1. First, an exercise about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator with Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

   (a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\text{def}}{=} \exp\bigl(\xi\hat{a}^\dagger - \xi^*\hat{a}\bigr)\,|0\rangle$. Show that

   $$|\xi\rangle \;=\; e^{-|\xi|^2/2}\, e^{\xi\hat{a}^\dagger}\,|0\rangle \quad\text{and}\quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{1}$$

   (b) Use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ to show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a Gaussian wave packet of the same width as the ground state $|0\rangle$. Also, show that the central position $\bar{x}$ and the central momentum $\bar{p}$ of this packet are related to the real and the imaginary parts of ξ,

   $$\bar{x} \;=\; \sqrt{\frac{2\hbar}{\omega m}}\times\operatorname{Re}\xi,\quad \bar{p} \;=\; \sqrt{2\hbar\omega m}\times\operatorname{Im}\xi,\quad \xi \;=\; \frac{m\omega\bar{x}\,+\,i\bar{p}}{\sqrt{2m\omega\hbar}}\,. \tag{2}$$

   (c) Use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$ to calculate the uncertainties Δx and Δp in a coherent state and verify their minimality: $\Delta x\Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle\hat{n}\rangle = |\xi|^2$.

The coherent states are not stationary, they evolve with time but stay coherent — a coherent state $|\xi_0\rangle$ at time $t=0$ becomes $|\xi(t)\rangle$ at later times — while the central position $\bar{x}$ and $\bar{p}$ of the wave packet moves according to the classical equations of motion for the harmonic oscillator.

   (d) Check that such classical motion calls for $\xi(t) = \xi_0\times e^{-i\omega t}$, then check that the corresponding coherent state $|\xi(t)\rangle$ obeys the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}\,|\xi(t)\rangle = \hat{H}\,|\xi(t)\rangle$.

   (e) The coherent states are not quite orthogonal to each other.
   Calculate their probability overlaps $|\langle\eta|\xi\rangle|^2$.

2. Now consider the coherent states of multi-oscillator systems such as quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

   (a) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

   $$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{3}$$

   for any given classical complex field $\Phi(\mathbf{x})$.

   (b) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

   $$\bar{N} \stackrel{\text{def}}{=} \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int\! d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{4}$$

   (c) Let the Hamiltonian for the quantum non-relativistic fields be

   $$\hat{H} \;=\; \int\! d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger(\mathbf{x})\cdot\nabla\hat{\Psi}(\mathbf{x}) \;+\; V(\mathbf{x})\times\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x})\right). \tag{5}$$

   Show that for any classical field configuration $\Phi(\mathbf{x},t)$ obeying the classical field equation

   $$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t), \tag{6}$$

   the time-dependent coherent state $|\Phi\rangle\,(t)$ satisfies the true Schrödinger equation

   $$i\hbar\frac{d}{dt}\,|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{7}$$

   (d) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.

3. And finally a reading assignment: my notes of the Bose–Einstein condensation, especially pages 16-25 where I prove the lemmas.