

This homework has 4 problems: Problems 1 and 2 are about the stress-energy tensor of the electromagnetic fields, while problems 3 and 4 are about the non-abelian $SU(N)$ gauge theories and their adjoint multiplets. Altogether, it's a pretty large homework set, so start working early.

1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L}. \quad (1)$$

For the scalar fields, real or complex, this Noether stress-energy tensor is properly symmetric, $T_{\text{Noether}}^{\mu\nu} = T_{\text{Noether}}^{\nu\mu}$. But for the vector, tensor, spinor, *etc.*, fields, the Noether stress-energy tensor (1) comes out asymmetric, so to make it properly symmetric one adds a total-divergence term of the form

$$T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu\nu}, \quad (2)$$

where $\mathcal{K}^{\lambda\mu\nu} \equiv -\mathcal{K}^{\mu\lambda\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (3)$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

- (a) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

- (b) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \quad (4)$$

Show that this expression indeed has form (2) for

$$\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu}A^\nu = -\mathcal{K}^{\mu\lambda,\nu}. \quad (5)$$

- (c) Write down the components of the stress-energy tensor (4) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^μ of some charged “matter” fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P_{EM}^μ and P_{mat}^μ . Consequently, we should have

$$\partial_\mu T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \quad (6)$$

but generally $\partial_\mu T_{\text{EM}}^{\mu\nu} \neq 0$ and $\partial_\mu T_{\text{mat}}^{\mu\nu} \neq 0$.

- (d) Use Maxwell’s equations to show that

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -F^{\nu\lambda}J_\lambda \quad (7)$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda}J_\lambda. \quad (8)$$

- (e) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ , Φ , and Φ^* fields is

$$\mathcal{L}_{\text{net}} = D^\mu \Phi^* D_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (9)$$

where

$$D_\mu \Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu \Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \quad (10)$$

are the *covariant* derivatives.

- (a) Write down the equation of motion for all fields in a covariant form. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial \mathcal{L}}{\partial A_\mu} \quad (11)$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

- (b) Write down the Noether stress-energy tensor for the whole system and show that

$$T_{\text{net}}^{\mu\nu} \equiv T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu\nu}, \quad (12)$$

where $T_{\text{EM}}^{\mu\nu}$ is exactly as in eq. (4) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in eq. (5), and

$$T_{\text{mat}}^{\mu\nu} = D^\mu \Phi^* D^\nu \Phi + D^\nu \Phi^* D^\mu \Phi - g^{\mu\nu} (D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi). \quad (13)$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM+matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda \mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

- (c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi = iqF_{\mu\nu}\Phi, \quad [D_\mu, D_\nu]\Phi^* = -iqF_{\mu\nu}\Phi^* \quad (14)$$

to show that

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda}J_\lambda \quad (15)$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem 1(d).

3. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N -vector,

$$\Psi'(x) = U(x)\Psi(x) \quad \text{i.e.} \quad \Psi^i(x) = \sum_j U_j^i(x)\Psi^j(x), \quad i, j = 1, 2, \dots, N \quad (16)$$

where $U(x)$ is an x -dependent unitary $N \times N$ matrix, $\det U(x) \equiv 1$. Now consider $N^2 - 1$ real fields $\Phi^a(x)$ forming an *adjoint multiplet*. In matrix form

$$\Phi(x) = \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \quad (17)$$

is a traceless hermitian $N \times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) = U(x)\Phi(x)U^\dagger(x). \quad (18)$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and the $\text{tr}(\Phi) = 0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet according to

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) + i[\mathcal{A}_\mu(x), \Phi(x)] \equiv \partial_\mu \Phi(x) + i\mathcal{A}_\mu(x)\Phi(x) - i\Phi(x)\mathcal{A}_\mu(x), \quad (19)$$

or in components

$$D_\mu \Phi^a(x) = \partial_\mu \Phi^a(x) - f^{abc}\mathcal{A}_\mu^b(x)\Phi^c(x). \quad (20)$$

- (a) Verify that these derivatives are indeed covariant under the finite gauge transforms (18).

- (b) Verify the Leibniz rule for the covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{21}$$

- (c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{22}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F_{\mu\nu}^b(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are the canonically normalized potential and tension fields, while $\mathcal{A}_\mu = gA_\mu$ is the connection in the covariant derivative and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ is the curvature of that connection.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (18). In other words, the $\mathcal{F}_{\mu\nu}^a(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

- (d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

4. In this problem we apply what we learned in problem 3 to the field equations of gauge theories. Let's start with the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu}. \tag{23}$$

- (a) Verify the covariant differential identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{24}$$

Note: the $\mathcal{F}_{\mu\nu}$ form an adjoint multiplet, so their covariant derivatives act as in eq. (19).

(b) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu \mathcal{F}^{\mu\nu} = 0$.

Hint: first show that for an infinitesimal variation $\delta \mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta \mathcal{F}_{\mu\nu}(x) = D_\mu \delta \mathcal{A}_\nu(x) - D_\nu \delta \mathcal{A}_\mu(x)$.

Next, consider an $SU(N)$ gauge theory in which $N^2 - 1$ vector fields $A_\mu^a(x)$ interact with some “matter” fields $\phi_\alpha(x)$,

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) + \mathcal{L}_{\text{mat}}(\phi, D_\mu \phi). \quad (25)$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu \phi$ that depend on the gauge fields A_μ^a , which give rise to the currents

$$J^{a\mu} = -\frac{\partial \mathcal{L}_{\text{mat}}}{\partial A_\mu^a} = -\sum_\phi \frac{\partial \mathcal{L}_{\text{mat}}}{\partial (D_\mu \phi)} \times ig \hat{T}^a \phi. \quad (26)$$

Collectively, these $N^2 - 1$ currents should form an adjoint multiplet of the local $SU(N)$ symmetry, meaning

$$J_\mu(x) \stackrel{\text{def}}{=} \sum_a J_\mu^a(x) \frac{\lambda^a}{2} \quad \text{transforms to} \quad J'_\mu(x) = U(x) J_\mu(x) U^\dagger(x), \quad (27)$$

or equivalently

$$J_\mu'^a(x) = \sum_b R_{\text{adj}}^{ab}(U(x)) \times J_\mu^b(x) \quad \text{for} \quad R_{\text{adj}}^{ab}(U) = \frac{1}{2} \text{tr}(\lambda^a U \lambda^b U^\dagger). \quad (28)$$

(c) Show that in this theory, the equation of motion for the A_μ^a fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires the currents to be *covariantly conserved*,

$$D_\mu J^\mu = \partial_\mu J^\mu + i[\mathcal{A}_\mu, J^\mu] = 0, \quad (29)$$

or in components, $\partial_\mu J^{a\mu} - f^{abc} \mathcal{A}_\mu^b J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge,

$$\frac{d}{dt} \int d^3\mathbf{x} J^{a0}(\mathbf{x}, t) \neq 0. \quad (30)$$

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\text{mat}} = D_\mu \Psi^\dagger D^\mu \Psi - m^2 \Psi^\dagger \Psi - \frac{\lambda}{4} (\Psi^\dagger \Psi)^2, \quad \mathcal{L}_{\text{net}} = \mathcal{L}_{\text{mat}} - \frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}), \quad (31)$$

and hence the $SU(N)$ currents are

$$J^{a\mu} = -g \text{Im} \left(\Psi^\dagger \lambda^a D^\mu \Psi \right) = -\frac{ig}{2} (D^\mu \Psi^\dagger) \lambda^a \Psi + \frac{ig}{2} \Psi^\dagger \lambda^a D^\mu \Psi. \quad (32)$$

- (d) Derive these $SU(N)$ currents, then verify that they do transform into each other as members of an adjoint multiplet of the gauge symmetry. In other words, they transform according to eqs. (27) and (28).

Hint: by definition of the $R_{\text{adj}}^{ab}(U)$ in eq. (28),

$$U \lambda^a U^\dagger = \sum_b R_{\text{adj}}^{ba}(U) \times \lambda^b, \quad U^\dagger \lambda^a U = \sum_b R_{\text{adj}}^{ab}(U) \times \lambda^b. \quad (33)$$

- (e) Finally, verify the covariant conservation $D_\mu J^{a\mu} = 0$ of the currents (32) when the scalar fields $\Psi^i(x)$ and $\Psi_i^\dagger(x)$ obey their equations of motion.