You have 2 extra days to finish this homework (not counting the break), so it's rather oversize. Please start working early, you would never finish it in just a couple of days.

Problems 1 and 2 are about Spontaneous Symmetry Breaking and the Higgs Mechanism. Problems 3 and 4 are about the electroweak interactions in the Standard Model. Finally, problem 5 is about charged pion decay, which involves both the Goldstone theorem and the weak interactions.

1. Let's start with the $SU(2)$ gauge symmetry coupled to a doublet of Higgs fields $\Phi^{1,2}(x)$,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; D_\mu\Phi^\dagger D^\mu\Phi \;-\; \frac{\lambda}{2}\left(\Phi^\dagger\Phi \,-\, \frac{v^2}{2}\right)^2. \tag{1}$$

As discussed in class (*cf.* my notes on the Higgs mechanism, pages 8–10), the Higgs doublet develops a vacuum expectation value (VEV)

$$\langle\Phi\rangle \;=\; \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

which completely breaks the $SU(2)$ gauge symmetry and gives the vector bosons equal masses $M_V = \frac{1}{2}gv$.

The reason all 3 vector fields get the same mass is a non-obvious global $SU(2)$ symmetry of the scalar fields. Indeed, the scalar potential is invariant under the $SO(4)$ symmetry which mixes the real and the imaginary parts of the scalar fields, and the $SO(4)$ or rather Spin(4) group happens to be isomorphic to $SU(2)\times SU(2)$. The gauge symmetry acts as one of these $SU(2)$ factors while the other $SU(2)$ factor of the $SO(4)$ remains a global symmetry.

To make both $SU(2)_{\text{local}}$ and $SU(2)_{\text{global}}$ symmetries manifest, consider a 2×2 matrix

$$W \;=\; \begin{pmatrix}+\Phi_2^* & +\Phi^1\\ -\Phi_1^* & +\Phi^2\end{pmatrix} \tag{3}$$

(a) Show that $\sigma_2 W^*\sigma_2 = W$, and that any matrix obeying this skewed reality condition has form (3) for come complex Φ^1 and Φ^2.

(b) Show that under a gauge transform of the $\Phi(x)$ doublet, the $W(x)$ matrix-valued field transforms in the similar manner,

$$\Phi'(x) \;=\; U(x)\Phi(x) \quad\Longrightarrow\quad W'(x) \;=\; U(x)W(x) \quad \forall U(x) \in SU(2). \tag{4}$$

(c) Show that in terms of the matrix-valued field $W(x)$ the Lagrangian (1) becomes

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \tfrac{1}{2}\,\mathrm{tr}\big(D_\mu W^\dagger D^\mu W\big) \;-\; \frac{\lambda}{8}\big(\mathrm{tr}(W^\dagger W) \;-\; v^2\big)^2, \tag{5}$$

and that it is invariant under the $SU(2)_{\rm local} \times SU(2)_{\rm global}$ symmetry which acts as

$$\begin{aligned} W'(x) \;&=\; U_L(x)W(x)U_G^\dagger, \\ A'_\mu(x) \;&=\; U_L(x)A_\mu(x)U_L^{-1}(x) \;+\; \frac{i}{g}(\partial_\mu U_L(x))U_L^{-1}(x), \\ &\text{local } U_L(x) \in SU(2), \quad \text{global } U_G \in SU(2). \end{aligned} \tag{6}$$

The vacuum expectation value (2) — which translates to

$$\langle W\rangle = \;\frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

— is not invariant under the $SU(2)_{\rm global}$ symmetry factor of (6). Nevertheless, there are no Goldstone bosons because $\langle W\rangle$ is invariant under a modified global $SU(2)$ symmetry

$$W'(x) \;=\; U_G W(x) U_G^\dagger, \quad \text{any } U_G \in SU(2). \tag{8}$$

Or terms of eq. (6), is a subgroup of the $SU(2)_{\rm local} \times SU(2)_{\rm global}$ parametrized by

- Any $U_G \in SU(2)$, accompanied by $U_L(x) = U_G\ \forall x$.

(d) Argue that the generators of this modified $SU(2)'_{\rm global}$ symmetry are

$$T^a[SU(2)'_{\rm global}] \;=\; T^a[SU(2)_{\rm global}] \;+\; T^a[SU(2)_{\rm local}] \tag{9}$$

and check that the VEV $\langle W\rangle$ is indeed invariant under the $SU(2)'_{\rm global}$.

Since the global $SU(2)$ symmetry is modified (by mixing with the gauge symmetry) rather than spontaneously broken, there are no Goldstone bosons for this symmetry. Moreover, the modification (9) make the symmetry act non-trivially on the massive vector fields.

(e) Show that the massive vector fields form a triplet of the $SU(2)'_{\text{global}}$, and that's why they all have the same mass.

2. In the previous homework (set#11, problem 4) we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

Thus, consider a theory of N^2 complex scalar fields $\Phi^i{}_j(x)$ organized into an $N \times N$ matrix Φ, and $2N^2-1$ real vector fields $B_\mu(x)$, $L^a_\mu(x)$, and $R^a_\mu(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L^a_\mu(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R^a_\mu(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2-1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} \;-\; \tfrac{1}{2}\,\mathrm{tr}\,(L_{\mu\nu}L^{\mu\nu}) \;-\; \tfrac{1}{2}\,\mathrm{tr}\,(R_{\mu\nu}R^{\mu\nu}) \;+\; \mathrm{tr}\,\Big(D^\mu\Phi^\dagger\, D_\mu\Phi\Big) \;-\; V(\Phi^\dagger\Phi), \quad (10)$$

where

$$\begin{aligned}
B_{\mu\nu} &= \partial_\mu B_\nu \;-\; \partial_\nu B_\mu\,, \\
L_{\mu\nu} &= \partial_\mu L_\nu \;-\; \partial_\nu L_\mu \;+\; ig[L_\mu, L_\nu], \\
R_{\mu\nu} &= \partial_\mu R_\nu \;-\; \partial_\nu R_\mu \;+\; ig[R_\mu, R_\nu], \\
D_\mu\Phi &= \partial_\mu\Phi \;+\; ig' B_\mu\Phi \;+\; igL_\mu\Phi \;-\; ig\Phi R_\mu\,, \\
D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger \;=\; \partial_\mu\Phi^\dagger \;-\; ig' B_\mu\Phi^\dagger \;+\; igR_\mu\Phi^\dagger \;-\; ig\Phi^\dagger L_\mu\,.
\end{aligned} \quad (11)$$

For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

Finally, the scalar potential V is precisely as in the previous homework,

$$V \;=\; \frac{\alpha}{2}\,\mathrm{tr}\big(\Phi^\dagger\Phi\Phi^\dagger\Phi\big) \;+\; \frac{\beta}{2}\,\mathrm{tr}^2\big(\Phi^\dagger\Phi\big) \;+\; m^2\,\mathrm{tr}\big(\Phi^\dagger\Phi\big), \qquad \alpha,\beta > 0, \quad m^2 < 0, \quad (12)$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle \Phi \rangle \;=\; C \times \mathbf{1}_{N\times N} \quad \text{where} \quad C \;=\; \sqrt{\frac{-m^2}{\alpha + N\beta}}\,, \tag{13}$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $H = SU(N)_V$ subgroup.

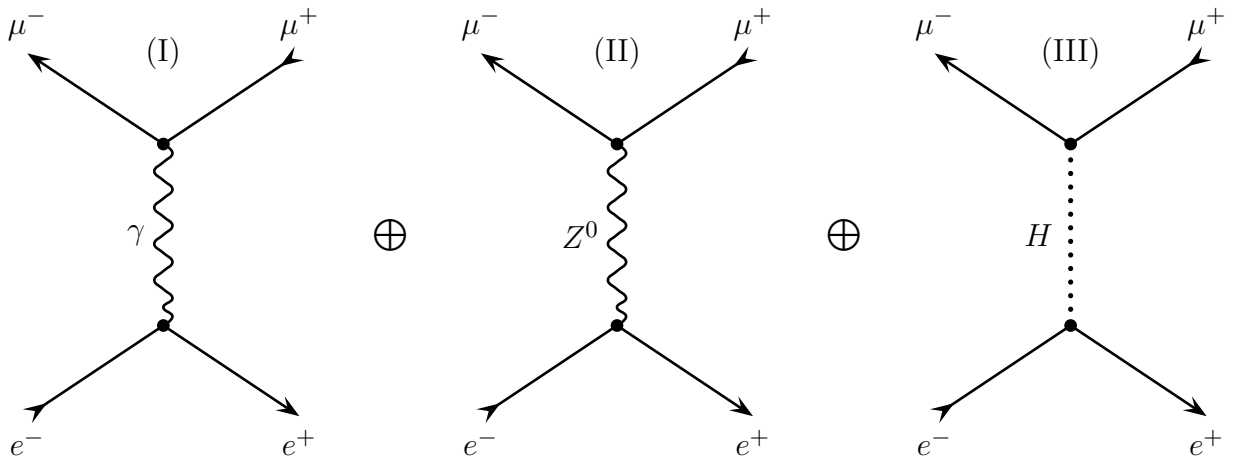
(a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle \Phi \rangle$ for the $\Phi(x)$ into the $\mathrm{tr}(D_\mu \Phi^\dagger\, D^\mu \Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ combinations of the $SU(N)$ gauge fields become massive, while the $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X^a_\mu(x) \equiv 0$, and $\Phi(x) \equiv \langle \Phi \rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

⋆ For extra challenge, allow for un-equal gauge couplings $g_L \neq g_R$. Find which combinations of the $L^a_\mu(x)$ and $R^a_\mu(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v \;=\; \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}\,. \tag{14}$$

3. Consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has three tree-level diagrams contributing to this process: one with the virtual photon in the s channel, one with the virtual Z^0 gauge boson, and one with the virtual Higgs scalar,

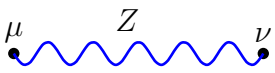


$$\mathcal{M}(e^-, e^+ \to \mu^-, \mu^+) \;=\; \mathcal{M}_\gamma \;+\; \mathcal{M}_Z \;+\; \mathcal{M}_H\,. \tag{15}$$

The first diagram was studied in class and also in the homework set#10. In this problem, we shall focus on the other two diagrams — especially the diagram (II) with a virtual Z^0 — and on their interference with the first diagram.

(a) Write down the amplitude $\mathcal{M}_H$ due to virtual Higgs scalar (diagram III). Also, relate the Yukawa couplings of the Higgs to the electrons and the muons to the fermion's masses, then argue that these couplings are so much smaller than the gauge couplings e or g' that the $\mathcal{M}_H$ is negligibly small compared to the $\mathcal{M}_Z$ or $\mathcal{M}_\gamma$ amplitudes.

Next, consider the amplitude due to a virtual Z^0 in the s channel (diagram II). For simplicity, let's work in the unitary gauge where the 'eaten-up' scalars do not have any vertices or propagators, while the massive gauge bosons like Z_0 have propagators



$$= \frac{i}{q^2 - M_Z^2 + iM_Z\Gamma_Z} \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right). \tag{16}$$

(b) Derive the electron-Z and muon-Z vertices in diagram (II) from the neutral week current,

$$\mathcal{L} \;\supset\; -g' Z_\lambda \times \sum_{\substack{\text{quarks and}\\ \text{leptons}}} \overline{\Psi}\gamma^\lambda \left(T^3 \frac{1-\gamma^5}{2} - Q\sin^2\theta\right)\Psi, \tag{17}$$

cf. my notes on the electroweak interactions of quarks and leptons, and write down the amplitude $\mathcal{M}_Z$. For simplicity, approximate $\sin^2\theta \approx \frac{1}{4}$ (experimentally, $\sin^2\theta \approx 0.233$) so that for the charged leptons like the electron or the muon

$$T^3\frac{1-\gamma^5}{2} - Q\sin^2\theta \;=\; \frac{-1+\gamma^5}{4} + \sin^2\theta \;\approx\; \frac{\gamma^5}{4}\,, \tag{18}$$

(c) Assume both the electrons and the muons to be ultra-relativistic ($E_{\text{c.m.}} = O(M_Z) \gg m_\mu, m_e$) and evaluate the amplitude $\mathcal{M}_Z$ for all possible particle helicities. (Use the center-of-mass frame.) I suggest you proceed exactly as in homework set#10 (problem 2) for the $\mathcal{M}_\gamma$ amplitude, but mind the γ^5 factors in the vertices and the massive vector propagator for the Z_0. Your answer should have form

$$\mathcal{M}_Z \;=\; \pm F(s)\times\mathcal{M}_\gamma \tag{19}$$

where the $\pm$ sign depends on the helicities while

$$F(s) \;=\; \left(\frac{g'}{4e}\right)^2 \frac{s}{s-M_Z^2+i\Gamma_Z M_Z}\,. \tag{20}$$

(d) Combine the amplitudes due to virtual Z and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+\to\mu^-\mu^+)/d\Omega$ as functions of CM energy$^2 = s$, scattering angle θ, and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned}
\frac{d\sigma(e^-_L + e^+_L \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_R \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} \;=\; 0,\\
\frac{d\sigma(e^-_{\text{any}} + e^+_{\text{any}} \to \mu^-_L + \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_{\text{any}} + e^+_{\text{any}} \to \mu^-_R + \mu^+_R)}{d\Omega_{\text{c.m.}}} \;=\; 0.
\end{aligned} \tag{21}$$

while

$$\begin{aligned}
\frac{d\sigma(e^-_L + e^+_R \to \mu^-_L + \mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_L \to \mu^-_R + \mu^+_L)}{d\Omega_{\text{c.m.}}}\\
&= \frac{\alpha^2}{4s}\times|1+F(s)|^2\times(1+\cos\theta)^2,\\
\frac{d\sigma(e^-_L + e^+_R \to \mu^-_R + \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_L \to \mu^-_L + \mu^+_R)}{d\Omega_{\text{c.m.}}}\\
&= \frac{\alpha^2}{4s}\times|1-F(s)|^2\times(1-\cos\theta)^2,
\end{aligned} \tag{22}$$

where the $|1\pm F(s)|^2$ factor stem from the interference between the virtual-photon and

virtual-Z diagrams.

(e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A \;=\; \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{23}$$

as functions of the total energy $E_{\rm c.m.}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \to \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-Z diagrams.

4. Next, another Standard Model problem. While most weak decays of quarks or leptons involve virtual W^+ or W^- gauge bosons, the top quark is so heavy that it decays into a real (*i.e.*, on-shell) W^+ and the bottom quark. The interactions relevant to this process are contained in

$$\mathcal{L} \;\supset\; -\frac{g}{2\sqrt{2}}\, W^-_\mu\, \overline{\Psi}_b \gamma^\mu (1-\gamma^5) \Psi_t \;-\; \frac{g}{2\sqrt{2}}\, W^+_\mu\, \overline{\Psi}_t \gamma^\mu (1-\gamma^5) \Psi_b \,. \tag{24}$$

For your information, $\alpha_w \equiv (g^2/4\pi) \approx 1/30$, $M_t \approx 173$ GeV, $M_W \approx 80.5$ GeV, and $M_b \approx 4.5$ GeV.

(a) Write down the Feynman vertices for the interactions (24), draw tree diagram(s) for the $t \to W^+ + b$ decay, and write down the tree-level decay amplitude.

(b) This amplitude does *not* satisfy the Ward identity. Write down a simple formula for $k^\mu_W \times \mathcal{M}_\mu$.

(c) The W^+ gauge boson is a massive particle, so it has 3 distinct spin/polarization states. Show that its polarization vectors $\mathcal{E}^\mu(k,\lambda)$ satisfy

$$\sum_\lambda \mathcal{E}^\mu(k,\lambda) \times \mathcal{E}^{*\nu}(k,\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu_W k^\nu_W}{M^2_W}\,, \tag{25}$$

and consequently, for the W emission amplitude of the form $\mathcal{M} = \mathcal{M}_\mu \times \mathcal{E}^\mu(k,\lambda)$,

summing $|\mathcal{M}|^2$ over the W polarizations yields

$$\sum_{\lambda} |\mathcal{M}|^2 \;=\; -\mathcal{M}^{\mu}\mathcal{M}^{*}_{\mu} \;+\; \frac{|\mathcal{M}_{\mu}k_W^{\mu}|^2}{M_W^2}\,. \tag{26}$$

(d) Going back to the top quark decay $t \to b + W^+$, sum the $|\mathcal{M}|^2$ over both final particles' spins, average over the initial top quark's spin, and calculate the decay rate.

For simplicity, you may neglect the bottom quark's mass compared to masses of the top quark and of the W boson.

5. The pions are pseudo-Goldstone bosons of the spontaneously broken chiral symmetry of QCD, so they can be created or annihilated by the axial isospin currents

$$J^a_{\mu 5}(x) \;=\; \overline{\Psi}(\bar{u},\bar{d})\gamma^{\mu}\gamma^5 \left(\frac{\tau^a}{2}\right)_{\text{isospin}} \Psi(u,d) = \; -f_{\pi}\partial_{\mu}\pi^a(x) \;+\; \text{multi-pion terms.} \tag{27}$$

The f_π in this formula is the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$. In this exercise, we shall see how this works. Experimentally, $f_\pi \approx 93$ MeV.

The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L}_{\text{Fermi}} \;=\; -2\sqrt{2}G_F J_L^{+\alpha} J^-_{L\alpha} \tag{28}$$

where $J_L^{\pm\alpha} = \frac{1}{2}(J_V^{\pm\alpha} - J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^{\alpha}\Psi(\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^{\alpha}\Psi(d) \;+\; \cdots, \\ J_L^{-\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^{\alpha}\Psi(\nu_\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^{\alpha}\Psi(u) \;+\; \cdots, \end{aligned} \tag{29}$$

where the $\cdots$ stand for other fermions of the Standard Model, and $\theta_c \approx 13^\circ$ is the Cabibbo angle.

For the pion decay process, the axial part of one of the charged currents annihilates the charged pion according to eq. (27) while the other charged current creates the lepton pair.

(a) Show that

$$\langle \text{vacuum}| \, \hat{J}_L^{-\alpha} \, \big|\pi^+\big\rangle \;=\; \frac{i f_\pi \cos\theta_c}{\sqrt{2}} \times p^\alpha(\pi^+) \tag{30}$$

and therefore the tree-level pion decay amplitude is

$$\mathcal{M} \;=\; \big\langle \mu^+, \bar{\nu}_\mu \big| \, \hat{\hat{L}}_{\text{Fermi}} \, \big|\pi^+\big\rangle \;=\; i G_f f_\pi \cos\theta_c \times p^\alpha(\pi^+) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{31}$$

(b) Sum $|\mathcal{M}|^2$ over the fermion spins, then calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$.

(c) Experimentally, $f_\pi \approx 93$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, $M_\nu \approx 0$, $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$, and $\theta_c \approx 13°$. Use these data to calculate the charged pion's lifetime and compare to the experimental value $\tau(\pi^\pm) = 2.6 \times 10^{-8}$ s.

(d) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} \;=\; \frac{M_e^2}{M_\mu^2} \, \frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \;\approx 1.2 \cdot 10^{-4}. \tag{32}$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of the mismatch between that lepton's chirality and helicity.