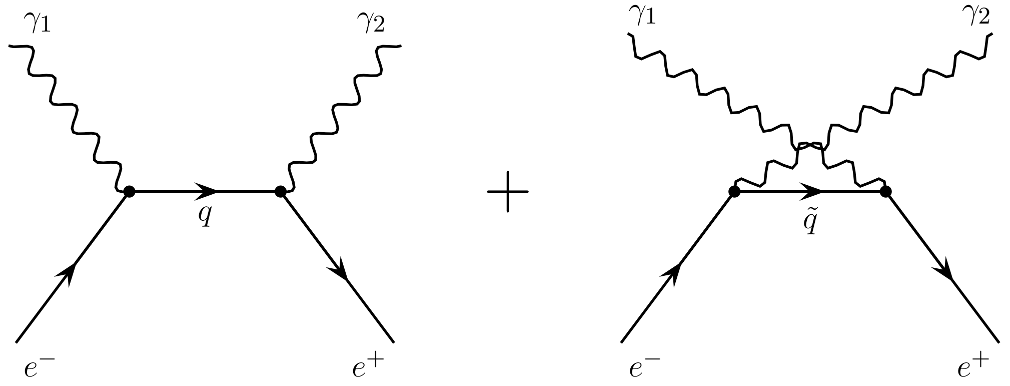
ANNIHILATION

In these notes I explain the $e^+e^- \to \gamma\gamma$ annihilation process. At the tree level of QED, there are two diagrams related by interchanging of the two photons in the final state:



(1)

The net amplitude due to these diagrams is

$$\begin{aligned}
\mathcal{M} &= \mathcal{E}^*_\mu(k_1,\lambda_1)\,\mathcal{E}^*_\nu(k_2,\lambda_2)\times\mathcal{M}^{\mu\nu},\\
\mathcal{M}^{\mu\nu} &= \mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu},\\
i\mathcal{M}_1^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\nu)\frac{i}{\not{q}-m}(ie\gamma^\mu)u(e^-),\\
i\mathcal{M}_2^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\mu)\frac{i}{\not{\tilde{q}}-m}(ie\gamma^\nu)u(e^-),
\end{aligned} \tag{2}$$

where $q = p_- - k_1 = k_2 - p_+$ and $\tilde{q} = p_- - k_2 = k_1 - p_+$. Note the opposite orders of the γ^μ and γ^ν vertices in the $\mathcal{M}_1$ and the $\mathcal{M}_2$ amplitudes since the two photons attach to the electron line in opposite order. Also note the bosonic symmetry between the two photons in the final state: exchanging the photons is equivalent to exchanging the two diagrams, thus

$$\mathcal{M}_1^{\mu\nu}(k_1,k_2;p_-,p_+) = \mathcal{M}_2^{\nu\mu}(k_1\leftrightarrow k_2;p_-,p_+) \quad\Longrightarrow\quad \mathcal{M}_{\rm net}^{\mu\nu} = \mathcal{M}_{\rm net}^{\nu\mu}(k_1\leftrightarrow k_2). \tag{3}$$

For calculation purposes, it is convenient to eliminate the matrix denominators from the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ using

$$\frac{1}{\not{q}-m} = \frac{\not{q}+m}{q^2-m^2} = \frac{\not{q}+m}{t-m^2} \quad\text{and}\quad \frac{1}{\not{\tilde{q}}-m} = \frac{\not{\tilde{q}}+m}{\tilde{q}^2-m^2} = \frac{\not{\tilde{q}}+m}{u-m^2}\,, \tag{4}$$

hence

$$\mathcal{M}_1^{\mu\nu} \;=\; \frac{-e^2}{t-m^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \quad\text{and}\quad \mathcal{M}_2^{\mu\nu} \;=\; \frac{-e^2}{u-m^2} \times \bar{v}\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu u. \tag{5}$$

Ward Identities

Before we go any further, lets check the Ward identities for the annihilation amplitude: For the first photon we should have $k_{1\mu}\mathcal{M}^{\mu\nu}=0$, and for the second photon $k_{2\nu}\mathcal{M}^{\mu\nu}=0$. Let's start with the first photon and the first diagram. Multiplying the second factor in the first eq. (5) by $k_{1\mu}$, we have

$$\begin{aligned}
\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times k_{1\mu} &= \bar{v}\gamma^\nu(\not{p}_- - \not{k}_1 + m)\not{k}_1 u \\
&= \bar{v}\gamma^\nu(\not{p}_- + m)\not{k}_1 u \quad \langle\!\langle\, \text{since } \not{k}_1\not{k}_1 = k_1^2 = 0 \,\rangle\!\rangle \\
&= \bar{v}\gamma^\nu\Big(2(p_-k_1) - \not{k}_1(\not{p}_- - m)\Big)u \quad \langle\!\langle\, \text{since } \not{p}_-\not{k}_1 = 2(p_-k_1) - \not{k}_1\not{p}_- \,\rangle\!\rangle \\
&= 2(p_-k_1)\times\bar{v}\gamma^\nu u \;-\; \bar{v}\gamma^\nu\not{k}_1\times(\not{p}_- - m)u \\
&= 2(p_-k_1)\times\bar{v}\gamma^\nu u \quad \langle\!\langle\, \text{since } (\not{p}_- - m)u = 0 \,\rangle\!\rangle
\end{aligned} \tag{6}$$

where

$$(2p_-k_1) \;=\; k_1^2 \;+\; p_-^2 \;-\; (p_- - k_1)^2 \;=\; 0 \;+\; m^2 \;-\; t. \tag{7}$$

This factor on the last line of eq. (6) cancels the denominator of the $\mathcal{M}_1^{\mu\nu}$ amplitude in eq. (5) (except for the overall sign), and we are left with

$$\mathcal{M}_1^{\mu\nu} \times k_{1\mu} \;=\; +e^2 \times \bar{v}\gamma^\nu u\,. \tag{8}$$

Note the non-zero right hand side — the first diagram does not obey the Ward identity all by itself. As for the second diagram, we have

$$\begin{aligned}
\bar{v}\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu u \times k_{1\mu} &= \bar{v}\,\not{k}_1(\not{k}_1 - \not{p}_+ + m)\gamma^\nu u \\
&= \bar{v}\,\not{k}_1(-\not{p}_+ + m)\gamma^\nu u \quad \langle\!\langle\, \text{since } \not{k}_1\not{k}_1 = k_1^2 = 0 \,\rangle\!\rangle \\
&= \bar{v}\Big(-2(p_+k_1) \;+\; (\not{p}_+ + m)\not{k}_1\Big)\gamma^\nu u \\
&\qquad\qquad \langle\!\langle\, \text{since } -\not{k}_1\not{p}_+ = -2(k_1p_+) + \not{p}_+\not{k}_1 \,\rangle\!\rangle \\
&= -2(p_+k_1)\times\bar{v}\gamma^\nu u \;+\; \bar{v}(\not{p}_+ + m)\times\not{k}_1\gamma^\nu u \\
&= -2(p_+k_1)\times\bar{v}\gamma^\nu u \quad \langle\!\langle\, \text{since } \bar{v}(\not{p}_+ + m) = 0 \,\rangle\!\rangle,
\end{aligned} \tag{9}$$

where

$$-2(p_+k_1) \;=\; (p_+ - k_1)^2 \;-\; p_+^2 \;-\; k_1^2 \;=\; u \;-\; 0 \;-\; m^2. \tag{10}$$

Again, this factor cancels the denominator of the $\mathcal{M}_2^{\mu\nu}$ amplitude in eq. (5) (but this time without an extra sign), and we are left with

$$\mathcal{M}_2^{\mu\nu} \times k_{1\mu} \;=\; -e^2 \times \bar{v}\gamma^\nu u\,. \tag{11}$$

Similar to the first diagram's amplitude $\mathcal{M}_1^{\mu\nu}$, the second diagram's amplitude $\mathcal{M}_2^{\mu\nu}$ also does not obey the Ward identity all by itself. However, the right hand sides of eqs. (8) and (11) cancel each other, so the *net amplitude* does obey the Ward identity,

$$\mathcal{M}_{\text{net}}^{\mu\nu} \times k_{1\mu} \;=\; \mathcal{M}_1^{\mu\nu} \times k_{1\mu} \;+\; \mathcal{M}_2^{\mu\nu} \times k_{1\mu} \;=\; 0. \tag{12}$$

This is an example of a general rule: The Ward identities does not work diagram by diagram, but only for sums of all diagrams related by permutations of photonic vertices on the same fermionic line — or for bigger sums, such as complete amplitudes to N-loop order for $N = 0, 1, 2, \ldots$.

The Ward identity $\mathcal{M}^{\mu\nu} \times k_{2\nu} = 0$ for the second photon works similarly to the first. In fact, thanks to the Bose symmetry (3) between the two photons, the two Ward identities are equivalent to each other,

$$\mathcal{M}^{\mu\nu} \;=\; \mathcal{M}^{\nu\mu}(k_1 \leftrightarrow k_2) \quad\Longrightarrow\quad \Big(\mathcal{M}^{\mu\nu} \times k_{1\mu} \;=\; 0 \quad\Longleftrightarrow\quad \mathcal{M}^{\mu\nu} \times k_{2\nu} \;=\; 0\Big). \tag{13}$$

Thus, for the second photon

$$\mathcal{M}_1^{\mu\nu} \times k_{2\nu} \;=\; -e^2 \times \bar{v}\gamma^\mu u \;\neq\; 0, \quad \mathcal{M}_2^{\mu\nu} \times k_{2\nu} \;=\; +e^2 \times \bar{v}\gamma^\mu u \;\neq\; 0, \quad \text{but} \quad \mathcal{M}_{\text{net}}^{\mu\nu} \times k_{2\nu} \;=\; 0. \tag{14}$$

Summing over the Spins and Polarizations

In a typical annihilation experiment, the initial electrons and positrons come from un-polarized beams where both spin states are equally likely. Likewise, the photon detector is sensitive to the outgoing photons' momenta but it does not care about their polarization states. To calculate the annihilation cross-section for such un-polarized process, we should sum the $|\mathcal{M}|^2$ over the final photon polarizations and average over the spins of the initial fermions.

Summing the $|\mathcal{M}|^2$ over the photon polarizations is explained in detail in my notes on Ward identities. Thanks to the Ward identities, we can do it in terms of the $\mathcal{M}^{\mu\nu}$ amplitude as

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu}\,. \tag{15}$$

For the annihilation process at hand $\mathcal{M}^{\mu\nu} = \mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$, so

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu} \;=\; \mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{1\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}\mathcal{M}^*_{2\mu\nu} \;+\; 2\,\mathrm{Re}\,\mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{2\mu\nu}\,. \tag{16}$$

Note that this formula does not need the $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ amplitudes to obey the Ward identities by themselves, it is enough that the net amplitude $\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$ obeys the identities. Specifically, for the $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ as in eqs. (5), we have

$$\begin{aligned}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\;& \frac{e^4}{(t-m^2)^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m)\gamma_\nu v \\ &+\; \frac{e^4}{(u-m^2)^2} \times \bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v \\ &+\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \mathrm{Re}\Big(\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v\Big).\end{aligned} \tag{17}$$

This formula takes care of summing over the photon polarizations, and now we need to average the result over the initial fermions' spins. As explained in my notes on Dirac traces,

$$\sum_{s_1,s_2} \bar{v}(p_+,s_+)\Gamma u(p_-,s_-) \times \bar{u}(p_-,s_-)\overline{\Gamma} v(p_+,s_+) \;=\; \mathrm{Tr}\Big((\not{p}_+ - m)\Gamma(\not{p}_- + m)\overline{\Gamma}\Big), \tag{18}$$

and in a similar way we may show that for $\Gamma' \neq \overline{\Gamma}$ we also have

$$\sum_{s_1,s_2} \bar{v}(p_+,s_+)\Gamma u(p_-,s_-) \times \bar{u}(p_-,s_-)\Gamma' v(p_+,s_+) \;=\; \mathrm{Tr}\Big((\not{p}_+ - m)\Gamma(\not{p}_- + m)\Gamma'\Big). \tag{19}$$

Applying these rules to averaging eq. (17) over the electron's and positron's spins gives us

$$\begin{aligned} \overline{|\mathcal{M}|^2} \;&\equiv\; \tfrac{1}{4}\sum_{s_-,s_+}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \\ &=\; \frac{e^4}{(t-m^2)^2}\times A_{11} \;+\; \frac{e^4}{(u-m^2)^2}\times A_{22} \;+\; \frac{2e^4}{(t-m^2)(u-m^2)}\times \mathrm{Re}\,A_{12}\,, \end{aligned} \tag{20}$$

where

$$\begin{aligned} A_{11} \;&=\; \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\nu(\not{q}+m)\gamma^\mu u(p_-,s_-) \times \bar{u}(p_-,s_-)\gamma_\mu(\not{q}+m)\gamma_\nu v(p_+,s_+) \\ &=\; \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\mu(\not{q}+m)\gamma_\nu\Big), \end{aligned} \tag{21}$$

$$\begin{aligned} A_{22} \;&=\; \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu u(p_-,s_-) \times \bar{u}(p_-,s_-)\gamma_\nu(\not{\tilde{q}}+m)\gamma_\mu v(p_+,s_+) \\ &=\; \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu(\not{p}_- + m)\gamma_\nu(\not{\tilde{q}}+m)\gamma_\mu\Big), \end{aligned} \tag{22}$$

$$\begin{aligned} A_{12} \;&=\; \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\nu(\not{q}+m)\gamma^\mu u(p_-,s_-) \times \bar{u}(p_-,s_-)\gamma_\nu(\not{\tilde{q}}+m)\gamma_\mu v(p_+,s_+) \\ &=\; \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\nu(\not{\tilde{q}}+m)\gamma_\mu\Big). \end{aligned} \tag{23}$$

And now we need to calculate these big traces. . .

Traceology 1

Let's start with the A_{11} trace. It looks rather formidable, but we may simplify it using formulae

$$\gamma^\mu\gamma_\mu \;=\; 4, \quad \gamma^\mu\not{a}\gamma_\mu \;=\; -2\,\not{a}, \quad \gamma^\mu\not{a}\not{b}\gamma_\mu \;=\; 4(ab), \quad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu \;=\; -2\,\not{c}\not{b}\not{a} \tag{24}$$

from the homework#7 (problem 3(b)). Indeed, a cyclic permutation of matrices inside the trace turns it into

$$A_{11} \;=\; \tfrac{1}{4}\,\mathrm{Tr}\Bigl(\gamma_\nu(\not{p}_+ - m)\gamma^\nu \times (\not{q}+m) \times \gamma^\mu(\not{p}_- + m)\gamma_\mu \times (\not{q}+m)\Bigr) \tag{25}$$

where thanks to eqs. (24) we have

$$\gamma^\mu(\not{p}_- + m)\gamma_\mu \;=\; \gamma^\mu \not{p}_-\gamma_\mu \;+\; m\gamma^\mu\gamma_\mu \;=\; -2\not{p}_- \;+\; m\times 4 \;=\; -2(\not{p}_- - 2m) \tag{26}$$

and likewise

$$\gamma_\nu(\not{p}_+ - m)\gamma^\nu \;=\; -2(\not{p}_+ + 2m), \tag{27}$$

hence

$$A_{11} \;=\; \mathrm{Tr}\Bigl((\not{p}_+ + 2m)(\not{q}+m)(\not{p}_- - 2m)(\not{q}+m)\Bigr). \tag{28}$$

Next, we expand the parentheses inside this trace and throw away terms with odd numbers of momenta $\not{p}$ or $\not{q}$. This eliminates 8 out of 16 terms, and leaves us with

$$\begin{aligned} A_{11} \;=&\; \mathrm{Tr}(\not{p}_+\not{q}\,\not{p}_-\not{q}) \;+\; m^2\times\mathrm{Tr}(\not{p}_+\not{p}_-) \;-\; 4m^2\times\mathrm{Tr}(\not{q}\,\not{q}) \\ &+\; 2\times 2m^2\times\mathrm{Tr}(\not{p}_-\not{q}) \;-\; 2\times 2m^2\times\mathrm{Tr}(\not{p}_+\not{q}) \;-\; 4m^4\times\mathrm{Tr}(1) \\ =&\; 2\times 4(p_+q)(p_-q) \;-\; 4(p_+p_-)(q^2) \;+\; m^2\times 4(p_+p_-) \;-\; 4m^2\times 4(q^2) \\ &+\; 4m^2\times 4(p_-q) \;-\; 4m^2\times 4(p_+q) \;-\; 4m^4\times 4. \\ =&\; 8(p_+q)(p_-q) \;-\; 4(p_+p_-)\times(q^2-m^2) \;-\; 16m^2\times\bigl(q^2 \;-\; (p_-q) \;+\; (p_+q) \;+\; m^2\bigr)\,. \end{aligned} \tag{29}$$

We may further simplify this formula by expressing all the momenta products in terms of the Mandelstam's variables s, t, and u. Using $p_-^2 = p_+^2 = m^2$ and $k_1^2 = k_2^2 = 0$, we have

$$q^2 \;=\; (p_- - k_1)^2 \;=\; t, \tag{30}$$

$$2p_-p_+ \;=\; (p_- + p_+)^2 \;-\; p_+^2 \;-\; p_-^2 \;=\; s \;-\; 2m^2, \tag{31}$$

$$2k_1p_- \;=\; k_1^2 \;+\; p_-^2 \;-\; (k_1 - p_-)^2 \;=\; 0 \;+\; m^2 \;-\; t, \tag{32}$$

$$2k_2p_+ \;=\; k_2^2 \;+\; p_+^2 \;-\; (k_2 - p_+)^2 \;=\; 0 \;+\; m^2 \;-\; t, \tag{33}$$

and hence

$$qp_- \;=\; (p_- - k_1)p_- \;=\; p_-^2 \,-\, p_-k_1 \;=\; m^2 \,+\, \tfrac{1}{2}(t-m^2) \;=\; +\tfrac{1}{2}(m^2+t), \tag{34}$$
$$qp_+ \;=\; (k_2 - p_+)p_+ \;=\; p_+k_2 \,-\, p_+^2 \;=\; -\tfrac{1}{2}(t-m^2) \,-\, m^2 \;=\; -\tfrac{1}{2}(t+m^2). \tag{35}$$

Consequently, on the last line of eq. (29), the last term vanishes —

$$q^2 \,-\, (p_-q) \,+\, (p_+q) \,+\, m^2 \;=\; t \,-\, \tfrac{1}{2}(t+m^2) \,-\, \tfrac{1}{2}(t+m^2) \,+\, m^2 \;=\; 0 \tag{36}$$

— while the remaining terms add up to

$$\begin{aligned} A_{11} &= 8(p_+q)(p_-q) \;-\; 4(q^2-m^2)\times(p_+p_-) \\ &= -2(t+m^2)^2 \;-\; 2(t-m^2)\times(s-2m^2 \;=\; -t-u) \\ &= -2t^2 \;-\; 4t\,m^2 \;-\; 2m^4 \;+\; 2t^2 \;+\; 2tu \;-\; 2t\,m^2 - \; 2u\,m^2 \\ &= 2tu \;-6tm^2 \;-\; 2um^2 \;-\; 2m^4 \\ &= 2(t-m^2)\,(u-3m^2) \;-\; 8m^4. \end{aligned} \tag{37}$$

This completes our evaluation of the first trace.

As to the second trace A_{22}, we could work it out through a similar calculation, but fortunately there is a shortcut. The two diagrams (1) for the annihilation process are related to each other by the Bose symmetry between the two final-state photons, so given the $\mathcal{M}_1^{\mu\nu}$ amplitude, the $\mathcal{M}_2^{\mu\nu}$ amplitude obtains by simply exchanging $\mu \leftrightarrow \nu$ and $k_1 \leftrightarrow k_2$. From the Mandelstam's s, t, u point of view, the $k_1 \leftrightarrow k_2$ exchange means $t \leftrightarrow u$, so after summing over all polarizations and spins we get

$$A_{22}(t,u) \;=\; A_{11}(t\leftrightarrow u) \;=\; 2(u-m^2)\,(t-3m^2) \;-\; 8m^4. \tag{38}$$

Traceology 2

Now consider the third trace

$$A_{12} \;=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_-+m)\times\gamma_\nu(\not{\tilde{q}}+m)\gamma_\mu(\not{p}_+-m)\Big) \tag{39}$$

which accounts for the interference between the two diagrams (1). Again, this is a rather formidable trace, but we may simplify it using the relations

$$\gamma^\mu\gamma_\mu \;=\; 4,\quad \gamma^\mu\not{a}\gamma_\mu \;=\; -2\,\not{a},\quad \gamma^\mu\not{a}\not{b}\gamma_\mu \;=\; 4(ab),\quad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu \;=\; -2\,\not{c}\not{b}\not{a}. \tag{24}$$

Indeed, consider the first 5 factors inside the trace A_{12}, from the first γ^ν to the second γ_ν through everything in-between:

$$\begin{aligned}\gamma^\nu\times(\not{q}+m)\gamma^\mu(\not{p}_-+m)\times\gamma_\nu \;&=\; m^2\times\gamma^\nu\gamma^\mu\gamma_\nu \;+\; m\times\gamma^\nu(\not{q}\gamma^\mu+\gamma^\mu\not{p}_-)\gamma_\nu \;+\; \gamma^\nu(\not{q}\gamma^\mu\not{p}_-)\gamma_\nu\\ &\qquad\langle\!\langle\,\text{in light of eqs. (24)}\,\rangle\!\rangle\\ &=\; -2m^2\gamma^\mu \;+\; 4m(q+p_-)^\mu \;-\; 2\,\not{p}_-\gamma^\mu\not{q}\,.\end{aligned} \tag{40}$$

Plugging this formula into eq. (39) for the A_{12}, we obtain

$$\begin{aligned}A_{12} \;&=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_-+m)\gamma_\nu\times(\not{\tilde{q}}+m)\gamma_\mu(\not{p}_+-m)\Big)\\ &=\; \mathrm{Tr}\left(\Big[m(q+p_-)^\mu\;-\;\tfrac{1}{2}\big(m^2\gamma^\mu+\not{p}_-\gamma^\mu\not{q}\big)\Big]\times\Big[m\big(\gamma_\mu\not{p}_+-\not{\tilde{q}}\gamma_\mu\big)+\big(\not{\tilde{q}}\gamma_\mu\not{p}_+\;-\;m^2\gamma_\mu\big)\Big]\right)\\ &\qquad\langle\!\langle\,\text{throwing away products of odd numbers of }\gamma\text{ matrices}\,\rangle\!\rangle\\ &=\; \mathrm{Tr}\Big(m(q+p_-)^\mu\times m\big(\gamma_\mu\,\not{p}_+-\not{\tilde{q}}\gamma_\mu\big)\Big)\;-\;\tfrac{1}{2}\,\mathrm{Tr}\Big(\big(m^2\gamma^\mu+\not{p}_-\gamma^\mu\not{q}\big)\times\big(\not{\tilde{q}}\gamma_\mu\not{p}_+\;-\;m^2\gamma_\mu\big)\Big)\end{aligned} \tag{41}$$

where the two traces on the bottom line evaluate to

$$\begin{aligned}\mathrm{Tr}\Big(m(q+p_-)^\mu\times m\big(\gamma_\mu\,\not{p}_+-\not{\tilde{q}}\gamma_\mu\big)\Big) \;&=\; m^2(q+p_-)^\mu\times\mathrm{Tr}\big((p_+-\tilde{q})\gamma_\mu\big)\\ &=\; m^2(q+p_-)^\mu\times 4(p_+-\tilde{q})_\mu\\ &=\; 4m^2\Big(-(q\tilde{q})\;+\;(qp_+)\;-\;(\tilde{q}p_-)\;+\;(p_-p_+)\Big)\end{aligned} \tag{42}$$

and

$$
\begin{aligned}
&\mathrm{Tr}\Big(\big(m^2\gamma^\mu + \not{p}_-\gamma^\mu\not{q}\big)\times\big(\not{\tilde{q}}\gamma_\mu\not{p}_+ \,-\, m^2\gamma_\mu\big)\Big) \;= \\
&\quad=\; \mathrm{Tr}\big(\not{p}_-\gamma^\mu\not{q}\not{\tilde{q}}\gamma_\mu\not{p}_+\big) \;+\; m^2\,\mathrm{Tr}\big(\gamma^\mu\not{\tilde{q}}\gamma_\mu\not{p}_+\big) \;-\; m^2\,\mathrm{Tr}\big(\not{p}_-\gamma^\mu\not{q}\gamma_\mu\big) \;-m^4\,\mathrm{Tr}\big(\gamma^\mu\gamma_\mu\big) \\
&\qquad\quad \langle\!\langle \text{ using } \gamma^\mu\not{q}\not{\tilde{q}}\gamma_\mu \,=\, 4(q\tilde{q}),\ \gamma^\mu\not{\tilde{q}}\gamma_\mu \,=\, -2\,\not{\tilde{q}},\ \gamma^\mu\not{q}\gamma_\mu \,=\, -2\,\not{q},\ \text{and } \gamma^\mu\gamma_\mu \,=\, 4\,\rangle\!\rangle \qquad (43)\\
&\quad=\; 4(q\tilde{q})\times\mathrm{Tr}(\not{p}_-\,\not{p}_+) \;-\; 2m^2\times\mathrm{Tr}(\not{\tilde{q}}\,\not{p}_+) \;+\; 2m^2\times\mathrm{Tr}(\not{p}_-\,\not{q}) \;-4m^4\times\mathrm{Tr}(1) \\
&\quad=\; 16(q\tilde{q})(p_-p_+) -\; 8m^2(\tilde{q}p_+) \;+\; 8m^2(qp_-) \;-\; 16m^4.
\end{aligned}
$$

Combining the two traces, we arrive at

$$
A_{12} \;=\; -8(q\tilde{q})(p_-p_+) \;+\; 4m^2\big(-(q\tilde{q}) \,+\, (q+\tilde{q})^\mu(p_+-p_-)_\mu \,+\, (p_-p_+)\big) \;+\; 8m^4. \qquad (44)
$$

To further simplify this rather messy formula, we note that

$$
q \;+\; \tilde{q} \;=\; (p_- - k_1) \;+\; (k_1 - p_+) \;=\; p_- \;-\; p_+, \qquad (45)
$$

hence

$$
(q+\tilde{q})^\mu(p_+-p_-)_\mu \;=\; -(p_- - p_+)^2 \;=\; 2(p_-p_+) \;-\; 2m^2 \;=\; s \;-\; 4m^2, \qquad (46)
$$

while

$$
\begin{aligned}
\tilde{q}q \;&=\; (p_- - k_2)(p_- - k_1) \;=\; p_-^2 \;-\; p_-(k_1+k_2 = p_- + p_+) \;+\; k_1k_2 \\
&=\; k_1k_2 \;-\; p_-p_+ \;=\; \tfrac{1}{2}s \;-\; \tfrac{1}{2}(s-2m^2) \;=\; m^2.
\end{aligned} \qquad (47)
$$

Consequently, the RHS of eq. (44) simplifies to

$$
\begin{aligned}
A_{12} \;&=\; -8m^2(\tfrac{1}{2}s - m^2) \;+\; 4m^2\big(-m^2 \,+\, (s-4m^2) \,+\, (\tfrac{1}{2}s - m^2)\big) \;+\; 8m^4 \\
&=\; +m^2s\times(-4+4+2) \;+\; m^4\times(+8-4-16-4+8) \\
&=\; 2m^2s \;-\; 8m^4,
\end{aligned} \qquad (48)
$$

Note the simplicity of this formula, after all the hard work we have done here. However, for combining the A_{12} trace with the other two traces A_{11} and A_{22}, it would be more convenient to re-express eq. (48) in terms of t and u rather than s. Thus, using $s = 2m^2 - t - u = (m^2 - t) + (m^2 - u)$ we write

$$
A_{12} \;=\; -2m^2(t-m^2) \;-\; 2m^2(u-m^2) \;-\; 8m^4. \qquad (49)
$$

Annihilation Summary

Having worked out the big traces, let's plug them back into eq. (20):

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4}{(t-m^2)^2}\times\Big(A_{11} = 2(t-m^2)(u-3m^2) - 8m^4\Big)\\
&\quad + \frac{e^4}{(u-m^2)^2}\times\Big(A_{22} = 2(u-m^2)(t-3m^2) - 8m^4\Big)\\
&\quad + \frac{2e^4}{(t-m^2)(u-m^2)}\times\Big(\mathrm{Re}\,A_{12} = -2m^2(t-m^2) - 2m^2(u-m^2) - 8m^4\Big)\\
&= 2e^4\left[\frac{u-3m^2}{t-m^2} + \frac{t-3m^2}{u-m^2} - \frac{2m^2}{u-m^2} - \frac{2m^2}{t-m^2}\right.\\
&\qquad\left. - \frac{4m^4}{(t-m^2)^2} - \frac{4m^4}{(u-m^2)^2} - \frac{8m^4}{(t-m^2)(u-m^2)}\right]\\
&= 2e^4\left[\frac{u-m^2}{t-m^2} + \frac{t-m^2}{u-m^2} - 4m^2\left(\frac{1}{t-m^2} + \frac{1}{u-m^2}\right)\right.\\
&\qquad\left. - 4m^4\left(\frac{1}{t-m^2} + \frac{1}{u-m^2}\right)^2\right],
\end{aligned} \tag{50}$$

or more compactly

$$\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{u-m^2}{t-m^2} + \frac{t-m^2}{u-m^2} + 1 - \left(1 + \frac{2m^2}{t-m^2} + \frac{2m^2}{u-m^2}\right)^2\right]. \tag{51}$$

This is our final result; the rest is kinematics.

Annihilation Kinematics

In the center of mass frame, $p_\mp^\mu = (E, \pm\mathbf{p})$ where $E = +\sqrt{\mathbf{p}^2 + m^2}$, and $k_{1,2}^\mu = (\omega, \pm\mathbf{k})$ where $\omega = |\mathbf{k}| = E$. Consequently,

$$\begin{aligned}
s &= 4E^2,\\
t &= -(\mathbf{p}-\mathbf{k})^2 = -\mathbf{p}^2 - E^2 + 2|\mathbf{p}|E\cos\theta,\\
u &= -(\mathbf{p}+\mathbf{k})^2 = -\mathbf{p}^2 - E^2 - 2|\mathbf{p}|E\cos\theta,\\
t-m^2 &= -2E^2 + 2|\mathbf{p}|E\cos\theta = -2E(E-|\mathbf{p}|\cos\theta),\\
u-m^2 &= -2E^2 - 2|\mathbf{p}|E\cos\theta = -2E(E+|\mathbf{p}|\cos\theta),
\end{aligned} \tag{52}$$

and hence

$$\frac{u-m^2}{t-m^2}+\frac{t-m^2}{u-m^2}+1 = \frac{E+|\mathbf{p}|\cos\theta}{E-|\mathbf{p}|\cos\theta}+\frac{E-|\mathbf{p}|\cos\theta}{E+|\mathbf{p}|\cos\theta}+1$$
$$= \frac{3E^2+\mathbf{p}^2\cos^2\theta}{E^2-\mathbf{p}^2\cos^2\theta}$$
$$= \frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta}\,, \tag{53}$$

$$\frac{1}{t-m^2}+\frac{1}{u-m^2} = \frac{-1}{2E}\left(\frac{1}{E-|\mathbf{p}|\cos\theta}+\frac{1}{E+|\mathbf{p}|\cos\theta}\right)$$
$$= \frac{-1}{2E}\times\frac{2E}{E^2-\mathbf{p}^2\cos^2\theta} = \frac{-1}{m^2+\mathbf{p}^2\sin^2\theta}\,, \tag{54}$$

$$1+\frac{2m^2}{t-m^2}+\frac{2m^2}{u-m^2} = \frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\,. \tag{55}$$

Plugging these formulae into eq. (51), we get

$$\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right], \tag{56}$$

and hence the partial cross section of annihilation

$$\frac{d\sigma(e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{|\mathbf{k}|}{|\mathbf{p}|}\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{\alpha^2}{8E|\mathbf{p}|}\times\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right]. \tag{57}$$

Now consider the behavior of this partial cross-section in two opposite limits: (1) non-relativistic electron and positron with $|\mathbf{p}|\ll m$, and (2) ultra-relativistic electron and positron with $|\mathbf{p}|\gg m$. In the non-relativistic limit (1), the expression inside square brackets in eq. (57) becomes approimately $3-(-1)^2=2$, hence *isotropic* partial cross section

$$\frac{d\sigma(\text{slow } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4m|\mathbf{p}|}\,. \tag{58}$$

And the total cross section in this limit is

$$\sigma_{\text{tot}}(\text{slow } e^+e^-\to\gamma\gamma) = \frac{4\pi}{2}\times\frac{\alpha^2}{4m|\mathbf{p}|} = \frac{\pi\alpha^2}{2m|\mathbf{p}|}\,, \tag{59}$$

where the total solid angle is $4\pi/2$ because of 2 identical photons in the final state.

In the opposite limit (2) of ultra-relativistic e^- and e^+, the expression inside square brackets in eq. (57) becomes

$$\Big[\cdots\Big] \;\approx\; \frac{3+\cos^2\theta}{\sin^2\theta}\;-\;1\;=\;\frac{2(1+\cos^2\theta)}{\sin^2\theta}\,, \tag{60}$$

which leads to the highly anisotropic partial cross section

$$\frac{d\sigma(\text{fast } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}}\;\approx\;\frac{\alpha^2}{4E^2}\times\frac{1+\cos^2\theta}{\sin^2\theta}\,. \tag{61}$$

Note how this cross-section is strongly peaked in the forward direction $\theta=0$ where one photon continues the electron's motion while the other continues the positron's motion.

According to eq. (61), the total annihilation cross-section

$$\sigma_{\text{tot}}(\text{fast } e^+e^-\to\gamma\gamma)\;=\;2\pi\int\limits_0^{\pi/2}d\theta\,\sin\theta\,\frac{d\sigma}{d\Omega_{\text{cm}}} \tag{62}$$

diverges at small angles, but that's an artefact of the approximation (60) becoming inaccurate at small angles where $\mathbf{p}^2\sin^2\theta\lesssim m^2$. Instead, for small angles we have

$$\Big[\cdots\Big]\;=\;\frac{4\mathbf{p}^2}{m^2+\mathbf{p}^2\theta^2}\;+\;O(1) \tag{63}$$

and consequently

$$\frac{d\sigma(\text{fast } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}}\;\approx\;\frac{\alpha^2}{2E^2}\times\left(\frac{\mathbf{p}^2}{m^2+\mathbf{p}^2\theta^2}\;+\;O(1)\right). \tag{64}$$

This cross-section is strongly peaked in the forward direction, but it does not diverge; instead, it integrates to

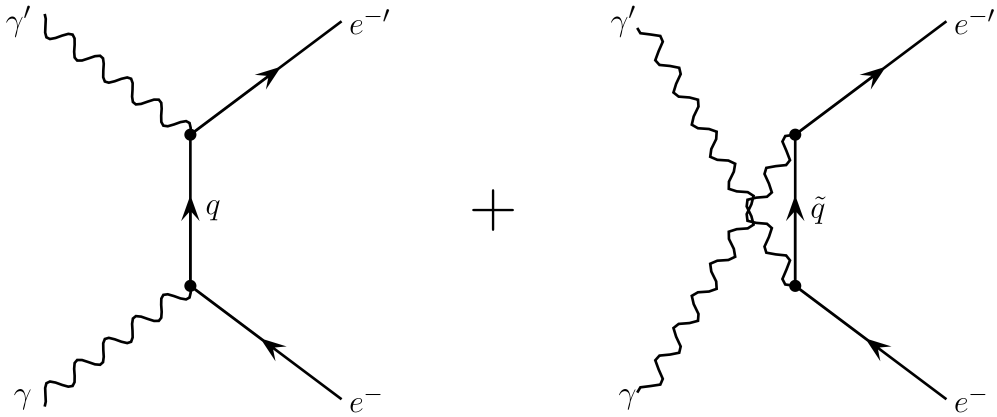
$$\sigma_{\text{tot}}(\text{fast } e^+e^-\to\gamma\gamma)\;=\;\frac{\pi\alpha^2}{E^2}\times\left(\log\frac{E}{m}\;+\;O(1)\right). \tag{65}$$

To get a more accurate formula, we need a better approximation for $d\sigma/d\Omega$ that's valid at both large and small angles θ. But let me skip the algebra and simply give you the result: The total annihilation cross-section for ultra-relativistic electron and positron is

$$\sigma_{\text{tot}}(\text{fast } e^+e^-\to\gamma\gamma)\;=\;\frac{\pi\alpha^2}{E^2}\times\left(\log\frac{2E}{m}\;-\;\frac{1}{2}\;+O(m^2/E^2)\right). \tag{66}$$

Compton Scattering

Compton scattering of an electron and a photon $e^-\gamma \to e^-\gamma$ is related by crossing symmetry to the $e^-e^+ \to \gamma\gamma$ annihilation. Indeed, at the tree level there are two diagrams



(67)

which are obviously related by the $s \leftrightarrow t$ crossing to the annihilation diagrams (1). Hence, given eq. (51) for the annihilation, we may immediately write down a similar formula for the Compton scattering without doing any work. All we need is to exchange $s \leftrightarrow t$ in eq. (51) and change the overall sign because we cross one fermion, thus

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} = 2e^4\left[-\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} - 1 + \left(1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2}\right)^2\right]. \tag{68}$$

This is it, except for the kinematics.

The Compton scattering is usually studied in the lab frame where the initial electron is at rest, $p^\mu = (m, \mathbf{0})$. In this frame, the initial and the final photon energies ω and ω' are related to photon's scattering angle θ via the Compton's formula

$$\frac{1}{\omega'} = \frac{1}{\omega} + \frac{1-\cos\theta}{m_e}, \tag{69}$$

originally written by Arthur Compton in terms of the photon's wavelengths as

$$\lambda' - \lambda = \frac{2\pi\hbar}{m_e c} \times (1-\cos\theta). \tag{70}$$

According to this formula, there is an upper limit on the energy of the final photon for any *fixed* scattering angle $\theta \neq 0$: Regardless of the initial energy ω, the final energy ω' can never exceed $m_e/(1-\cos\theta)$.

The Compton's formula follows from the energy–momentum conservation $p' = p + k - k'$, which leads to

$$p'^2 \;=\; (p + k - k')^2 \;=\; p^2 \;+\; k^2 \;+\; k'^2 \;+\; 2pk \;-\; 2pk' \;-\; 2kk'. \tag{71}$$

In light of the mass-shell conditions $p'^2 = p^2 = m^2$ and $k'^2 = k^2 = 0$, this means

$$2pk \;-\; 2pk' \;-\; 2kk \;=\; 0. \tag{72}$$

In the lab frame $pk = m\omega$, $pk' = m\omega'$, while $kk' = \omega\omega' - \mathbf{k} \cdot \mathbf{k}' = \omega\omega'(1 - \cos\theta)$, so eq. (72) becomes

$$2m\omega \;-\; 2m\omega' \;-\; 2\omega\omega'(1 - \cos\theta) \;=\; 0, \tag{73}$$

and after dividing every term by $2\omega\omega' m$ we get the Compton formula

$$\frac{1}{\omega'} \;-\; \frac{1}{\omega} \;-\; \frac{1 - \cos\theta}{m} \;=\; 0. \tag{69}$$

The Mandelstam variables s and u in the lab frame are

$$\begin{aligned} s \;&\equiv\; (k + p)^2 \;=\; (\omega + m)^2 \;-\; (\mathbf{k} + \mathbf{0})^2 \;=\; 2\omega m \;+\; m^2, \\ u \;&\equiv\; (k' - p)^2 \;=\; (\omega' - m)^2 \;-\; (\mathbf{k}' - \mathbf{0})^2 \;=\; -2\omega' m \;+\; m^2, \end{aligned} \tag{74}$$

and hence

$$s \;-\; m^2 \;=\; +2m\omega, \qquad u \;-\; m^2 \;=\; -2m\omega'. \tag{75}$$

Plugging these values into eq. (68), we have

$$-\frac{u - m^2}{s - m^2} - \frac{s - m^2}{u - m^2} \;=\; +\frac{\omega'}{\omega} + \frac{\omega}{\omega'}\,, \tag{76}$$

$$\frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2} = \frac{m}{\omega} - \frac{m}{\omega'}$$

$$\langle\!\langle \text{ by the Compton formula } \rangle\!\rangle$$

$$= -(1-\cos\theta), \tag{77}$$

$$-1 + \left(1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2}\right)^2 = -1 + \big(1 - 1 + \cos\theta\big)^2$$

$$= -1 + \cos^2\theta = -\sin^2\theta, \tag{78}$$

and therefore

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} = 2e^4 \times \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right). \tag{79}$$

Finally, we need the phase space factor for the lab frame. We may calculate it directly from the general phase space rules — see my notes on the subject, — but there is a shortcut relating the lab-frame and the center-of-mass-frame scattering angles to each other via Mandelstam variables t and s. In the lab frame

$$t = 2m^2 - s - u = 2m\omega' - 2m\omega, \tag{80}$$

hence for fixed ω,

$$dt = 2m \times d\omega'(\theta_{\text{lab}}) = -2m\omega'^2 \times d\left(\frac{1}{\omega'} = \frac{1}{\omega} + \frac{1-\cos\theta_{\text{lab}}}{m}\right) = 2\omega'^2 \times d\cos\theta_{\text{lab}}. \tag{81}$$

On the other hand, in the center-of-mass frame,

$$t = -2\omega_{\text{cm}}^2(1-\cos\theta_{\text{cm}}) \implies dt = 2\omega_{\text{cm}}^2 \times d\cos\theta_{\text{cm}}. \tag{82}$$

Comparing eqs. (81) and (82), we immediately see that

$$\omega_{\text{lab}}'^2 \times d\cos\theta_{\text{lab}} = \omega_{\text{cm}}^2 \times d\cos\theta_{\text{cm}} \tag{83}$$

and consequently

$$\omega_{\text{lab}}'^2 \times d\Omega_{\text{lab}} = \omega_{\text{cm}}^2 \times d\Omega_{\text{cm}}. \tag{84}$$

The ω_{cm} in this formula is the photon's energy in the center-of-mass frame, which obtains from the s invariant as

$$\begin{aligned}\sqrt{s} &= \omega_{\rm cm} + \sqrt{\omega_{\rm cm}^2 + m^2} \\ &\Downarrow \\ \omega_{\rm cm}^2 + m^2 &= (\sqrt{s} - \omega_{\rm cm})^2 = s + \omega_{\rm cm}^2 - 2\sqrt{s}\omega_{\rm cm} \\ &\Downarrow \\ \omega_{\rm cm} &= \frac{s - m^2}{2\sqrt{s}} = \frac{m\omega_{\rm lab}}{\sqrt{s}}.\end{aligned} \tag{85}$$

Plugging this formula into eq. (84), we arrive at

$$\omega_{\rm lab}'^2 \times d\Omega_{\rm lab} = \frac{m^2\omega_{\rm lab}^2}{s} \times d\Omega_{\rm cm} \tag{86}$$

and hence

$$\frac{d\Omega_{\rm cm}}{s} = \left(\frac{\omega'}{m\omega}\right)_{\rm lab}^2 \times d\Omega_{\rm lab}. \tag{87}$$

In the center of mass frame the partial cross-section obtains as

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2} \times \frac{d\Omega_{\rm cm}}{s}. \tag{88}$$

In the lab frame, the cross-section is exactly the same, but it's relation to the lab-frame scattering angle follows from eq. (87),

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2} \times \frac{\omega'^2}{m^2\omega^2} \times d\Omega_{\rm lab}, \tag{89}$$

hence

$$\frac{d\sigma}{d\Omega_{\rm lab}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2} \times \frac{\omega'^2}{m^2\omega^2}. \tag{90}$$

Finally, plugging the mod-squared amplitude (79) into this formula gives us the *Klein–Nishina formula*:

$$\frac{d\sigma^{\rm Compton}}{d\Omega_{\rm lab}} = \frac{\alpha^2}{2m_e^2} \times \frac{\omega'^2}{\omega^2} \times \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right) \tag{91}$$

where ω' is given by the Compton formula (69).

For low photon energies $\omega \ll m_e$, the Compton's formula gives $\omega' \approx \omega$, and the Klein–Nishina cross-section (91) becomes the good old Thompson cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \rightarrow \frac{d\sigma^{\text{Thompson}}}{d\Omega_{\text{lab}}} = \frac{\alpha^2}{2m_e^2} \times (2 - \sin^2\theta \;=\; 1 + \cos^2\theta), \tag{92}$$

and the total cross-section is

$$\sigma_{\text{total}}^{\text{Thompson}} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.663\,\text{barn}. \tag{93}$$

On the other hand, for very high photon energies $\omega \gg m_e$ and $\theta \not\approx 0$, we have

$$\omega' \ll \omega \quad \Longrightarrow \quad \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \approx \frac{\omega}{\omega'}\,, \tag{94}$$

and the Klein–Nishina formula becomes

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \approx \frac{\alpha^2}{2m_e^2} \times \frac{\omega'}{\omega} \approx \frac{\alpha^2}{2m_e \times \omega} \times \frac{1}{1 - \cos\theta}\,. \tag{95}$$

This approximation is not accurate at small angles $\theta \lesssim \sqrt{2m_e/\omega}$ for which $\omega' \not\ll \omega$, so the cross section does not really diverge for $\theta \to 0$. Instead, at small angles we have large but finite partial cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \approx \frac{\alpha^2}{m_e \times \omega} \times \frac{\theta^4 - 2\theta^2(2m_e/\omega) + 2(2m_e/\omega)^2}{(\theta^2 + (2m_e/\omega))^3} \not\to \infty \tag{96}$$

and hence finite total cross-section

$$\sigma_{\text{total}}^{\text{Compton}} \approx \frac{\pi\alpha^2}{m_e \times \omega} \times \left(\log\frac{2\omega}{m_e} + \frac{1}{2}\right). \tag{97}$$