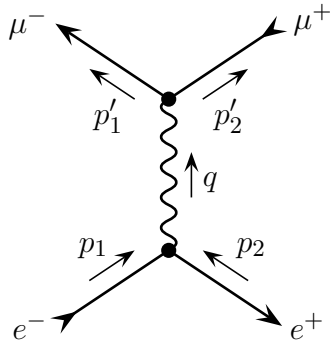


1. First, a reading assignment: §4.7 of the *Peskin&Schroeder* textbook about the Feynman rules of the Yukawa theory. Find out where the sign rules for the fermionic lines come from. Also find out the origin of the Yukawa potential $V(r) \propto e^{-mr}/r$. (There is also a much shorter explanation of the Yukawa theory and the Yukawa potential on the last three pages of [my notes on QED Feynman rules](#).
2. Second, a simple QED problem: pair production of muons in electron-positron collisions, $e^-e^+ \rightarrow \mu^-\mu^+$. As I explained in class, there is only one tree diagram for this process,



which yields the amplitude

$$\langle \mu^-, \mu^+ | \mathcal{M} | e^-, e^+ \rangle = \frac{e^2}{s} \times \bar{u}(\mu^-) \gamma^\nu v(\mu^+) \times \bar{v}(e^+) \gamma_\nu u(e^-). \quad (1)$$

In class I have focused on the un-polarized pair-production cross-section — see [my notes on the subject](#), — but in this exercise you should focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u_R &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v_L &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v_R &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \quad (2)$$

cf. [homework set#7](#), eq. (31).

(a) Show that in the approximation (2),

$$\bar{v}(e_L^+)\gamma_\nu u(e_L^-) = \bar{v}(e_R^+)\gamma_\nu u(e_R^-) = 0, \quad (3)$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

This fact is of great practical importance for the electron-positron colliders. Any kind of particle production which proceeds through a virtual vector particle — a photon, or Z^0 , or even something not yet discovered — would have the $\bar{v}(e^+)\gamma_\nu u(e^-)$ factor in the amplitude, so the electron and the positron must have opposite helicities, or they would not annihilate each other and make pairs.

Now suppose we have a longitudinally polarized electron beam — say $\lambda = +\frac{1}{2}$ only — but the positron beam is un-polarized. Because of eq. (5), only the left-handed positrons would collide with the right-handed electrons and produce pairs, while the left-handed positrons would do something else. Likewise, if the electron beam has the $\lambda = -\frac{1}{2}$ polarization, then only the right-handed positrons would collide with our left-handed electrons and make pairs, while the left-handed positrons would do something else. Thus, as far as the pair-production is concerned, the positron beam could just as well be longitudinally polarized with $\lambda(e^+) = -\lambda(e^-)$.

In other words, if we want to study polarization effects in fermion pair production, it's enough to longitudinally polarize just the electron beam. We do not need to polarize the positron beam — which is much harder to do — because the electrons of a definite helicity would automatically select positrons of the opposite helicity.

(b) Show that the μ^- and the μ^+ produced in electron-positron collision must also have *opposite helicities* because otherwise

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_L^+) = \bar{u}(\mu_R^-)\gamma^\nu v(\mu_R^+) = 0. \quad (4)$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4-vector $\bar{v}(e^+)\gamma^\nu u(e^-)$

in this frame and show that

$$\bar{v}(e_L^+)\gamma_\nu u(e_R^-) = 2E \times (0, -i, +1, 0)_\nu, \quad \bar{v}(e_R^+)\gamma_\nu u(e_L^-) = 2E \times (0, +i, +1, 0)_\nu. \quad (5)$$

- (d) In the center-of-mass frame, the muons fly away in opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the xz plane, thus

$$p_1'^\nu = (E, +E \sin \theta, 0, +E \cos \theta), \quad p_1'^\nu = (E, -E \sin \theta, 0, -E \cos \theta) \quad (6)$$

Calculate the 4-vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) &= 2E \times (0, -i \cos \theta, -1, +i \sin \theta), \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) &= 2E \times (0, +i \cos \theta, -1, -i \sin \theta). \end{aligned} \quad (7)$$

- (e) Now calculate the amplitudes (1) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1 + \cos \theta)^2, \\ \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1 - \cos \theta)^2, \\ \frac{d\sigma(e_L^- + e_L^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \quad (8)$$

- (f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.

3. Next, a harder QED problem: the elastic scattering $e^-e^+ \rightarrow e^-e^+$ of ultra-relativistic electrons and positrons. This process is called the *Bhabha scattering* after Homi Bhabha who has calculated its cross-section back in 1935. His calculation was the leading order in perturbation theory; in modern terms, it corresponds to the three-level of QED. Today, the Bhabha cross-section is known to very high precision, so the observed rate of Bhabha scatterings at electron-positron colliders is used to monitor the collider's luminosity.

At the tree level of QED, there are two diagrams contributing to the Bhabha scattering, namely

$$(9)$$

- (a) Evaluate the two diagrams and write down the amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. Mind the sign rules for the fermions.

Now comes the real work: calculating the un-polarized partial cross-section

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \quad (10)$$

where $\overline{|\mathcal{M}|^2}$ stands for $|\mathcal{M}|^2$ summed over final particle spins and averaged over the spins of the initial particles. Note the two diagrams (9) must be added together before squaring the amplitude, because

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\text{Re}(\mathcal{M}_1^* \mathcal{M}_2) \neq |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2. \quad (11)$$

For simplicity, assume $E \gg m_e$ and neglect the electron's mass throughout your calculation. You may find it convenient to express products of momenta in terms of Mandelstam's variables s , t , and u . In the $m_e \approx 0$ approximation, $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m_e^2 \approx 0$ while

$$(p_1 p_2) = (p_1' p_2') \approx \frac{1}{2}s, \quad (p_1 p_1') = (p_2 p_2') \approx -\frac{1}{2}t, \quad (p_1 p_2') = (p_2 p_1') \approx -\frac{1}{2}u. \quad (12)$$

- (b) Start with the second diagram's amplitude $\mathcal{M}_2$. Sum / average the $|\mathcal{M}_2|^2$ over all

spins and show that

$$\frac{1}{4} \sum_{\text{all spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2 + u^2}{s^2}. \quad (13)$$

(c) Similarly, show that for the first diagram

$$\frac{1}{4} \sum_{\text{all spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2 + u^2}{t^2}. \quad (14)$$

(d) Now consider the interference $\mathcal{M}_1^* \times \mathcal{M}_2$ between the two diagrams. Show that

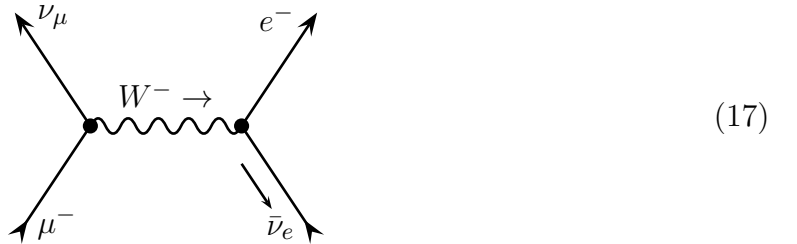
$$\frac{1}{4} \sum_{\text{all spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = 2e^4 \times \frac{u^2}{st}. \quad (15)$$

(e) Finally assemble all the terms together and show that for the Bhabha scattering

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{\alpha^2}{2s} \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2} = \frac{\alpha^2}{4s} \times \left(\frac{3 + \cos^2 \theta}{1 - \cos \theta} \right)^2. \quad (16)$$

4. Finally, another problem on spin averaging; it involves weak interactions rather than QED, but uses the same techniques.

Consider the muon decay into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$. At the tree level of the Standard model, this decay proceeds through a single Feynman diagram



Since I have not yet explained the Standard Model in class — although I plan to do it just before the Thanksgiving break, — let me simply spell out the Feynman rules relevant to this diagram.

- The vertices and the external fermionic legs attached to them are

$$\begin{aligned}
\begin{array}{c} \nearrow \nu_\mu \\ \bullet \\ \nwarrow \mu^- \end{array} &= \bar{u}(\nu_\mu) \left(\frac{-ig_2}{\sqrt{2}} \gamma^\kappa \frac{1-\gamma^5}{2} \right) u(\mu^-), & \begin{array}{c} \nearrow e^- \\ \bullet \\ \nwarrow \bar{\nu}_e \end{array} &= \bar{u}(e^-) \left(\frac{-ig_2}{\sqrt{2}} \gamma^\lambda \frac{1-\gamma^5}{2} \right) v(\bar{\nu}_e),
\end{aligned} \tag{18}$$

where g_2 is the $SU(2)_W$ gauge coupling.

- W^- is a massive vector particle, so its propagator is

$$\bullet \text{---} \text{wavy line} \text{---} \bullet = \frac{-i}{q^2 - M_W^2} \left(g_{\kappa\lambda} - \frac{k_\kappa k_\lambda}{M_W^2} \right) \xrightarrow{|q| \ll M_W} \frac{ig_{\kappa\lambda}}{M_W^2}. \tag{19}$$

The approximation here corresponds to the effective Fermi theory of weak interactions. It is valid for all nuclear beta decays as well as weak decays of all particles much lighter than $M_W \approx 80$ GeV. In particular, it is valid for the muon decay in question.

- (a) Assemble the muon decay amplitude (in the Fermi theory approximation) to

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} \times [\bar{u}(\nu_\mu) \gamma^\lambda (1-\gamma^5) u(\mu^-)] \times [\bar{u}(e^-) \gamma_\lambda (1-\gamma^5) v(\bar{\nu}_e)] \tag{20}$$

where

$$G_F = \frac{g_2^2}{4\sqrt{2}M_W^2} \approx 1.17 \cdot 10^{-5} \text{ GeV}^{-2} \tag{21}$$

is the Fermi constant of low-energy weak interactions.

- (b) Sum the absolute square of the amplitude (20) over the final particle spins, average over the initial muon's spin, and write the result as a product of two Dirac traces,

$$\overline{|\mathcal{M}|^2} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\text{all spins}} |\mathcal{M}|^2 = \frac{G_F^2}{4} \times \text{tr}(\text{matrix product \#1}) \times \text{tr}(\text{matrix product \#2}), \tag{22}$$

Since the neutrino, the antineutrino, and even the electron are much lighter than the initial muon, you may neglect their masses.

Both matrix products in eq. (22) carry 2 Lorentz indices, eventually to be summed over. But it's more convenient to do that summation after we evaluated the traces of these matrix products.

(c) Evaluate the traces in eq. (22).

(d) Sum the product of two traces over the Lorentz indices, and show that altogether

$$\overline{|\mathcal{M}|^2} = 64G_F^2(p_\mu \cdot p_{\bar{\nu}})(p_e \cdot p_\nu). \quad (23)$$

Given this formula, calculating the muon's decay rate is a straightforward phase space exercise, and we have already done this exercise in the previous homework (set#9, problem 5). The results were

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2}{12\pi^3} M_\mu E_e^2 (3M_\mu - 4E_e) \times \Theta(\tfrac{1}{2}M_\mu - E_e), \quad (24)$$

$$\Gamma_{\text{net}} = \frac{G_F^2 M_\mu^5}{192\pi^3}. \quad (25)$$