

The first problem of this set continues the story of the massive vector field. The other two problems (2) and (3) are about the $SO(N)$ symmetry of the quantum theory of N scalar fields.

1. Consider the massive vector field $\hat{A}^\mu(x)$ from all the previous homework sets 1–3. In particular, in problem 1 from the [last homework#3](#) you (should have) expanded the free vector field into creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \left(e^{-ikx} \times f_{\mathbf{k},\lambda}^\mu \times \hat{a}_{\mathbf{k},\lambda} + e^{+ikx} \times f_{\mathbf{k},\lambda}^{*\mu} \times \hat{a}_{\mathbf{k},\lambda}^\dagger \right)^{k^0 = +\omega_{\mathbf{k}}}. \quad (1)$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda = -1, 0, +1$), while $f_{\mathbf{k},\lambda}^\mu$ are the polarization vectors obeying $k_\mu f^\mu(\mathbf{k}, \lambda) = 0$. Specifically, in the helicity basis

$$\begin{aligned} \text{for } \lambda = \pm 1 : \quad & f_{\mathbf{k},\lambda}^0 = 0, \quad \mathbf{f}_{\mathbf{k},\lambda} = \mathbf{e}_\lambda(\mathbf{k}), \\ \text{for } \lambda = 0 : \quad & f_{\mathbf{k},\lambda}^0 = \frac{|\mathbf{k}|}{m}, \quad \mathbf{f}_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}}}{m} \frac{\mathbf{k}}{|\mathbf{k}|}. \end{aligned} \quad (2)$$

In this problem, we are going to calculate the Feynman propagator for the massive vector field (1).

- (a) First, a lemma about the polarization 4-vectors (2). Show that these 4-vectors obtain by Lorentz boosting of the purely-spatial vectors $(0, \mathbf{e}_\lambda(\mathbf{k}))$ into the frame of the wave moving with the velocity $\mathbf{v} = \mathbf{k}/\omega_{\mathbf{k}}$,

$$f_{\mathbf{k},\lambda}^\mu = B^\mu_\nu(\mathbf{v})(0, \mathbf{e}_\lambda(\mathbf{k}))^\nu. \quad (3)$$

Also, verify that the $f_{\mathbf{k},\lambda}^\mu$ are normalized to

$$\langle\langle \text{for the same } \mathbf{k} \rangle\rangle : \quad g_{\mu\nu} f_{\mathbf{k},\lambda}^\mu f_{\mathbf{k},\lambda'}^{*\nu} = -\delta_{\lambda,\lambda'}. \quad (4)$$

(b) Next, another lemma: show that

$$\sum_{\lambda} f_{\mathbf{k},\lambda}^{\mu} f_{\mathbf{k},\lambda}^{*\nu} = -g^{\mu\nu} + \frac{k^{\mu} k^{\nu}}{m^2}. \quad (5)$$

(c) Now use these lemmas to calculate the “vacuum sandwich” of two vector fields (1) and show that

$$\begin{aligned} \langle 0 | \hat{A}^{\mu}(x) \hat{A}^{\nu}(y) | 0 \rangle &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left[\left(-g^{\mu\nu} + \frac{k^{\mu} k^{\nu}}{m^2} \right) e^{-ik(x-y)} \right]_{k^0 = +\omega_{\mathbf{k}}} \\ &= \left(-g^{\mu\nu} - \frac{\partial^{\mu} \partial^{\nu}}{m^2} \right) D(x-y) \end{aligned} \quad (6)$$

where

$$D(x-y) \stackrel{\text{def}}{=} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left[e^{-ik(x-y)} \right]_{k^0 = +\omega_{\mathbf{k}}}. \quad (7)$$

Please note: no time ordering in the “vacuum sandwich” (6).

(d) Next, consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\text{scalar}}(x-y)$. Show that

$$\left(-g^{\mu\nu} - \frac{\partial^{\mu} \partial^{\nu}}{m^2} \right) G_F^{\text{scalar}}(x-y) = \langle 0 | \mathbf{T} \hat{A}^{\mu}(x) \hat{A}^{\nu}(y) | 0 \rangle + \frac{i}{m^2} \delta^{\mu 0} \delta^{\nu 0} \delta^{(4)}(x-y). \quad (8)$$

To avoid the δ -function singularity in formulae like (8), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified*[★] according to

$$\mathbf{T}^* \hat{A}^{\mu}(x) \hat{A}^{\nu}(y) = \mathbf{T} \hat{A}^{\mu}(x) \hat{A}^{\nu}(y) + \frac{i}{m^2} \delta^{\mu 0} \delta^{\nu 0} \delta^{(4)}(x-y). \quad (9)$$

Consequently, the Feynman propagator for the massive vector field is defined using the

★ See *Quantum Field Theory* by Claude Itzykson and Jean-Bernard Zuber.

modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \stackrel{\text{def}}{=} \langle 0 | \mathbf{T}^* \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle \quad (10)$$

(e) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) = \int \frac{d^4 \mathbf{k}}{(2\pi)^4} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \times \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i0}. \quad (11)$$

The classical action for the free vector field can be written as

$$S = \frac{1}{2} \int d^4 x A_\mu(x) \mathcal{D}^{\mu\nu} A_\nu(x) \quad (12)$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator

$$\mathcal{D}^{\mu\nu} \stackrel{\text{def}}{=} (\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu. \quad (13)$$

(f) Check that the action (12) is correct, then show that the Feynman propagator (11) is a Green's function of the operator (13),

$$\mathcal{D}_x^{\mu\nu} G_{\nu\lambda}^F(x-y) = +i \delta_\lambda^\mu \delta^{(4)}(x-y). \quad (14)$$

2. Now let's change the subject and consider N interacting real scalar fields $\Phi_1, \dots, \Phi_N$ with the $O(N)$ symmetric Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^N (\partial_\mu \Phi_a)^2 - \frac{m^2}{2} \sum_{a=1}^N \Phi_a^2 - \frac{\lambda}{24} \left(\sum_{a=1}^N \Phi_a^2 \right)^2. \quad (15)$$

Next week, we shall learn the Noether theorem — here is the [link for the impatient](#), — according to which the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J_{ab}^\mu(x) = -J_{ba}^\mu(x) = \Phi_a(x) \partial^\mu \Phi_b(x) - \Phi_b(x) \partial^\mu \Phi_a(x). \quad (16)$$

(a) Verify that all these currents are conserved, $\partial_\mu J_{ab}^\mu = 0$, when the fields $\Phi^c(x)$ obey their classical equations of motion.

In the quantum field theory, the currents (16) become operators

$$\begin{aligned}\hat{\mathbf{J}}_{ab}(\mathbf{x}, t) &= -\hat{\mathbf{J}}_{ba}(\mathbf{x}, t) = -\hat{\Phi}_a(\mathbf{x}, t)\nabla\hat{\Phi}_b(\mathbf{x}, t) + \hat{\Phi}_b(\mathbf{x}, t)\nabla\hat{\Phi}_a(\mathbf{x}, t), \\ \hat{J}_{ab}^0(\mathbf{x}, t) &= -\hat{J}_{ba}^0(\mathbf{x}, t) = \hat{\Phi}_a(\mathbf{x}, t)\hat{\Pi}_b(\mathbf{x}, t) - \hat{\Phi}_b(\mathbf{x}, t)\hat{\Pi}_a(\mathbf{x}, t),\end{aligned}\tag{17}$$

and the rest of this problem is about the net charge operators

$$\hat{Q}_{ab}(t) = -\hat{Q}_{ba}(t) = \int d^3\mathbf{x} \hat{J}_{ab}^0(\mathbf{x}, t) = \int d^3\mathbf{x} \left(\hat{\Phi}_a(\mathbf{x}, t)\hat{\Pi}_b(\mathbf{x}, t) - \hat{\Phi}_b(\mathbf{x}, t)\hat{\Pi}_a(\mathbf{x}, t) \right).\tag{18}$$

- (b) Write down the equal-time commutation relations for the quantum $\hat{\Phi}_a$ and $\hat{\Pi}_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.
- (c) Show that

$$\begin{aligned}\left[\hat{Q}_{ab}(t), \hat{\Phi}_c(\mathbf{x}, \text{same } t) \right] &= -i\delta_{bc}\hat{\Phi}_a(\mathbf{x}, t) + i\delta_{ac}\hat{\Phi}_b(\mathbf{x}, t), \\ \left[\hat{Q}_{ab}(t), \hat{\Pi}_c(\mathbf{x}, \text{same } t) \right] &= -i\delta_{bc}\hat{\Pi}_a(\mathbf{x}, t) + i\delta_{ac}\hat{\Pi}_b(\mathbf{x}, t),\end{aligned}\tag{19}$$

- (d) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.
- (e) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab}, \hat{Q}_{cd} \right] = -i\delta_{[c[b}\hat{Q}_{a]d]} \equiv -i\delta_{bc}\hat{Q}_{ad} + i\delta_{ac}\hat{Q}_{bd} + i\delta_{bd}\hat{Q}_{ac} - i\delta_{ad}\hat{Q}_{bc}.\tag{20}$$

3. Continuing the previous problem, let's turn off the interactions (*i.e.*, take $\lambda = 0$) and focus on the free fields.

- (a) Expand all the fields into linear combinations of the creation and annihilation operators $\hat{a}_{\mathbf{p},a}^\dagger$ and $\hat{a}_{\mathbf{p},a}$ ($a = 1, \dots, N$), then show that in terms of these operators the charges (18) become

$$\hat{Q}_{ab} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(-i\hat{a}_{\mathbf{p},a}^\dagger \hat{a}_{\mathbf{p},b} + i\hat{a}_{\mathbf{p},b}^\dagger \hat{a}_{\mathbf{p},a} \right).\tag{21}$$

For $N = 2$, the two real scalar fields combine into one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$, while the $SO(2)$ symmetry becomes the $U(1)$ phase

symmetry

$$\Phi(x) \rightarrow e^{-i\theta}\Phi(x), \quad \Phi^*(x) \rightarrow e^{+i\theta}\Phi^*(x). \quad (22)$$

In the Fock space, the corresponding quantum fields $\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$ give rise to particles and anti-particles of opposite charges; the creation and annihilation operators for such particles and antiparticles are

$$\begin{aligned} \hat{a}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle annihilation operators,} \\ \hat{b}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle annihilation operators,} \\ \hat{a}_{\mathbf{p}}^\dagger &= \frac{\hat{a}_{\mathbf{p},1}^\dagger - i\hat{a}_{\mathbf{p},2}^\dagger}{\sqrt{2}} && \text{are particle creation operators,} \\ \hat{b}_{\mathbf{p}}^\dagger &= \frac{\hat{a}_{\mathbf{p},1}^\dagger + i\hat{a}_{\mathbf{p},2}^\dagger}{\sqrt{2}} && \text{are antiparticle creation operators.} \end{aligned} \quad (23)$$

(b) Show that in terms of the operators (23),

$$\hat{Q}_{21} = -\hat{Q}_{12} = \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right). \quad (24)$$

(c) In terms of $\hat{\Phi}$ and $\hat{\Phi}^\dagger$, the commutation relations (19) become

$$[\hat{Q}_{21}, \hat{\Phi}(x)] = -\hat{\Phi}(x), \quad [\hat{Q}_{21}, \hat{\Phi}^\dagger(x)] = +\hat{\Phi}^\dagger(x). \quad (25)$$

Verify these commutators, then use the Campbell identity

$$\begin{aligned} e^{\hat{A}} \hat{B} e^{-\hat{A}} &= \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{A}, \dots, [\hat{A}, \hat{B}] \dots]_{n \text{ times}} \\ &= \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{6} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \end{aligned} \quad (26)$$

to show that the charge $\hat{Q}_{21}$ *generates* the phase symmetry (22) according to

$$\begin{aligned} \exp(+i\theta\hat{Q}_{21}) \hat{\Phi}(x) \exp(-i\theta\hat{Q}_{21}) &= e^{-i\theta} \hat{\Phi}(x), \\ \exp(+i\theta\hat{Q}_{21}) \hat{\Phi}^\dagger(x) \exp(-i\theta\hat{Q}_{21}) &= e^{+i\theta} \hat{\Phi}^\dagger(x). \end{aligned} \quad (27)$$

Now let's go back to $N > 2$ and show that the charges $\hat{Q}_{ab}$ generate the $SO(N)$ symmetry of the quantum fields. Any $SO(N)$ rotation matrix R can be written as a matrix exponential

of an antisymmetric matrix, $R = \exp(A)$ for $A^\top = -A$. For this matrix A , let's define a unitary operator in the Fock space

$$\hat{U} = \exp\left(-\frac{i}{2} \sum_{ab} A_{ab} \hat{Q}_{ab}\right). \quad (28)$$

(d) Verify that this operator is indeed unitary for any real antisymmetric matrix A .

Hint: check and use the hermiticity of the generators $\hat{Q}_{ab}$.

(e) Show that $\hat{U}$ implements the $SO(N)$ rotation R in the scalar field space,

$$\hat{U} \hat{\Phi}_a(x) \hat{U}^\dagger = \sum_b R_{ab} \hat{\Phi}_b. \quad (29)$$

Hint: use the commutation relations (19) and the Campbell identity (26).

(f) Argue that $[\hat{Q}_{ab}, \hat{H}] = 0$ and eq. (29) for the action of the $\hat{U}$ symmetries on the quantum fields together imply similar transformation laws for the creation and the annihilation operators

$$\hat{U} \hat{a}_{\mathbf{p},a} \hat{U}^\dagger = \sum_b R_{ab} \hat{a}_{\mathbf{p},b} \quad \text{and} \quad \hat{U} \hat{a}_{\mathbf{p},a}^\dagger \hat{U}^\dagger = \sum_b R_{ab} \hat{a}_{\mathbf{p},b}^\dagger. \quad (30)$$

(g) Finally, show that when $\hat{U}$ acts on a multi-particle state, it rotates the species index of each particle by R ,

$$\hat{U} |n : (\mathbf{p}_1, a_1), \dots, (\mathbf{p}_n, a_n)\rangle = \sum_{b_1, \dots, b_n} R_{a_1, b_1} \cdots R_{a_n, b_n} |n : (\mathbf{p}_1, b_1), \dots, (\mathbf{p}_n, b_n)\rangle. \quad (31)$$

Note: for simplicity assume that all particles have different momenta, $\mathbf{p}_1 \neq \mathbf{p}_2$, etc., then use part (f).