

LOOP COUNTING

Earlier in class — *cf.* [my notes](#) — we have developed the Feynman rules for the $\lambda\Phi^4$ theory. In those rules, each vertex of a Feynman graph carries a factor λ , so the graphs contributing to the order V of the perturbation theory have V vertices. In a moment, we shall see that the same order V can be related to the number L of the closed loops in the graph and the number E of the graphs external legs,

$$\mathcal{M} \propto \lambda^V \quad \text{for} \quad V = \frac{1}{2}E - 1 + L. \quad (1)$$

Since E is fixed for any particular process, it follows that the lowest order of the perturbation theory comes from the graphs with $L = 0$, *i.e.* no loops at all. Such graphs are called *tree graphs* and the corresponding lowest order of the perturbation theory is called *the tree level*.

In this semester we shall stick to the tree-level calculations because the tree graphs are much easier to evaluate — they do not involve any momentum integrals over the loop momenta. The loop graphs will have to wait until the Spring semester.

Meanwhile, let me derive eq. (1) for the order of the perturbation theory. Consider a completely general connected Feynman graph for the $\lambda\Phi^4$ theory, and let's count its basic features:

E external lines,

P propagators (internal lines),

V vertices,

L closed loops.

There are two graph-theoretical constraints on these four counts. First, the **Euler Theorem**: *for any connected graph*

$$V - P + L = 1. \quad (2)$$

Second, let's count the number of line ends on vertices. Since each propagator has both ends on vertices while each external line has one end on a vertex (while the other end is outside

the graph),

$$\# \text{line ends} = 2P + E. \quad (3)$$

On the other hand, all vertices of the $\lambda\Phi^4$ theory have valence = 4, the number of line ends on all the vertices is

$$\# \text{line ends} = 4V, \quad (4)$$

hence

$$2P + E = 4V. \quad (5)$$

Now let's combine the constraints (5) and (2):

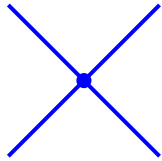
$$\begin{aligned} E + 2P &= 4V, \\ 2L - 2P &= 2 - 2V, \\ E + 2L &= 2 + 2V, \end{aligned} \quad (6)$$

and hence

$$V = \frac{1}{2}E - 1 + L, \quad (7)$$

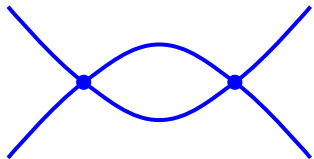
exactly as in eq. (1).

Now let me illustrate eq. (1) with a few examples. Let's start with the elastic scattering amplitude $\mathcal{M}(2 \rightarrow 2)$ of two particles. That is, two incoming particles and two outgoing particles, for the total of $E = 4$ external legs, so we should have $V = L + 1$. And indeed, at the tree $L = 0$ level



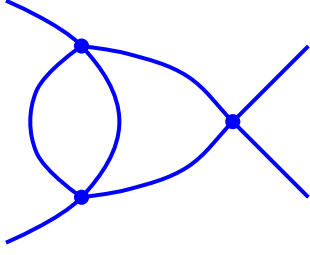
$V = 1, \quad \mathcal{M}_{\text{tree}} = -\lambda. \quad (8)$

Next, an one-loop example:



$V = L + 1 = 2, \quad \mathcal{M}_{1\text{loop}} = O(\lambda^2). \quad (9)$

And now a two-loop example:



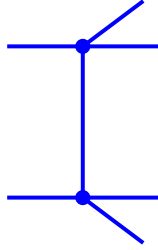
$$V = L + 1 = 3, \quad \mathcal{M}_{2\text{loops}} = O(\lambda^3). \quad (10)$$

Note that in all these examples of $E = 4$ graphs

$$\mathcal{M}_{L\text{loops}} = O(\lambda^{L+1}). \quad (11)$$

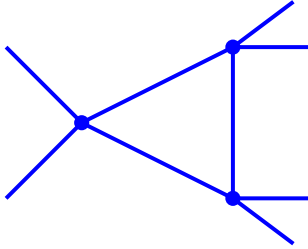
in agreement with eq. (1).

Next, consider the pair-creation process $2 \rightarrow 4$: two particles come in but four particles come out. Any graph contributing to this amplitude has $E = 6$ external legs, so it should have $V = L + 2$. And indeed, here are a couple of examples: (1) tree level



$$V = L + 2 = 2, \quad \mathcal{M}_{\text{tree}} = O(\lambda^2); \quad (12)$$

(2) one loop



$$V = L + 2 = 3, \quad \mathcal{M}_{1\text{loop}} = O(\lambda^3); \quad (13)$$

and likewise, in all higher-loop examples we should have

$$\mathcal{M}_{L\text{loops}} = O(\lambda^{L+2}). \quad (14)$$

Note that the order of the perturbation theory increases not only with the number of loops but also with the numbers of external legs, *i.e.* the net number of incoming and outgoing

particles. Thus, at the lowest-order of the perturbation theory for any particular process,

$$\mathcal{M}_{\text{tree}} \propto \lambda^{(E-2)/2}. \quad (15)$$

ADDING A CUBIC COUPLING

The $\lambda\Phi^4$ theory has a $\Phi \rightarrow -\Phi$ discrete symmetry that forbids a cubic self-coupling $\kappa\Phi^3$ of the scalar field. Let's drop that symmetry and add the cubic coupling to the theory, thus

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \frac{\kappa}{3!}\Phi^3 - \frac{\lambda}{4!}\Phi^4 \quad (16)$$

where the blue terms comprise the free Lagrangian while the red terms are perturbations. Since the free Lagrangian is unaffected by the cubic coupling, the Feynman rules of this theory have the same propagators as in the pure $\lambda\Phi^4$ theory,

$$\text{---}\bullet\text{---} = \frac{i}{q^2 - m^2 + i0}. \quad (17)$$

But the two perturbation terms in the Lagrangian (16) — and hence in the quantum Hamiltonian,

$$\hat{V} = \int d^3\mathbf{x} \left(\frac{\kappa}{3!}\hat{\Phi}^3 + \frac{\lambda}{4!}\hat{\Phi}^4 \right) \quad (18)$$

— give rise to two different vertex types: the valence = 3 vertices

$$\text{---}\bullet\text{---} \quad \text{---}\bullet\text{---} \quad \text{---}\bullet\text{---} = -i\kappa \times (2\pi)^4 \delta^{(4)}(\pm p_1 \pm p_2 \pm p_3), \quad (19)$$

and the valence = 4 vertices

$$\text{---}\bullet\text{---} \quad \text{---}\bullet\text{---} = -i\lambda \times (2\pi)^4 \delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm p_4). \quad (20)$$

Two vertex types with different valences changes the loop counting formula (1) for the order of the perturbation theory. To derive the new formula, consider a general connected

Feynman graphs with

V_3 vertices of valence 3,

V_4 vertices of valence 4,

L closed loops,

E external lines,

P propagators.

For this graph, the Euler theorem requires

$$L - P + V_{\text{net}} = 1 \quad (21)$$

and hence

$$L - P = 1 - V_3 - V_4. \quad (22)$$

At the same time, counting line ends on vertices tells us that

$$E + 2P = \# \text{line ends} = 3V_3 + 4V_4. \quad (23)$$

Combining this formula with the Euler theorem (22), we get

$$\begin{aligned} E + 2L &= (E + 2P) + 2(L - P) \\ &= 3V_3 + 4V_4 + 2(1 - V_3 - V_4) \\ &= 2 + V_3 + 2V_4 \end{aligned} \quad (24)$$

and hence

$$V_3 + 2V_4 = (E - 2) + 2L. \quad (25)$$

Eq. (25) does not fix the separate powers of the two couplings κ and λ in the amplitude of a graph with given numbers of loops and external legs. Instead, it fixes the net power of κ^2 **or** λ ,

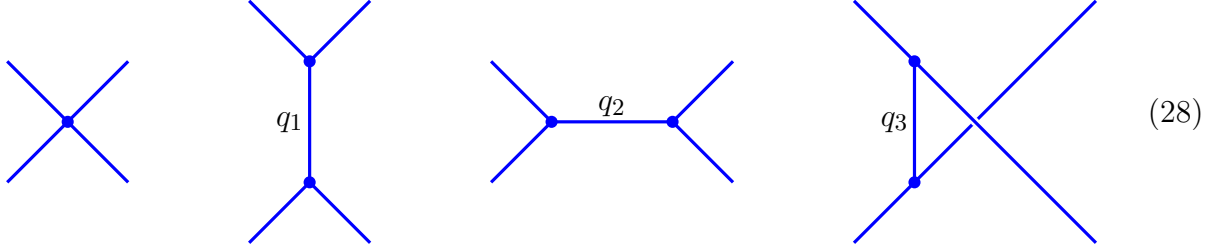
$$\mathcal{M} \propto (\kappa^2 \text{ or } \lambda)^{L+(E-2)/2}, \quad (26)$$

but we may trade one power of λ for two powers of κ or vice verse.

Let me illustrate this rule with a few Feynman graphs. Let's start with the tree graphs for the elastic $2 \rightarrow 2$ scattering: For $E = 4$ and $L = 0$ we have

$$L + \frac{E - 2}{2} = 1 \implies \text{expect } \mathcal{M} \propto \lambda \text{ or } \kappa^2, \quad (27)$$

and indeed, there are 4 tree diagrams



which together evaluate to

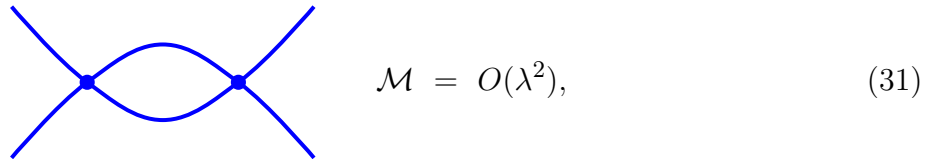
$$i\mathcal{M}_{\text{tree}} = -i\lambda + (-i\kappa)^2 \frac{i}{q_1^2 - m^2} + (-i\kappa)^2 \frac{i}{q_2^2 - m^2} + (-i\kappa)^2 \frac{i}{q_3^2 - m^2}. \quad (29)$$

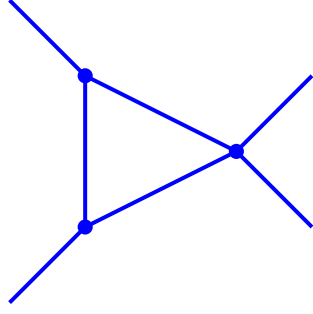
By inspection, every term in this amplitude is either $O(\lambda)$ or $O(\kappa^2)$, in agreement with eq. (26) for $E = 4$ and $L = 0$.

Now consider the one-loop amplitude for the same $2 \rightarrow 2$ elastic scattering process. According to eq. (26) we should have

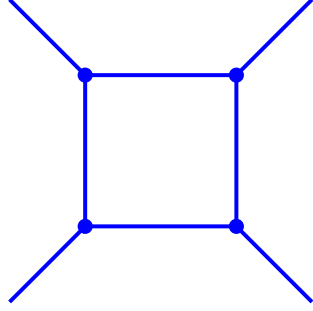
$$\begin{aligned} \mathcal{M}_{1\text{loop}} &= O\left((\lambda \text{ or } \kappa^2)^2\right) \\ &= O(\lambda^2) \text{ or } O(\lambda\kappa^2) \text{ or } O(\kappa^4), \end{aligned} \quad (30)$$

and indeed, all the one-loop graphs for this amplitude carry these powers of the couplings. Here are some examples:





$$\mathcal{M} = O(\lambda\kappa^2), \quad (32)$$



$$\mathcal{M} = O(\kappa^2). \quad (33)$$

Likewise, at higher loop orders we should get

$$\begin{aligned} \mathcal{M}_{L \text{ loops}}(2 \rightarrow 2) &= O\left((\lambda \text{ or } \kappa^2)^{L+1}\right) \\ &= O(\lambda^{L+1}) \text{ or } O(\lambda^L \kappa^2) \text{ or } \cdots \text{ or } O(\lambda \kappa^{2L}) \text{ or } O(\kappa^{2L+2}), \end{aligned} \quad (34)$$

but let me skip the examples.

MULTIPLE SCALAR FIELDS

Thus far, we have worked out the Feynman rules for the quantum theory of a single scalar field $\hat{\Phi}(x)$. In this section, we shall see the Feynman rules for theories with multiple scalar fields. The theories with non-scalar fields — especially the Dirac spinors and the vectors — will be explained in later classes.

Let's start with a theory of two scalar fields $\hat{A}(x)$ and $\hat{B}(x)$ with the classical Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \quad (35)$$

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\partial_\mu A)^2 - \frac{M_A^2}{2}A^2 + \frac{1}{2}(\partial_\mu B)^2 - \frac{M_B^2}{2}B^2, \quad (36)$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda_A}{4!}A^4 - \frac{\lambda_B}{4!}B^4 - \frac{\lambda_{AB}}{(2!)^2}A^2B^2. \quad (37)$$

In the Feynman rules for this theory, we need a graphic distinction between lines belonging to the A field and lines belonging to the B field, so let me use single lines for the A and double lines for the B ,

$$\text{---}^A \quad \text{and} \quad \text{=}=^B$$

The *internal* lines — the propagators — for the two field species obtain from the free Lagrangian (36):

$$\begin{aligned} \text{---}^A \text{---}^A &= \frac{i}{q^2 - M_A^2 + i0}, \\ \text{=}=^B \text{=}=^B &= \frac{i}{q^2 - M_B^2 + i0}, \end{aligned} \tag{38}$$

but no mixed AB propagators.

Finally, the vertices follow from the interaction terms (37). Since all the interaction terms are quartic, all vertices have valence = 4, but there 3 different vertex types with different types of the 4 lines:

- All 4 lines of type A ,

$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} = -i\lambda_A. \tag{39}$$

- All 4 lines of type B ,

$$\begin{array}{c} \text{=}= \quad \text{=}= \\ \bullet \\ \text{=}= \quad \text{=}= \end{array} = -i\lambda_B. \tag{40}$$

- Two of the lines of type A and the other two lines of type B ,

$$\begin{array}{c} \diagup \quad \text{=}= \\ \bullet \\ \text{=}= \quad \diagdown \end{array} = -i\lambda_{AB}. \tag{41}$$

★ Make sure to keep track of the line types at each vertex!

Note: the vertex factors λ_A , λ_B , and λ_{AB} obtain from the coefficients of the corresponding interactions terms in eq. (37) times the combinatorial factors counting permutations of *similar* lines, but only of the similar lines: A with A or B with B but not A with B . That's why the denominators in the three terms in eq. (37) are what they are in order to cancel these combinatorial factors: $4! = 24$ for the 4-A-line or 4-B-line vertices, but $(2!)^2 = 4$ for the mixed 2-A, 2-B vertex.

★ ★ ★

Now consider a more general theory of N scalar fields $\hat{\Phi}_a(x)$, $a = 1, 2, \dots, N$. The free classical Lagrangian of this theory is

$$\mathcal{L}_{\text{free}} = \sum_{a=1}^N \left[\frac{1}{2} (\partial_\mu \Phi_a)^2 - \frac{M_a^2}{2} \Phi_a^2 \right], \quad (42)$$

while the interaction terms comprise

$$\mathcal{L}_{\text{int}} = -\frac{1}{3!} \sum_{a,b,c} \kappa_{abc} \Phi_a \Phi_b \Phi_c - \frac{1}{4!} \sum_{a,b,c,d} \lambda_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d. \quad (43)$$

Without loss of generality I assume the cubic couplings κ_{abc} to be totally symmetric WRT permutations of the 3 indices a, b, c indices while the quartic couplings λ_{abcd} are totally symmetric WRT permutations of all 4 indices a, b, c, d .

For small N we may use different line types to graphically distinguish between the scalar species, but for larger N this becomes impractical. Instead, we may use the same line type for all the scalar fields, but *label each line by the field species a* ,

$$\begin{array}{c} \bullet \text{---} \xrightarrow{a} \bullet \\ q \end{array} = \frac{i}{q^2 - M_a^2 + i0}. \quad (44)$$

Both the internal lines and the external lines should be labeled by the field species. For an external line, the label is fixed by the species of the incoming or the outgoing particle. But for an internal line, the label is variable, to be summed over (after we multiply together all other factors of the Feynman diagram).

As to the vertices of the Feynman graphs, they follow from the interaction terms (43):
The valence = 3 vertex

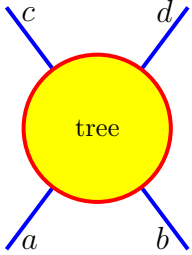
$$\begin{array}{c} a \\ \diagdown \\ \bullet \\ \diagup \\ b \end{array} \begin{array}{c} c \\ \hline \end{array} = -i\kappa_{abc}, \quad (45)$$

and the valence = 4 vertex

$$\begin{array}{cc} a & c \\ \diagdown & \diagup \\ \bullet & \\ \diagup & \diagdown \\ b & d \end{array} = -i\lambda_{abcd}, \quad (46)$$

and both vertex types must be totally symmetric in the species indices. So if we are given the interaction Lagrangian (43) where the κ_{abc} and the λ_{abcd} couplings are not totally symmetric, we must symmetrize them; We shall see an example of such symmetrization later in these notes.

Meanwhile, let me illustrate the above Feynman rules of the multi-scalar QFT by calculating the tree-level amplitude for a 2 body $\rightarrow$ 2 body scattering where the initial particles are of species a and b and the final particles are of species c and d ,

$$i\mathcal{M}_{\text{tree}}(a + b \rightarrow c + d) = \text{tree} \quad (47)$$


There are 4 tree diagrams contributing to this amplitude, namely

$$\begin{array}{cccc}
 \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} d \\ \diagup \\ \bullet \\ \diagdown \\ b \end{array} &
 \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} d \\ \diagup \\ \bullet \\ \diagdown \\ b \end{array} &
 \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} d \\ \diagup \\ \bullet \\ \diagdown \\ b \end{array} &
 \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} d \\ \diagup \\ \bullet \\ \diagdown \\ b \end{array} \\
 & e \quad q_1 & e \quad q_2 & e \quad q_3
 \end{array} \quad (48)$$

Evaluating them according to the above Feynman rules, we get

$$i\mathcal{M}_1 = -i\lambda_{abcd}, \quad (49)$$

$$i\mathcal{M}_2 = \sum_e (-i\kappa_{abe}) \times \frac{i}{q_1^2 - m_e^2} \times (-i\kappa_{ecd}), \quad (50)$$

$$i\mathcal{M}_3 = \sum_e (-i\kappa_{ace}) \times \frac{i}{q_2^2 - m_e^2} \times (-i\kappa_{ebd}), \quad (51)$$

$$i\mathcal{M}_4 = \sum_e (-i\kappa_{ade}) \times \frac{i}{q_3^2 - m_e^2} \times (-i\kappa_{ebc}), \quad (52)$$

where the propagator momenta q_1, q_2, q_3 follow from the momentum conservation in the vertices:

$$\begin{aligned} q_1 &= p_a + p_b = p_c + p_d, \\ q_2 &= p_a - p_c = p_d - p_b, \\ q_3 &= p_a - p_d = p_c - p_b. \end{aligned} \quad (53)$$

In terms of the [Mandelstam's variables](#) s, t, u — the Lorentz invariant parameters of the $2 \rightarrow 2$ scattering kinematics, —

$$q_1^2 = s, \quad q_2^2 = t, \quad q_3^2 = u. \quad (54)$$

Altogether, the net tree amplitude of the $a + b \rightarrow c + d$ process amounts to

$$\begin{aligned} \mathcal{M}_{\text{tree}}(a + b \rightarrow c + d) &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 \\ &= -\lambda_{abcd} - \sum_e \left(\frac{\kappa_{abe}\kappa_{ecd}}{s - M_e^2} + \frac{\kappa_{ace}\kappa_{ebd}}{t - M_e^2} + \frac{\kappa_{ade}\kappa_{ebc}}{u - M_e^2} \right). \end{aligned} \quad (55)$$

Note: the Feynman vertices (45) and (46) presume totally symmetric 3-scalar couplings κ_{abc} and 4-scalar couplings λ_{abcd} . So if we are given a multi-scalar theory where the interaction Lagrangian does not have form

$$\mathcal{L}_{\text{int}}(\Phi) = -\frac{1}{3!} \sum_{abc} \kappa_{abc} \times \Phi_a \Phi_b \Phi_c - \frac{1}{4!} \sum_{abcd} \lambda_{abcd} \times \Phi_a \Phi_b \Phi_c \Phi_d \quad (56)$$

with totally symmetric couplings, we must rewrite it in this form to obtain the coefficients of the 3-scalar and the 4-scalar vertices.

For example, suppose we are given

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{8} \left(\sum_a \Phi_a^2 \right)^2 = -\frac{\lambda}{8} \sum_a \Phi_a^4 - \frac{\lambda}{4} \sum_{a < b} \Phi_a^2 \Phi_b^2. \quad (57)$$

The quartic couplings here are far from being totally symmetric, so we need to symmetrize them. Starting with

$$\sum_a \Phi_a^2 = \sum_{a,b} \Phi_a \Phi_b \times \delta_{ab}, \quad (58)$$

we get

$$\left(\sum_a \Phi_a^2 \right)^2 = \sum_{ab} \Phi_a \Phi_b \delta_{ab} \times \sum_{c,d} \Phi_c \Phi_d \delta_{cd} = \sum_{a,b,c,d} \Phi_a \Phi_b \Phi_c \Phi_d \times \delta_{ab} \delta_{cd}, \quad (59)$$

and then we totally symmetrize the $\delta_{ab} \delta_{cd}$ factor here:

$$\delta_{ab} \delta_{cd} \mapsto \frac{1}{24} \delta_{(ab} \delta_{cd)} = \frac{1}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}). \quad (60)$$

Consequently,

$$\begin{aligned} \mathcal{L}_{\text{int}} &= -\frac{\lambda}{8} \sum_{a,b,c,d} \Phi_a \Phi_b \Phi_c \Phi_d \times \delta_{ab} \delta_{cd} \\ &= -\frac{\lambda}{8} \sum_{a,b,c,d} \Phi_a \Phi_b \Phi_c \Phi_d \times (\text{symmetrized } \delta_{ab} \delta_{cd}) \\ &= -\frac{\lambda}{24} \sum_{a,b,c,d} \Phi_a \Phi_b \Phi_c \Phi_d \times (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}). \end{aligned} \quad (61)$$

Or in other words,

$$\mathcal{L}_{\text{int}} = -\frac{1}{24} \sum_{abcd} \lambda_{abcd} \times \Phi_a \Phi_b \Phi_c \Phi_d \quad (62)$$

for a totally symmetric coupling

$$\lambda_{abcd} = \lambda \times (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}). \quad (63)$$

Therefore, the 4-scalar vertex of this theory is

$$\begin{array}{c} a \quad c \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ b \quad d \end{array} = -i\lambda \times (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \quad (64)$$

Finally, consider the perturbation theory of the multi-scalar QFT. Let's organize the Feynman graphs for any particular scattering amplitude according to the net number L of closed loops in the graph. Then, similar to the single- Φ theory, a graphs with L loops and E external legs has

$$V_3 + 2V_4 = (E - 2) + 2L, \quad (65)$$

where V_3 and V_4 are respectively the net numbers of 3-scalar and 4-scalar vertices. The specific coefficients κ_{abc} and λ_{abcd} of these vertices depend on the field species involved in each particular vertex, but *for the sake of perturbative expansion in powers of several couplings, let us presume*

$$\begin{aligned} \text{all the } \kappa_{abc} &= O(\kappa) = O(\sqrt{\lambda}), \\ \text{all the } \lambda_{abcd} &= O(\lambda). \end{aligned} \quad (66)$$

Under these assumptions, an amplitude of any graph with V_3 3-scalar and V_4 4-scalar vertices is

$$\mathcal{M} = O(\kappa^{V_3} \times \lambda^{V_4}) = O(\lambda^{\frac{1}{2}V_3 + V_4}), \quad (67)$$

hence in light of eq (65)

$$\mathcal{M} = O(\lambda^{(E-2)/2} \times \lambda^L). \quad (68)$$

In particular, the tree-level amplitude for 2×2 scattering should be $O(\lambda)$, and indeed the amplitude (55) is $O(\lambda)$, provided all the couplings scale with λ as in eq. (66).