

1. Consider the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

(a) In 4D terms, the commutation relations of the Lorentz generators are

$$[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \quad (1)$$

Show that in 3D terms, these relations become

$$[\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{J}^k, \quad [\hat{J}^i, \hat{K}^j] = i\epsilon^{ijk}\hat{K}^k, \quad [\hat{K}^i, \hat{K}^j] = -i\epsilon^{ijk}\hat{J}^k. \quad (2)$$

The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$[\hat{V}^\lambda, \hat{J}^{\mu\nu}] = ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \quad (3)$$

- (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.
- (c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian operator of a QFT or a quantum system of several particles.

Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$, or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

2. Next, consider the little group $G(p)$ of Lorentz symmetries preserving some momentum 4-vector p^μ . For the moment, allow the p^μ to be time-like, light-like, or even space-like — anything goes as long as $p \neq 0$.

(a) Show that the little group $G(p)$ is generated by the 3 components of the vector

$$\hat{\mathbf{R}} = p^0 \hat{\mathbf{J}} + \mathbf{p} \times \hat{\mathbf{K}} \quad (4)$$

after a suitable component-by-component rescaling.

Suppose the momentum p^μ belongs to a massive particle, thus $p^\mu p_\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β , thus $p^\mu = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta \gamma m$. In this case, the properly normalized generators of the little group $G(p)$ are the

$$\begin{aligned} \tilde{J}^x &= \frac{1}{m} \hat{R}^x = \gamma \hat{J}^x - \beta \gamma \hat{K}^y, \\ \tilde{J}^y &= \frac{1}{m} \hat{R}^y = \gamma \hat{J}^y + \beta \gamma \hat{K}^x, \\ \tilde{J}^z &= \frac{1}{\gamma m} \hat{R}^z = \hat{J}^z, \quad \text{the helicity.} \end{aligned} \quad (5)$$

- (b) Show that these generators have angular-momentum-like commutators with each other, $[\tilde{J}^i, \tilde{J}^j] = i\epsilon^{ijk} \tilde{J}^k$. Consequently, the little group $G(p)$ is isomorphic to the rotation group $SO(3)$.

Now suppose the momentum p^μ belongs to a massless particle, $p^\mu p_\mu = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^\mu = (E, 0, 0, E)$. In this case, we cannot normalize the generators of the little group as in eq. (5); instead, let's normalize them according to

$$\hat{\mathbf{I}} = \frac{1}{E} \hat{\mathbf{R}} = \hat{\mathbf{J}} + \vec{\beta} \times \hat{\mathbf{K}}, \quad (6)$$

or in components,

$$\hat{I}^x = \hat{J}^x - \hat{K}^y, \quad \hat{I}^y = \hat{J}^y + \hat{K}^x, \quad \hat{I}^z = \hat{J}^z. \quad (7)$$

- (c) Show that these generators obey similar commutation relations to the $\hat{p}^x$, $\hat{p}^y$, and $\hat{J}^z$

operators, namely

$$[\hat{J}^z, \hat{I}^x] = +i\hat{I}^y, \quad [\hat{J}^z, \hat{I}^y] = -i\hat{I}^x, \quad [\hat{I}^x, \hat{I}^y] = 0. \quad (8)$$

Consequently, the little group $G(p)$ is isomorphic to the $\text{ISO}(2)$ group of *rotations and translations* in the xy plane.

- (d) Finally, show that for a tachyonic momentum with $p^\mu p_\mu < 0$, the properly normalized generators of the little group have similar commutation relations to the $\hat{K}^x$, $\hat{K}^y$, and $\hat{J}^z$ operators. Consequently, the little group $G(p)$ is isomorphic to the $SO^+(2, 1)$, the continuous Lorentz group in $2 + 1$ spacetime dimensions.

3. Now let's focus on the little group of a *lightlike* momentum p^μ . As I explained in class, the finite unitary representations of this little group are singlets of definite helicity, specifically the states $|p, \lambda\rangle$ obeying

$$\hat{I}_\parallel |p, \lambda\rangle = \lambda |p, \lambda\rangle \quad \text{and} \quad \hat{\mathbf{I}}_\perp |p, \lambda\rangle = 0. \quad (9)$$

- (a) Show that in 4D terms, the conditions (9) amount to

$$\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma |p, \lambda\rangle = \lambda\hat{P}_\mu |p, \lambda\rangle. \quad (10)$$

- (b) Use eq. (10) to show that the *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \text{SO}^+(3, 1), \quad \hat{\mathcal{D}}(L) |p, \lambda\rangle = |Lp, \text{same } \lambda\rangle \times e^{i \text{phase}}. \quad (11)$$

4. The $\text{Spin}(3, 1)$ group — the double cover of the continuous Lorentz group $SO^+(3, 1)$ — is isomorphic to $SL(2, \mathbf{C})$, the group of complex (but not necessarily unitary) 2×2 matrices of unit determinant. The relations between such matrices and the Lorentz symmetries are explained in [my notes on Lorentz representations for the fields](#), but some technical details were left out as exercises for the students. This problem collects these exercises.

(a) Show that the components of the two 3-vectors

$$\hat{\mathbf{J}}_+ = \frac{1}{2}(\hat{\mathbf{J}} + i\hat{\mathbf{K}}) \quad \text{and} \quad \hat{\mathbf{J}}_- = \frac{1}{2}(\hat{\mathbf{J}} - i\hat{\mathbf{K}}) = \hat{\mathbf{J}}_+^\dagger. \quad (12)$$

obey commutation relations

$$[\hat{J}_+^i, \hat{J}_+^j] = i\epsilon^{ijk} \hat{J}_+^k, \quad [\hat{J}_-^i, \hat{J}_-^j] = i\epsilon^{ijk} \hat{J}_-^k, \quad \text{but} \quad [\hat{J}_+^i, \hat{J}_-^j] = 0. \quad (13)$$

Now consider the $\mathbf{2}$ ($j_+ = \frac{1}{2}, j_- = 0$) and the $\bar{\mathbf{2}}$ ($j_+ = 0, j_- = \frac{1}{2}$) representations of the Lorentz or rather $\text{Spin}(3, 1)$ group. In the $\mathbf{2}$ representation $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{K} = -\frac{i}{2}\boldsymbol{\sigma}$ while in the $\bar{\mathbf{2}}$ representation $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{K} = +\frac{i}{2}\boldsymbol{\sigma}$, hence for a 3-space rotation R through angle ϕ around axis $\mathbf{n}$

$$M_{\mathbf{2}}(R) = M_{\bar{\mathbf{2}}} = \exp\left(-\frac{i}{2}\phi \mathbf{n} \cdot \boldsymbol{\sigma}\right) \quad (14)$$

while for a pure Lorentz boost B of rapidity r in the direction $\mathbf{n}$

$$M_{\mathbf{2}}(B) = \exp\left(-\frac{1}{2}r \mathbf{n} \cdot \boldsymbol{\sigma}\right), \quad M_{\bar{\mathbf{2}}}(B) = \exp\left(+\frac{1}{2}r \mathbf{n} \cdot \boldsymbol{\sigma}\right). \quad (15)$$

(b) The more familiar β and γ parameters of a boost are related to its rapidity r as

$$\beta = \tanh(r), \quad \gamma = \cosh(r), \quad \beta\gamma = \sinh(r). \quad (16)$$

Show that in terms of these parameters, eqs. (15) translate to

$$M_{\mathbf{2}}(B) = \sqrt{\gamma} \times \sqrt{1 - \beta \mathbf{n} \cdot \boldsymbol{\sigma}}, \quad M_{\bar{\mathbf{2}}}(B) = \sqrt{\gamma} \times \sqrt{1 + \beta \mathbf{n} \cdot \boldsymbol{\sigma}}. \quad (17)$$

Later in class, we shall need these formulae to find the plane-wave solutions of the Dirac equation.

- (c) Let $M = M_{\mathbf{2}}(L)$ and $\overline{M} = M_{\overline{\mathbf{2}}}(L)$ be matrices representing the same continuous Lorentz symmetry $L \in SO^+(3, 1)$ in the $\mathbf{2}$ and the $\overline{\mathbf{2}}$ spinor representations. Use eqs. (14) and (15) to show that for any such L ,

$$\overline{M} = \sigma_2 M^* \sigma_2 \quad \text{and} \quad M = \sigma_2 \overline{M}^* \sigma_2. \quad (18)$$

Hint: prove and use $\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}$.

Next, consider the vector representation of the Lorentz symmetry and the equivalent bi-spinor representation of the $SL(2, \mathbf{C})$. In the matrix form, the $(j_+ = j_- = \frac{1}{2})$ bi-spinor multiplet of $SL(2, \mathbf{C})$ is a complex 2×2 matrix V which transforms according to

$$V \mapsto V' = M \times V \times M^\dagger \quad \text{for } M \in SL(2, \mathbf{C}). \quad (19)$$

Let's identify this bi-spinor with a Lorentz vector V^μ according to

$$V = V^\mu \overline{\sigma}_\mu = V^0 \mathbf{1}_{2 \times 2} + \mathbf{V} \cdot \boldsymbol{\sigma}, \quad (20)$$

where I follow the Peskin&Schroeder convention for the 2×2 matrices σ^μ and $\overline{\sigma}^\mu$:

$$\sigma^\mu = \overline{\sigma}_\mu = (1_{2 \times 2}, +\boldsymbol{\sigma}), \quad \overline{\sigma}^\mu = \sigma_\mu = (1_{2 \times 2}, -\boldsymbol{\sigma}). \quad (21)$$

The bi-spinor transform (19) defines a linear transform

$$V^\mu \mapsto V'^\mu = L^\mu_\nu V^\nu \quad (22)$$

of the vector V^μ .

- (d) Show that this transform is real (real V'^μ for real V^ν) and Lorentzian (preserves $V'^\mu V'_\mu = V^\nu V_\nu$). Hint: show that $\det(V) = V_\mu V^\mu$.

- (e) Show that the Lorentz transform (22) is orthochronous.

For extra challenge, show that it is also proper ($\det(L) = +1$) and therefore continuous, $L \in SO^+(3, 1)$.

- (f) Verify that this $SL(2, \mathbf{C}) \rightarrow SO^+(3, 1)$ map respects the group law, $L^\mu_\nu(M_2 M_1) = L^\mu_\lambda(M_2) L^\lambda_\nu(M_1)$.

Finally, consider the tensor representations of the Lorentz symmetry.

- (g) Show that the $(j_+ = 1, j_- = 1)$ representation is equivalent to a 2-index symmetric traceless tensor, $T^{\mu\nu} = T^{\nu\mu}$, $g_{\mu\nu} T^{\mu\nu} = 0$.

Also, show that the reducible $(j_+ = 1, j_- = 0) + (j_+ = 0, j_- = 1)$ representation is equivalent to a 2-index antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$.

Hint: For any kind of angular momentum — Hermitian or not, — the tensor product of two doublets is a triplet plus a singlet, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.