

Some theories of fundamental interaction predict the existence of *dyons* — magnetic monopoles that also have electric charges. Dyons are usually very heavy compared to ordinary particles, so when an ordinary charged particle orbits a dyon, the latter can be thought as a static source of the electric and the magnetic fields: In Gauss units,

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{r^2} \mathbf{n}, \quad \mathbf{B}(\mathbf{x}) = \frac{M}{r^2} \mathbf{n}. \quad (1)$$

So let's consider the motion of a spinless non-relativistic particle of mass m and electric charge q in these static fields.

Let's start with the classical motion of the particle in question. It's net angular momentum is

$$\mathbf{J} = \mathbf{L}_{\text{mech}} + \mathbf{J}_{\text{EM}} = \mathbf{x} \times \vec{\pi} - \frac{qM}{c} \mathbf{n} \quad (2)$$

where $\vec{\pi} = m\mathbf{v}$ is the kinematic momentum of the particle rather than its canonical momentum.

- (a) Verify that it is this net angular momentum that is conserved by the classical motion of the particle, $d\mathbf{J}/dt = 0$.

In quantum mechanics, we have a similar formula for the net angular momentum,

$$\hat{\mathbf{J}} = \hat{\mathbf{x}} \times \vec{\hat{\pi}} - \frac{qM}{c} \frac{\hat{\mathbf{x}}}{\hat{r}} \quad (3)$$

where

$$\vec{\hat{\pi}} = \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}(\hat{\mathbf{x}}). \quad (4)$$

In light of eq. (4), the (equal time) commutation relations for the position and *kinematic*

momentum operators are

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_i, \hat{\pi}_j] = i\hbar\delta_{ij}, \quad (5)$$

but

$$[\hat{\pi}_i, \hat{\pi}_j] = \frac{iq\hbar}{c}\epsilon_{ijk}B_k(\hat{\mathbf{x}}) \xrightarrow{\text{in the dyon field}} \frac{iqM\hbar}{c}\epsilon_{ijk}\frac{\hat{x}_k}{\hat{r}^3}. \quad (6)$$

- (b) Use these commutation relation to show that the components of the angular momentum operator (3) indeed commute with each other — and with the other vectors — as legitimate angular momentum operators. Specifically,

$$[\hat{x}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{x}_k, \quad (7)$$

$$[\hat{\pi}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{\pi}_k, \quad (8)$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k. \quad (9)$$

- (c) Show that the operators $\hat{J}_i$ are conserved, *i.e.*, that they commute with the particle's Hamiltonian

$$\hat{H} = \frac{\hat{\pi}^2}{2m} + \frac{Qq}{\hat{r}}. \quad (10)$$

The vector potential due to the magnetic charge of the dyon can be written in spherical coordinates as

$$\mathbf{A}_{N,S}(r, \theta, \phi) = M \frac{\pm 1 - \cos \theta}{r \sin \theta} \cdot \mathbf{e}_\phi, \quad (11)$$

where $\mathbf{e}_\phi$ is the unit vector in the ϕ direction while the two signs correspond to the two different gauge choices for the Dirac monopole: ‘+’ for the $\mathbf{A}_N$ potential on the Northern side of the dyon ($0 \leq \theta < \pi - \epsilon$), and ‘−’ for the $\mathbf{A}_S$ potential on the Southern side ($\epsilon < \theta \leq \pi$).

- (d) Show that for these gauge choices, the $\hat{J}_z$ operator acts in the spherical coordinate basis as

$$\hat{J}_z \psi = -i\hbar \frac{\partial \psi}{\partial \phi} \mp \frac{qM}{c} \psi. \quad (12)$$

Note that thanks to the Dirac's charge quantization rule, the $\mp(qM/c)$ factor in the second term here is always an integer or half-integer multiple of $\hbar$.

- (e) Likewise, show that the other two components of the angular momentum have form

$$\begin{aligned} \hat{J}_+ &= \hat{J}_x + i\hat{J}_y = \hbar e^{+i\phi} \left[+\frac{\partial}{\partial \theta} + i \cot \theta \times \frac{\partial}{\partial \phi} - \frac{qM}{\hbar c} \frac{1 \mp \cos \theta}{\sin \theta} \right], \\ \hat{J}_- &= \hat{J}_x - i\hat{J}_y = \hbar e^{-i\phi} \left[-\frac{\partial}{\partial \theta} + i \cot \theta \times \frac{\partial}{\partial \phi} - \frac{qM}{\hbar c} \frac{1 \mp \cos \theta}{\sin \theta} \right], \end{aligned} \quad (13)$$

Now let's look for the simultaneous eigenstates $|n, j, m\rangle$ of the $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ operators. By the usual rules of the angular momenta, for each given n and j , m runs from $-j$ to $+j$ by 1. However, in presence of the dyon, the spectrum of j is different from the spectrum of ℓ for the ordinary orbital angular momentum: Instead of $\ell = 0, 1, 2, 3, \dots$, we now have

$$j = j_{\min}, j_{\min} + 1, j_{\min} + 2, \dots \quad \text{where} \quad j_{\min} = \frac{|qM|}{\hbar c}. \quad (14)$$

In particular, for a half-integral $qM/\hbar c$, we have j running over half-integral rather than integral values.

- (f) Use eqs. (12) and (13) to obtain this spectrum of allowed values of j .

Now let's diagonalize the Hamiltonian (10). As a first step, let's separate the radial and the angular directions of the operator $\vec{\pi}^2$.

- (g) Use the commutation relations (5) through (9) to show that

$$\vec{\pi}^2 = \hat{\pi}_r^2 + \frac{1}{\hat{r}^2} \left(\vec{J}^2 - \left(\frac{qM}{c} \right)^2 \right) \quad (15)$$

where

$$\hat{\pi}_r = \frac{1}{2} \{ \hat{n}_i, \hat{\pi}_i \} \xrightarrow{\text{coordinate basis}} -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right). \quad (16)$$

- (h) Finally, write down the radial Schrödinger equation for a given j and show that for $qQ < 0$ the bound state energies are

$$E(n_r, j) = -\frac{m(qQ)^2}{2\hbar^2} \times \frac{1}{(n_r + \lambda)^2} \quad (17)$$

where n_r is a positive integer $1, 2, 3, \dots$ while λ is the positive root of

$$\lambda(\lambda + 1) = j(j + 1) - (qM/\hbar c)^2. \quad (18)$$

By comparison, in the absence of the magnetic charge j is $\ell = 0, 1, 2, 3, \dots$, hence $\lambda = \ell$, and $n_r + \lambda = n_r + \ell$ is the principle quantum number N .