PHY–396 K. Mid-term Exam. Due Thursday, October 22 (2026)

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Please do not waste your time — and also my time — by copying posted homework solutions or class notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead; likewise for the class notes or anything else explicitly derived in class. Similarly, you may quote the textbook or any other book you have happened to read, but in that case *make sure to spell out all the intermediate steps.*

The first three problems of the exam are about the topologically massive gauge fields in $2+1$ spacetime dimensions: Problem 1 deals with the classical abelian fields, problem 2 with the classical non-abelian fields, and problem 3 with the quantum abelian fields. Finally, problem 4 takes you back to $3+1$ dimensions and deals with the isospin symmetry of nucleons and pions.

1. [20 points] In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ is equivalent to an axial Lorentz vector, $F^{\mu\nu} = \epsilon^{\mu\nu\lambda} F_\lambda$. Consequently, in 3D one can make the photons massive without breaking the gauge invariance of the electromagnetic field $A_\mu(x)$. Indeed, consider the following Lagrangian:

$$\mathcal{L} \;=\; -\frac{1}{2}\, F_\lambda F^\lambda \;+\; \frac{m}{2}\, F_\lambda A^\lambda \tag{1}$$

where

$$F_\lambda(x) \;=\; \tfrac{1}{2}\epsilon_{\lambda\mu\nu} F^{\mu\nu}(x) \;=\; \epsilon_{\lambda\mu\nu}\, \partial^\mu A^\nu(x), \tag{2}$$

or in components, $F^1 = -E^2$, $F^2 = +E^1$, and $F^0 = -B$. (In $2+1$ dimensions, the magnetic field is a pseudoscalar rather than an axial vector.) The parity-breaking mass term in the Lagrangian (1) is called the *topological mass term.*

(a) Although the topological mass term is not gauge invariant, the action $S = \int d^3x\, \mathcal{L}$ is gauge invariant. (Assuming the $A^\mu(x)$ vanish fast enough when $x \to \infty$.)
Prove this.

(b) The massive analogues of the 3D Maxwell equations are

$$\partial_\lambda F^\lambda \;=\; 0 \quad \text{and} \quad \epsilon^{\lambda\mu\nu}\partial_\mu F_\nu \;=\; mF^\lambda. \tag{3}$$

Derive these equations. Also, spell them out in 2D vector notations.

(c) Show that eqs. (3) imply the Klein–Gordon equations for the tension fields, $(\partial^2 + m^2)F_\lambda(x) = 0$.
Hint: in $2+1$ dimension $\epsilon^{\alpha\beta\gamma}\epsilon_\alpha^{\ \mu\nu} = g^{\beta\mu}g^{\gamma\nu} - g^{\beta\nu}g^{\gamma\mu}$.

(d) Write down the plane-wave solutions to eqs. (3) and show that for each on-shell momentum $k^\mu = (+\omega_{\mathbf{k}}, \mathbf{k})$ there is only one physical polarization. Or rather, one polarization for the positive-frequency wave and another polarization for the negative frequency wave.

(e) Show that the polarization of the $\mathbf{k} = 0$ wave is circular: $\mathbf{E}(t)$ rotates left for $m > 0$ but right for $m < 0$. Then argue that this means that the 2D spin state $|s\rangle$ of a topologically massive photon follows the sign of the topological mass m, namely $s = 1 \times \text{sign}(m)$.

Note: don't actually quantize the massive EM fields and/or expand them into creation and annihilation operators. Instead, simply infer the particle's spin states from the plane waves' polarizations.

2. [20 points] The non-abelian analogue of the topological mass term for the 3D gauge fields is the Chern–Simons term. Combining it with the usual Yang–Mills Lagrangian for the non-abelian gauge fields, we get

$$\begin{aligned}\mathcal{L} \;=\; \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{CS}} \;&=\; -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{4\pi}\,\epsilon^{\lambda\mu\nu}\,\text{tr}\left(\mathcal{A}_\lambda\,\partial_\mu\mathcal{A}_\nu \;+\; \frac{2i}{3}\,\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right) \\ &=\; -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{8\pi}\,\epsilon^{\lambda\mu\nu}\,\text{tr}\left(\mathcal{A}_\lambda\,\mathcal{F}_{\mu\nu} \;-\; \frac{2i}{3}\,\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right).\end{aligned} \tag{4}$$

For simplicity, let's assume an $SU(N)$ gauge symmetry group, and all the traces are in the fundamental representation $\mathbf{N}$ of the group.

(a) Expand the Lagrangian (4) in terms of the canonically normalized component fields $A^a_\mu(x)$. The Yang–Mills part was done in class, so focus on the Chern–Simons term: Expand it into a topological mass term $m = kg^2/4\pi$ for each gluon and a derivative-less 3–gluon interaction term.

Note that g^2 has dimensionality of mass in 3D while the k coefficient — called the *Chern–Simons level* — is dimensionless. In fact, k must be integer (positive, negative, or zero) to assure gauge invariance of the quantum theory.

(b) Verify invariance of the action $\int d^3x\, \mathcal{L}$ under the *infinitesimal* gauge transformations.

(b*) For extra credit (10 points), *instead of part (b)*:
Show that under a finite gauge transformation $U(x)$, the action changes by a field-independent amount

$$\Delta S \;=\; \frac{-k}{12\pi}\int d^3x\, \epsilon^{\lambda\mu\nu}\; \mathrm{tr}\Big(U^{-1}\partial_\lambda U\cdot U^{-1}\partial_\mu U\cdot U^{-1}\partial_\nu U\Big). \tag{5}$$

FYI — but don't try to prove this during the exam — the integral here depends only on the topological properties of the $U(x)$, and its values are always integer $\times\, 24\pi^2$. Consequently, for integer k — and only for integer k — $\Delta S = 2\pi \times$ an integer, which makes e^{iS} gauge invariant and assures the gauge invariance of the path integral of the quantum theory.

(c) Derive the non-abelian analogues of eqs (3).

3. [20 points] Next, consider the quantum theory of the massive EM fields in 3D; for simplicity, let's focus on the abelian fields of problem 1.

(a) Rewrite the classical Lagrangian (1) in 2D vector notations; for clarity, use the $\mathbf{E}$ and B fields instead of $\mathbf{F}$ and F^0. Than show that the canonical conjugate fields to the A^0 and $\mathbf{A}$ potentials are the

$$\Pi^0 \;=\; 0 \quad\text{and}\quad \Pi^i \;=\; -E^i \;+\frac{m}{2}\,\epsilon^{ij}A^j. \tag{6}$$

(b) Use eqs. (6) to argue that the quantum $\hat{E}^i$ and $\hat{B}$ fields should obey the *equal-time* commutation relations

$$\Big[\hat{E}^i(\mathbf{x},t),\hat{E}^j(\mathbf{y},\text{same } t)\Big] \;=\; -im\epsilon^{ij}\,\delta^{(2)}(\mathbf{x}-\mathbf{y}), \tag{7}$$

$$\Big[\hat{B}(\mathbf{x},t),\hat{E}^i(\mathbf{y},\text{same } t)\Big] \;=\; +i\epsilon^{ij}\nabla^j_x\delta^{(2)}(\mathbf{x}-\mathbf{y}), \tag{8}$$

$$\Big[\hat{B}(\mathbf{x},t),\hat{B}(\mathbf{y},\text{same } t)\Big] \;=\; 0. \tag{9}$$

Also, show that these commutation relations are consistent with the time-independent field equation

$$m\hat{B} \;=\; \nabla\cdot\hat{\mathbf{E}} \tag{10}$$

that is realized in the quantum theory as an operatorial identity.

(c) Derive the Hamiltonian of the quantum theory from the classical Lagrangian — *cf.* part (a) — and show that *regardless of the mass* m,

$$\hat{H} \;=\int d^2\mathbf{x}\Big(\tfrac{1}{2}\hat{\mathbf{E}}^2 \,+\, \tfrac{1}{2}\hat{B}^2\Big). \tag{11}$$

Instead, the mass enters the quantum theory via the commutation relation (7) and the operatorial identity (10).

(d) Finally, use the Hamiltonian (11) and the commutation relations from part (b) to show that in the Heisenberg picture, the quantum $\hat{\mathbf{E}}(x)$ and $\hat{B}(x)$ fields obey similar equations of motion to the classical fields.

4. [40 points] For the last problem of this exam, let's go back to 3+1 dimensions and consider a theory of two Dirac fields Ψ^i $(i=1,2)$ representing the nucleons and three real pseudoscalar fields ϕ^a $(a=1,2,3)$ representing the pions (pi-mesons). The classical Lagrangian of this theory is

$$\mathcal{L} \;=\; \tfrac{1}{2}\,\partial_\mu\phi^a\,\partial^\mu\phi^a \;-\; \tfrac{1}{2}m^2\,\phi^a\phi^a \;+\; \overline{\Psi}_j(i\,\partial\!\!\!/ - M)\Psi^j \;-\; ig\phi^a\times\overline{\Psi}_i\big(\tfrac{1}{2}\tau^a\big)^i{}_j\gamma^5\Psi^j \;-\; \tfrac{1}{24}\lambda\,(\phi^a\phi^a)^2 \tag{12}$$

where τ^a are the Pauli matrices for the isospin (denoted τ rather than σ to avoid confusion with the spin matrices) and two repeated indices should be summed over.

In real life, there are many other fields interacting with the pions and/or the nucleons — for example, the EM fields, — but please disregard them for this this exam. Instead, please focus on the fields and the interactions in the Lagrangian (12) only. In particular, take the $SU(2)$ isospin symmetry — under which the nucleon fields form a doublet while the pion fields form a triplet — to be an exact symmetry of the theory.

(a) Spell out how the Ψ^i, $\overline{\Psi}_i$, and ϕ^a fields transform under the $SU(2)$ isospin symmetry. Also, check that every term in the Lagrangian (12) is indeed invariant under this symmetry.

(b) Write down the classical equation of motion for all the fields.

(c) The classical Noether currents of the isospin are

$$J^a_\mu \;=\; \epsilon^{abc}\phi^b\partial_\mu\phi^c \;+\; \overline{\Psi}_i\gamma^\mu(\tfrac{1}{2}\tau^a)^i{}_j\Psi^j. \tag{13}$$

Derive these currents from the Noether theorem, then verify that they are indeed conserved when all the fields obey their equations of motion. Note: because of the Yukawa interactions between the pion and the nucleon fields, the two terms in eqs. (13) are not conserved by themselves, but the net currents J^a_μ are indeed conserved.

Now let's quantize the field theory at hand and focus on the isospin charges

$$\hat{T}^a \;=\; \int \hat{j}^{0a}\, d^3\mathbf{x} \;=\; \hat{T}^a_{\text{bosonic}} \;+\; \hat{T}^b_{\text{fermionic}} \tag{14}$$

(d) Spell out the the bosonic and the fermionic parts of the $\hat{T}^a$ charges in terms of the appropriate quantum fields. Then evaluate the equal-time commutators of the charges with the quantum fields and show that the charges act as the isospin generators, namely

$$\big[\hat{T}^a(t),\hat{\phi}^b(\mathbf{x},t)\big] \;=\; i\epsilon^{abc}\phi^c(\mathbf{x},t), \qquad \big[\hat{T}^a(t),\hat{\pi}^b(\mathbf{x},t)\big] \;=\; i\epsilon^{abc}\pi^c(\mathbf{x},t), \tag{15}$$

$$\big[\hat{T}^a(t),\hat{\Psi}^i(\mathbf{x},t)\big] \;=\; -\tfrac{1}{2}(\tau^a)^i{}_j\hat{\Psi}^j(\mathbf{x},t), \qquad \big[\hat{T}^a(t),\hat{\Psi}^\dagger_i(\mathbf{x},t)\big] \;=\; +\tfrac{1}{2}(\tau^a)^j{}_i\hat{\Psi}^\dagger_j(\mathbf{x},t). \tag{16}$$

For the $\hat{T}^a_{\text{bosonic}}$ operators and their commutators (15), adapt the $SO(N)$ formulae from the homework#4 (problem 2) to the $SO(3)$ notations instead of redoing the work. But the $\hat{T}^a_{\text{fermionic}}$ and their commutators (16) must be derived from scratch the hard way.

(e) Verify the $SU(2)$ commutation relations $\big[\hat{T}^a,\hat{T}^b\big] = i\epsilon^{abc}\hat{T}^c$ of the charge operators. Again, the commutators of the bosonic charges were done back in homework#4 (problem 2), so simply adapt the results to the $SO(3)$ notations. But for the fermionic charges, you must calculate their commutators the hard way from eqs. (16).

(f) Verify that the charge operators $\hat{T}^a$ commute with the Hamiltonian of the theory.

Note: The Yukawa-coupling term in the Hamiltonian does not commute with the separate $\hat{T}^a_{\text{bosonic}}$ or $\hat{T}^a_{\text{fermionic}}$ parts of the isospin charges but only with the net isospin charges (14).

(g) Go to the Schrödinger picture, expand all the fields into creation and annihilation operators as if the fields were free (*i.e.*, in the $g, \lambda \to 0$ limit), and write down the isospin generators $\hat{T}^a$ in terms of the creation and the annihilation operators. To avoid confusion, reserve the $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ notations for the nucleon/antinucleon creation/annihilation operators and some other notations for the pions, for example $\hat{c}, \hat{c}^\dagger$.

(h) Finally, spell out how the $\hat{T}^a$ charges — or rather the $\hat{T}^3$ and the $\hat{T}^\pm = \hat{T}^1 \pm i\hat{T}^2$ operators — act on the one-particle states $|p(\mathbf{p}, s)\rangle$, $|n(\mathbf{p}, s)\rangle$, $|\bar{p}(\mathbf{p}, s)\rangle$, $|\bar{n}(\mathbf{p}, s)\rangle$, and $\left|\pi^{-1,0,+1}(\mathbf{p})\right\rangle$.