

1. First, an exercise in Dirac matrices  $\gamma^\mu$ . In this problem, you should not assume any explicit matrices for the  $\gamma^\mu$  but simply use the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (1)$$

When necessary, you may also assume that the Dirac matrices are  $4 \times 4$ , and that the  $\gamma^0$  matrix is hermitian while the  $\gamma^1, \gamma^2, \gamma^3$  matrices are antihermitian,  $(\gamma^0)^\dagger = +\gamma^0$  while  $(\gamma^i)^\dagger = -\gamma^i$  for  $i = 1, 2, 3$ .

\* My conventions here are:  $\epsilon^{0123} = -1$ ,  $\epsilon_{0123} = +1$ ,  $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$ ,  
 $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$ , etc.

- (a) The original Dirac equation used  $\beta = \gamma^0$  and  $\alpha^i = \gamma^0\gamma^i$  (for  $i = 1, 2, 3$ ) instead of the  $\gamma^\mu$ . Show that eqs. (1) are equivalent to requiring all 4 matrices  $\beta, \alpha^1, \alpha^2, \alpha^3$  to anticommute with each other and to square to 1.
- (b) Show that  $\gamma^\alpha\gamma_\alpha = 4$ ,  $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$ ,  $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ ,  
 and  $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$ .  
 Hint: use  $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$  repeatedly.

- (c) The electron field in the EM background obeys the *covariant Dirac equation*  
 $(i\gamma^\mu D_\mu - m)\Psi(x) = 0$  where  $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$ . Show that this equation implies

$$(D_\mu D^\mu + m^2 - eF_{\mu\nu}S^{\mu\nu})\Psi(x) = 0. \quad (2)$$

Besides the 4 Dirac matrices  $\gamma^\mu$ , there is another useful matrix  $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ .

- (d) Show that the  $\gamma^5$  anticommutes with each of the  $\gamma^\mu$  matrices —  $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$  — and commutes with all the spin matrices,  $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$ .
- (e) Show that the  $\gamma^5$  is hermitian and that  $(\gamma^5)^2 = 1$ .
- (f) Show that  $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$  and that  $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = +24i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$ .

(g) Show that  $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = +6i\epsilon^{\kappa\lambda\mu\nu}\gamma_\kappa\gamma^5$ .

(h) Show that any  $4 \times 4$  matrix  $\Gamma$  is a unique linear combination of the following 16 matrices:  $1$ ,  $\gamma^\mu$ ,  $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$ ,  $\gamma^5\gamma^\mu$ , and  $\gamma^5$ .

2. Since all the spin matrices  $S^{\mu\nu}$  commute with the  $\gamma^5$ , for all *continuous* Lorentz symmetries  $L^\mu_\nu$  their Dirac-spinor representations  $M_D(L) = \exp(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta})$  are block-diagonal in the eigenbasis of the  $\gamma^5$ . This makes the Dirac spinor  $\Psi$  a *reducible* multiplet of the continuous Lorentz group  $SO^+(3,1)$  — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor  $\psi_L$  and the right-handed Weyl spinor  $\psi_R$ .

This decomposition becomes clear in the Weyl convention for the Dirac matrices where in  $2 \times 2$  block notations

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (3)$$

for  $\sigma^\mu$  and  $\bar{\sigma}^\mu$  as in the Peskin & Schroeder convention:

$$\sigma^\mu = \bar{\sigma}_\mu = (1_{2 \times 2}, +\boldsymbol{\sigma}), \quad \bar{\sigma}^\mu = \sigma_\mu = (1_{2 \times 2}, -\boldsymbol{\sigma}). \quad (4)$$

(a) Show that in the Weyl convention (3), the  $\gamma^5$  matrix is diagonal, specifically

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}. \quad (5)$$

(b) Write down explicit  $S^{\mu\nu}$  matrices in the Weyl convention and show that

$$S^{\mu\nu} = \begin{pmatrix} S_L^{\mu\nu} & 0 \\ 0 & S_R^{\mu\nu} \end{pmatrix} \quad (6)$$

where  $S_L^{\mu\nu} = S_{\mathbf{2}}^{\mu\nu}$  and  $S_R^{\mu\nu} = S_{\bar{\mathbf{2}}}^{\mu\nu}$  are respectively the  $\mathbf{2}$  and  $\bar{\mathbf{2}}$  representations of the Lorentz generators, *cf.* problem 4 from the [previous homework](#).

In light of eqs. (6), the Dirac spinor is a reducible  $\mathbf{2} + \overline{\mathbf{2}}$  multiplet of the  $\text{Spin}(3, 1)$  Lorentz group, and for any continuous Lorentz transform  $L$  we have

$$M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix} \quad \text{for } M_L(L) = M_{\mathbf{2}}(L) \text{ and } M_R(L) = M_{\overline{\mathbf{2}}}(L). \quad (7)$$

Consequently, in the Weyl convention the 4-components Dirac spinor field  $\Psi(x)$  splits into two 2-component Weyl spinor fields — the left-handed Weyl spinor field  $\psi_L(x)$  and the right-handed Weyl spinor field  $\psi_R(x)$  — which transform independently (from each other) under the continuous Lorentz symmetries,

$$\Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') &= M_L(L)\psi_L(x), \\ \psi'_R(x') &= M_R(L)\psi_R(x). \end{aligned} \quad (8)$$

In problem 4(c) from the [previous homework](#) you saw (or at least should have seen) that

$$M_R(L) = \sigma_2 M_L^*(L) \sigma_2 \quad \text{and} \quad M_L(L) = \sigma_2 M_R^*(L) \sigma_2. \quad (9)$$

- (c) Use these eqs. to show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the  $\sigma_2 \times \psi_L^*(x)$  transforms under continuous Lorentz symmetries like the  $\psi_R(x)$ , while the  $\sigma_2 \times \psi_R^*(x)$  transforms like the  $\psi_L(x)$ . Note: the  $*$  superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the components, thus

$$\psi_L = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi_L^* = \begin{pmatrix} \psi_{L1}^\dagger \\ \psi_{L2}^\dagger \end{pmatrix}, \quad \text{while} \quad \psi_L^\dagger = (\psi_{L1}^\dagger \quad \psi_{L2}^\dagger), \quad (10)$$

and likewise for the  $\psi_R$  and its conjugates.

Next, consider the Dirac Lagrangian  $\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$ .

- (d) Express this Lagrangian in terms of the Weyl spinor fields  $\psi_L(x)$  and  $\psi_R(x)$  (and their conjugates  $\psi_L^\dagger(x)$  and  $\psi_R^\dagger(x)$ ).
- (e) Show that for  $m = 0$  — and only for  $m = 0$  — the two Weyl spinor fields become independent from each other.

3. Next, consider the plane-wave solutions of the Dirac equation,  $\Psi_\alpha(x) = u_\alpha \times e^{-ipx}$  and  $\Psi_\alpha(x) = v_\alpha \times e^{+ipx}$  for some  $x$ -independent Dirac spinors  $u_\alpha(p, s)$  and  $v_\alpha(p, s)$ .

(a) Check that these waves indeed solve the Dirac equation, provided  $p^2 = m^2$  while

$$(\not{p} - m)u(p, s) = 0, \quad (\not{p} + m)v(p, s) = 0 \quad (11)$$

where  $\not{p}$  is the Dirac slash notation for the  $\gamma^\mu p_\mu$ . Likewise, for any Lorentz vector  $a^\mu$ , we may write  $\not{a}$  to denote  $\gamma^\mu a_\mu$ .

By convention, we always take  $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$  — that's why we have separate positive-frequency waves  $e^{-ipx}u_\alpha$  and negative-frequency waves  $e^{+ipx}v_\alpha$  — while the spinor coefficients  $u(p, s)$  and  $v(p, s)$  are normalized to

$$u^\dagger(p, s)u(p, s') = v^\dagger(p, s)v(p, s') = 2E\delta_{s,s'}. \quad (12)$$

In this problem we shall write down explicit formulae for these spinors in the Weyl convention for the  $\gamma^\mu$  matrices.

(b) Show that for  $\mathbf{p} = 0$ ,

$$u(\mathbf{p} = \mathbf{0}, s) = \begin{pmatrix} \sqrt{m} \xi_s \\ \sqrt{m} \xi_s \end{pmatrix} \quad (13)$$

where  $\xi_s$  is a two-component  $SO(3)$  spinor encoding the electron's spin state. The  $\xi_s$  are normalized to  $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$ .

(c) For other momenta,  $u(p, s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$  for the boost that turns  $(m, \vec{0})$  into  $p^\mu$ . In the previous homework, you saw (or should have seen) that for any boost  $B$

$$M_L(B) = \sqrt{\gamma} \times \sqrt{1 - \boldsymbol{\beta} \cdot \boldsymbol{\sigma}}, \quad M_R(B) = \sqrt{\gamma} \times \sqrt{1 + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}}. \quad (14)$$

Use this formula to show that

$$u(p, s) = \begin{pmatrix} \sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ \sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi_s \end{pmatrix}. \quad (15)$$

(d) Use similar arguments to show that

$$v(p, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{p_\mu \sigma^\mu} \eta_s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta_s \end{pmatrix} \quad (16)$$

where  $\eta_s$  are two-component  $SO(3)$  spinors normalized to  $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$ .

Physically, the  $\eta_s$  should have opposite spins from the  $\xi_s$  — the holes in the Dirac sea have opposite spins (as well as  $p^\mu$ ) from the missing negative-energy particles. Mathematically, this requires  $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$ ; we may solve this condition by letting  $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$ .

(e) Check that  $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$  indeed provides for the  $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$ , then show that this leads to

$$v(p, s) = \gamma^2 u^*(p, s) \quad \text{and} \quad u(p, s) = \gamma^2 v^*(p, s). \quad (17)$$

We shall need these relations for the charge conjugation symmetry of Dirac fields.

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity  $\lambda = \pm \frac{1}{2}$ , the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components  $\psi_L(x)$  or  $\psi_R(x)$  of the Dirac spinor  $\Psi(x)$ , while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p, -\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v(p, -\tfrac{1}{2}) &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \quad (18)$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

In the previous problem we saw that for  $m = 0$ , the LH and the RH Weyl spinor fields decouple from each other. Now this exercise shows us which particle modes comprise each Weyl spinor: *The  $\psi_L(x)$  and its hermitian conjugate  $\psi_L^\dagger(x)$  contain the left-handed fermions and the right-handed antifermions, while the  $\psi_R(x)$  and the  $\psi_R^\dagger(x)$  contain the right-handed fermions and the left-handed antifermions.*

4. In problem 3 we have worked out the  $u(p, s)$  and  $v(p, s)$  spinors in the Weyl basis for the Dirac matrices. In this problem we are going to establish some convention-independent properties of these Dirac spinors, — although you may use the Weyl-basis formulae to verify them. We shall use these properties in class when we get to the Quantum Electrodynamics (QED).

(a) Dirac spinors  $u(p, s)$  and  $v(p, s)$  are normalized to

$$u^\dagger(p, s)u(p, s') = 2E_p\delta_{s,s'}, \quad v^\dagger(p, s)v(p, s') = 2E_p\delta_{s,s'}. \quad (19)$$

Show that the combinations  $\bar{u}u$  and  $\bar{v}v$  have a different normalization, namely

$$\bar{u}(p, s)u(p, s') = +2m\delta_{s,s'}, \quad \bar{v}(p, s)v(p, s') = -2m\delta_{s,s'}. \quad (20)$$

(b) There are only two independent  $SO(3)$  spinors, hence  $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s \eta_s^\dagger = \mathbf{1}_{2 \times 2}$ . Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p, s) \bar{u}_\beta(p, s) = (\not{p} + m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p, s) \bar{v}_\beta(p, s) = (\not{p} - m)_{\alpha\beta}. \quad (21)$$