⋆ This homework set has 1 reading assignment and 3 calculational problems. The first 2 problems (including the reading assignment) concern QED (or QED with an extra neutral scalar). But the last 2 problems concern a very different subject — the spontaneous symmetry breakdown and the resulting Goldstone or pseudo-Goldstone bosons.

1. First, a reading assignment: my notes on annihilation and Compton scattering. Although I explained this subject in class, it would help you to *carefully* re-read the notes and pay attention to the algebra. Make sure you understand and can follow all the calculations.

   In particular, make sure you understand: (1) how the Ward identity works for the two-diagram amplitude; (2) how you sum/average over all spins and polarizations and get to the 3 traces in eqs. (20–23); (3) the techniques used for calculating these traces; (4) the crossing symmetry between the annihilation and the Compton scattering; (5) the lab-frame kinematics for the Compton scattering.

2. Next, a QED problem, or rather a QED + Yukawa problem. Suppose there exists a heavy neutral scalar particle with a Yukawa coupling to the electron. We have not discovered it yet, but a future electron-positron collider would be able to create it, or rather co-create it with a photon through the annihilation-like process $e^- + e^+ \to \text{scalar} + \gamma$.

   To see how this might work, consider the QED-like theory comprised of the EM field $A^\mu(x)$, the electron field $\Psi(x)$, and a real scalar field $\varphi(x)$. The φ field is neutral but it has Yukawa coupling g to the electron field $\Psi(x)$ — which also couples to the EM field $A^\mu(x)$ according to the usual QED rules. Altogether,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D} - m_e)\Psi + \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi \;-\; \tfrac{1}{2}M_s^2\varphi^2\right] \;+\; g\varphi\times\overline{\Psi}\Psi. \qquad (1)$$

   The scalar particles S are much heavier than electrons or positrons, $M_s \gg m_e$. However, relativistic electron and positron colliding with each other at CM energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

   (a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level amplitude $\langle \gamma + S|\,\mathcal{M}\,|e^- + e^+\rangle$.

(b) Verify the Ward identity for the photon. Note: the Ward identity does not have to work for individual diagrams, but it must work for the net tree amplitude.

(c) Sum $|\mathcal{M}|^2$ over the photon's polarizations and average over the fermion's spins. Show that

$$\overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4}\sum_{s_-,s_+}\sum_{\lambda}|\mathcal{M}|^2 \;=\; e^2g^2\left(\frac{A_{11}}{(t-m_e^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{2}$$

where

$$\begin{aligned} A_{11} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q} + m_e)\Big), \\ A_{22} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\not{\tilde{q}} + m_e)(\not{p}_- + m_e)(\not{\tilde{q}} + m_e)\gamma_\mu\Big), \\ A_{12} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)(\not{\tilde{q}} + m_e)\gamma_\mu\Big). \end{aligned} \tag{3}$$

Since $M_s \gg m_e$, the initial electron and positron must be ultra-relativistic. So let's simplify our calculation by neglecting the electron's mass both in the traces (3) and in the denominators in eq. (2).

(d) Evaluate the Dirac traces (3) in the $m_e \approx 0$ approximation and express them in terms of the Mandelstam variables s, t, u. Show that

$$\text{for } m_e \approx 0, \quad A_{11} \;\approx\; A_{22} \;\approx\; tu, \quad A_{12} \;\approx\; (t-M_s^2)(u-M_s^2). \tag{4}$$

Note: because of the scalar's mass, the kinematic relations between various momentum products such as $(k_\gamma p_\mp)$ and between the Mandelstam's s, t, and u are different from the $e^+e^- \to \gamma\gamma$ annihilation.

(e) Finally, assemble the net $\overline{|\mathcal{M}|^2}$ (in the $m_e \approx 0$ approximation), work out the kinematics in the CM frame, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}\,.$$

3. Now let's change the subject to the spontaneous symmetry breaking. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to an exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone boson is no longer exactly massless but only relatively light. The best-known real-life examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2)_R \cong \text{Spin}(4)$. This symmetry would be exact if the two lightest quark flavors u and d were massless; in real life, the masses m_u and m_d are small but non quite zero, and the symmetry is only approximate. Somehow (and people are still arguing how), the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2)_V \cong \text{Spin}(3)$, and the 3 generators of the broken Spin(4)/Spin(3) give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

   As a toy model of *approximate* $SO(N+1)$ symmetry spontaneously broken down to $SO(N)$, consider the *linear sigma model* of $N+1$ scalar fields ϕ_i with the Lagrangian

$$\mathcal{L} \;=\; \sum\nolimits_i \tfrac{1}{2}(\partial_\mu \phi_i)^2 \;-\; \frac{\lambda}{8}\left(\sum\nolimits_i \phi_i^2 \;-\; f^2\right)^2 \;+\; \beta\lambda f^2 \times \phi_{N+1}\,. \tag{5}$$

   For $\beta = 0$ this Lagrangian has exact $O(N+1)$ symmetry, which would be spontaneously broken down to $O(N)$ by non-zero vacuum expectation values of the scalar fields. For a non-zero β, the last term in the Lagrangian (5) *explicitly* breaks the $O(N+1)$ symmetry, but for $\beta \ll f$ we may treat the $O(N+1)$ as *approximate* symmetry.

   (a) Assume $\beta > 0$ and $\beta \ll f$. Show that the scalar potential of the linear sigma model has a unique minimum at

$$\langle \phi_1 \rangle \;=\; \cdots \langle \phi_N \rangle \;=\; 0, \quad \big\langle \phi_{N+1} \big\rangle \;=\; f \;+\; \beta \;+\; O(\beta^2/f). \tag{6}$$

   (b) Re-express the Lagrangian (5) in terms of the shifted fields

$$\sigma(x) \;=\; \phi_{N+1}(x) \;-\; \big\langle \phi_{N+1} \big\rangle\,, \qquad \pi^i(x) \;=\; \phi_i(x) \quad \text{for } i = 1, \ldots, N. \tag{7}$$

   and show that the π^i fields are massive but much lighter than the σ field. Specifically, $M_\pi^2 \approx \lambda f \times \beta$ while $M_\sigma^2 \approx \lambda f(f + 3\beta) \approx \lambda f^2 \gg M_\pi^2$.

In QCD terms, $N = 3$, the three $\pi^{1,2,3}$ fields (or rather the $\pi^0 = \pi^3$ and the $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$) correspond to the three pi-mesons of rather small mass $m_\pi \approx 140$ MeV, and the σ corresponds to the very broad sigma resonance at about 500 MeV.

(c) Spell out the cubic and the quartic couplings of the σ and π^i fields to each other and show that

$$(\text{cubic coupling})^2 \;=\; (\text{quartic coupling}) \times \bigl(M_\sigma^2 - M_\pi^2\bigr). \tag{8}$$

For $\beta = 0$ and hence $M_\pi^2 = 0$, these couplings are precisely as in problem 4 of homework set#9 (eq. (HW9.3)). Therefore — as we saw in that homework — for low-energy pions with $E \ll M_\sigma$, the scattering amplitudes $\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(\lambda E_{\text{cm}}^2/M_\sigma^2)$ or smaller.

For small $\beta \neq 0$ and hence small but non-zero pion mass, the coupling relation (8) is slightly different from what we had in homework#9, so the several tree diagrams contributing to the scattering of low-energy pions do not quite cancel each other.

(d) Recalculate the pion scattering amplitudes to allow for eq. (8) for $M_\pi^2 > 0$. Basically, go over solutions to homework#9, parts 4(c–d), and correct a few formulae.

In particular, show that to the leading order in β, for $s, t, u \ll M_\sigma$,

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;\approx\; \frac{1}{f^2}\left(\begin{array}{r}(s - m_\pi^2) \times \delta^{jk}\delta^{\ell m} \;+\; (t - m_\pi^2) \times \delta^{j\ell}\delta^{km} \\ +\; (u - m_\pi^2) \times \delta^{jm}\delta^{k\ell}\end{array}\right), \tag{9}$$

which does not vanish when any of the pion's momenta becomes small. Instead, for *slow* pions with $|\mathbf{p}| \ll m_\pi$, this amplitude becomes

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;\approx\; \bigl(3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}\bigr) \times \left(\frac{m_\pi^2}{f^2} \;\approx\; \frac{\lambda\beta}{f}\right) \;\neq\; 0. \tag{10}$$

4. Finally, a more complicated problem on spontaneous symmetry breaking. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi^i{}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \mathrm{tr}\left(\partial^\mu \Phi^\dagger\, \partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{11}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\,\mathrm{tr}\left(\Phi^\dagger \Phi \Phi^\dagger \Phi\right) \;+\; \frac{\beta}{2}\left(\mathrm{tr}\left(\Phi^\dagger \Phi\right)\right)^2 \;+\; m^2\,\mathrm{tr}\left(\Phi^\dagger \Phi\right). \tag{12}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{13}$$

For $\alpha = 0$ the theory would have a much larger continuous symmetry group $SO(2N^2)$ which mixes up the real and the imaginary parts of all the $\Phi^i{}_j$ regardless of their complex structure or the $N \times N$ matrix structure of Φ. But for $\alpha \neq 0$ this huge symmetry group is reduced to $G = SU(N)_L \times SU(N)_R \times U(1)$.*

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger \Phi$. Express the potential (12) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0. \tag{14}$$

In terms of the matrix Φ, eq. (14) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (13) to $\Phi = C \times$ the unit matrix, so without loss of generality we

$\star$ It is easy to see that the α-term is invariant under the $SU(N)_L \times SU(N)_R \times U(1)$ subgroup of the $SO(2N^2)$, but showing that this term has no other continuous symmetries is a hard exercise in group theory. But since this is a homework on QFT rather than group theory, let's accept this statement without a proof.

may assume that the vacuum lies at

$$\langle\Phi\rangle \;=\; C\times\mathbf{1}_{N\times N} \quad i.e. \quad \langle\Phi^i{}_j\rangle \;=\; C\times\delta^i_j\,. \tag{15}$$

(c) Show that the symmetries (13) preserving these VEVs are limited to the $U_L = U_R \in SU(N)$ and $\theta = 0$. In other words, the $SU(N)_L \times SU(N)_R \times U(1)$ symmetry of the theory is spontaneously broken down to the $SU(N)_V$ subgroup.

Now that we know the vacuum state of the theory and its symmetry, let's find the particle spectrum of the theory.

(d) Expand the scalar potential in powers of the $\delta\Phi(x) = \Phi(x) - \langle\Phi\rangle$ and the $\delta\Phi^\dagger(x)$,

$$V(\delta\Phi^\dagger,\delta\Phi) \;=\; \text{const} \;+\; V_1 \;+\; V_2 \;+\; V_3 \;+\; V_4\,. \tag{16}$$

Show that $V_1 = 0$ while

$$V_2 \;=\; \frac{\alpha C^2}{2}\,\mathrm{tr}\big((\delta\Phi^\dagger+\delta\Phi)^2\big) \;+\; \frac{\beta C^2}{2}\,\mathrm{tr}^2(\delta\Phi^\dagger+\delta\Phi). \tag{17}$$

(e) Altogether, the N^2 complex scalar fields give rise to $2N^2$ particle species. Find the masses of all those particles from eq. (17).
Hint: Split the complex matrix Φ into its hermitian and antihermitian parts, and also into trace and traceless parts,

$$\delta\Phi(x) \;=\; \frac{\chi_1(x)+i\chi_2(x)}{\sqrt{2N}}\times\mathbf{1}_{N\times N} + \;\frac{\varphi_1(x)+i\varphi_2(x)}{\sqrt{2}} \tag{18}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are traceless hermitian matrices (or rather matrix-valued fields) while $\chi_1(x)$ and $\chi_2(x)$ are ordinary real fields.

(f) Finally, organize the $2N^2$ particles into multiplets of the unbroken $SU(N)_V$ symmetry and make sure that all members of each multiplet have the same mass.

Also, check the Nambu–Goldstone theorem for this model — verify that for each *spontaneously broken* generator of the symmetry (13) there is a massless particle with similar quantum numbers WRT the unbroken $SU(N)_V$ subgroup.