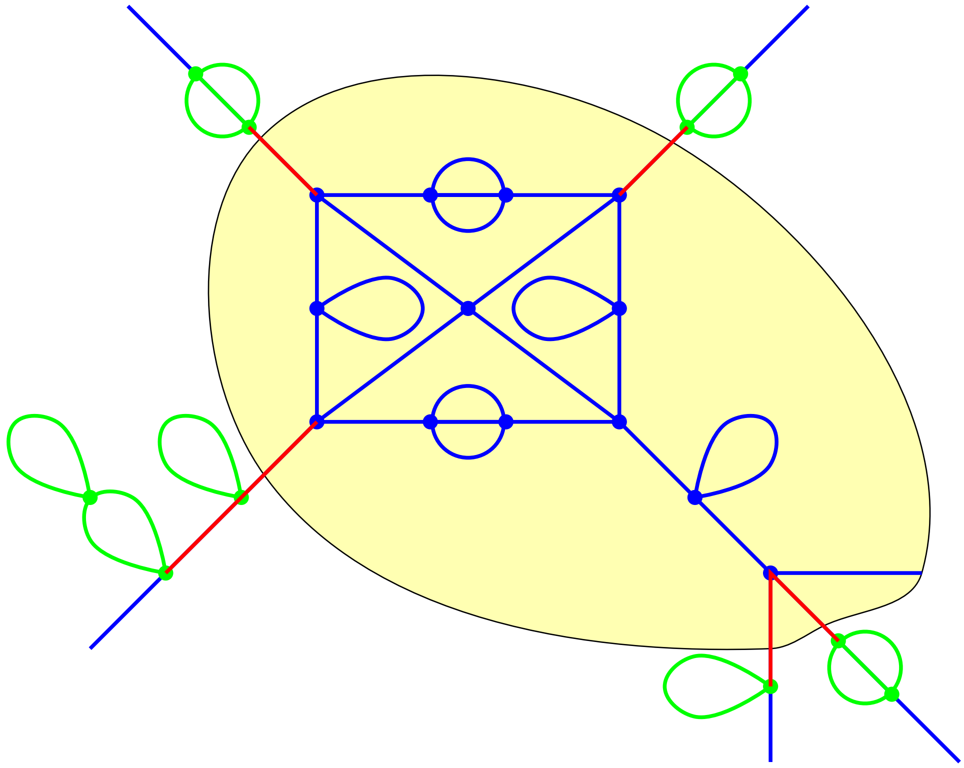
In all these diagrams, the red propagators carry momenta equal to the on-shell external particles' momenta regardless of any other momenta in the diagram, hence

$$\text{red propagator} \;=\; \frac{i}{q^2-m^2+i0} \;=\; \frac{1}{0} \tag{8}$$

and the amplitude blows up.

All these examples of troublesome diagrams have *external leg bubbles,* highlighted in green on figures (4) and (7). The general definition of such an external leg bubble is: A sub-diagram, which is [1] can be severed from the rest of the diagram by cutting a single propagator (shown in red on figures (4) and (7)), and [2] connected to only one external leg. By momentum conservation, the red propagator connecting an external leg bubble to the rest of the diagram has the same momentum as the external leg, so this propagator is stuck on-shell regardless of any loop momenta or scattering angles, and that's what causes the diagram's amplitude to blow up before any momentum integrals.

The practical solution to the external leg bubble problem is to *amputate* all the external leg bubbles. That is, remove them from the diagram, and treat the (red) propagators connecting them to the rest of the diagram as new external legs. For example:



(9)

where the yellow 'egg' is the surviving core of the diagram once all the external leg bubbles have been amputated. Note that many internal lines of the surviving core include sub-diagrams similar to the external leg bubbles we amputate, but we do not cut those internal lines because the momenta they carry are NOT stuck on-shell. Instead, such internal-line bubbles lead to quantum corrections of the particle's mass and of the field strength that creates and annihilates the particles. We shall address this issue in detail in a few lectures. But for the moment, we simply notice that only the external leg bubbles should be amputated but not any bubbles in the internal lines.

When amputating diagrams, one should be careful not to double-count (or multiple-count) similar surviving cores of different diagrams. That is, when several diagrams start with different external leg bubbles but end up with similar surviving cores after the external leg bubbles have been amputated, we should count this surviving core only once. That is, *when calculating the scattering amplitudes, we should include only the diagrams without any* *external leg bubbles*,

$$i\mathcal{M}(1,2\to 1',\ldots,n')\times(2\pi)^4\delta^{(4)}(p_1+p_2-p'_1-\cdots-p'_n) \;=\; \sum\begin{bmatrix}\text{Connected Feynman diagrams}\\ \text{with } n+2 \text{ external legs}\\ \text{and no external leg bubbles}\end{bmatrix} \tag{10}$$

How do I justify this pragmatic rule? Back in late October, I argued that the sum of *all* Feynman diagrams — including the diagrams with vacuum bubbles, or other undesirable features — yields S-matrix elements between eigenstates of the free Hamiltonian $\hat{H}_0$ rather than between the physical incoming and outgoing multi-particle states,

$$\begin{aligned}\sum\begin{bmatrix}\text{all Feynman diagrams}\\ \text{with } n+2 \text{ external legs}\end{bmatrix} &= \left\langle \text{free}: 1',\ldots,n'\right|\hat{S}\left|\text{free}: 1,2\right\rangle\\ &= \langle 0|\,\hat{a}'_1\cdots\hat{a}'_n\,\hat{S}\,\hat{a}^\dagger_1\hat{a}^\dagger_2\,|0\rangle\end{aligned} \tag{11}$$

where $|0\rangle$ is the ground state of the $\hat{H}_0$ rather than the true vacuum state $|\Omega\rangle$. I have also argued (without a proof) that the matrix elements between physical incoming and outgoing states are related to the matrix elements (11) as

$$\begin{aligned}\left\langle \text{out}: p'_1,p'_2\ldots\right|\hat{S}\left|\text{in}: p_1,p_2,\ldots\right\rangle \;=\; \lim_{\text{reg}\to\text{away}}&\left\langle\text{free}: p'_1,p'_2\ldots\right|\hat{S}\left|\text{free}: p_1,p_2,\ldots\right\rangle_{\text{reg}}\times\\ &\times C_{\text{vac}}(\text{reg})\times\prod_{\substack{\text{incoming or}\\ \text{outgoing}\\ \text{particles}}} F(p_i\text{ or }p'_i,\text{reg}),\end{aligned} \tag{12}$$

where C_{vac} is the same factor for all Feynman diagrams reflecting quantum corrections to the QFT vacuum state, while $F(p_i)$ or $F(p'_i)$ factors — stemming from quantum corrections to the physical particle states — are the same for all diagrams with the same incoming or outgoing particle momenta. Moreover, back in October we have seen that the C_{vac} factor can be canceled by simply skipping all Feynman diagrams with vacuum bubbles, thus

$$\left\langle \text{out}: p_1', p_2' \ldots \right| \hat{S} \left| \text{in}: p_1, p_2, \ldots \right\rangle = \lim_{\text{reg}\to\text{away}} \sum \begin{bmatrix} \text{all Feynman diagrams} \\ \text{with } n+2 \text{ external legs} \\ \text{and without vacuum bubbles} \end{bmatrix} \times \prod_{\substack{\text{incoming or} \\ \text{outgoing} \\ \text{particles}}} F(p_i \text{ or } p_i'). \tag{13}$$

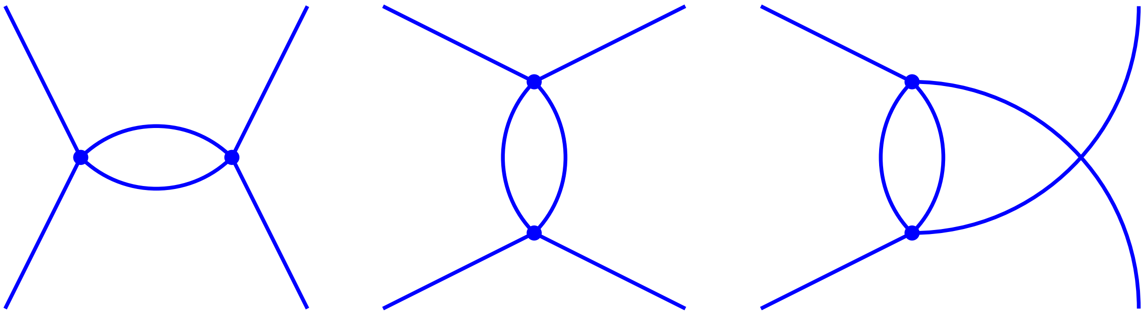
Today I am making the next step in this argument and claim that the external leg factors $F(p)$ or $F(p')$ cancel all the leg bubbles which might attach to the corresponding external leg. A few classes later I shall prove this claim using multi-field correlation functions and the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula, but for the moment let me use this claim to write a simpler formula for the physical matrix elements:

$$\left\langle \text{out}: p_1', p_2' \ldots \right| \hat{S} \left| \text{in}: p_1, p_2, \ldots \right\rangle = \sum \begin{bmatrix} \text{all Feynman diagrams} \\ \text{with } n+2 \text{ external legs,} \\ \text{without vacuum bubbles,} \\ \text{and without external leg bubbles} \end{bmatrix}. \tag{14}$$

Or in terms of the scattering amplitudes,

$$i\mathcal{M}(1,2\to 1',\ldots,n') \times (2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-\cdots-p_n') = \\ = \sum \begin{bmatrix} \text{Connected Feynman diagrams} \\ \text{with } n+2 \text{ external legs and} \\ \text{without external leg bubbles} \end{bmatrix} \tag{10}$$

In particular, at the one-loop level, the elastic scattering amplitude $\mathcal{M}(1+2\to 1'+2')$ in the $\lambda\phi^4$ theory comes from just 3 diagrams without external leg bubbles, namely



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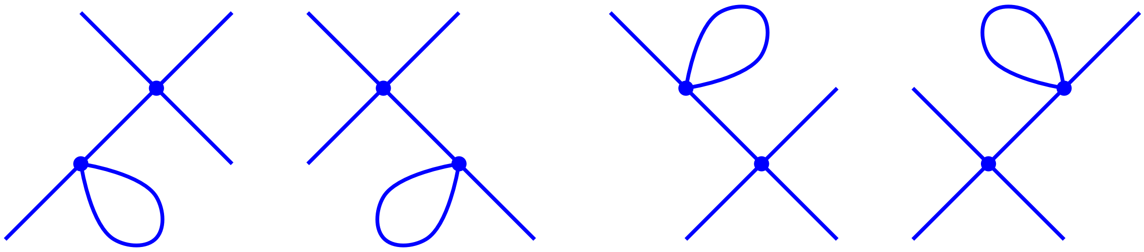
while the diagrams which do have external leg bubbles —



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— do not contribute.