

## Multiplets

In Physics terminology, the *multiplets* of some symmetry are often confused with the *representations* of the symmetry group. The two concepts are related but not synonymous, so let me explain the difference.

A *multiplet* is a set of objects — quantum states, operators, fields, whatever — which transform into each other by the symmetry action. For example, consider the  $|n, \ell, m\rangle$  states of a hydrogen atom (for simplicity, let's ignore the spin). With respect to the  $SO(3)$  rotational symmetry group, the states with the same  $n$  and  $\ell$  but different  $m$  form a multiplet  $(n, \ell)$ : they transform into each other according to

$$R : |n, \ell, m\rangle \rightarrow \sum_{m'} \hat{\mathcal{D}}_{m, m'}^{(\ell)}(R) |n, \ell, m'\rangle, \quad (1)$$

but the transformation law does not involve any states with different  $n$  or  $\ell$ . Instead, such states form different multiplets  $(n', \ell')$ . In particular, the multiplets  $(n, \ell)$  and  $(n', \ell)$  — with the same  $\ell$  but different  $n' \neq n$  — are different multiplets comprising different states, even though they transform in a similar way under the  $SO(3)$  symmetry.

By comparison, a *representation* is a mathematical object — a homomorphism from the symmetry group  $G$  into a space of matrices or linear operators. For example, the homomorphisms

$$R \in SO(3) \mapsto \text{matrix} \hat{\mathcal{D}}_{m, m'}^{(\ell)}(R) \quad (2)$$

for  $\ell = 0, 1, 2, 3, \dots$  form finite representations of the  $SO(3)$  group, but for any such homomorphism the representation does not care what the  $\hat{\mathcal{D}}_{m, m'}^{(\ell)}(R)$  matrices act upon. Thus, for each  $\ell$  there is only one representation of the  $SO(3)$  by  $(2\ell + 1) \times (2\ell + 1)$  matrices.

From the multiplet point of view, a *representation is the type of a multiplet*: it specifies how the members of a multiplet transform into each other, regardless of what these members happen to be. In the hydrogen atom example, the multiplets  $(n, \ell)$  and  $(n', \ell)$  (with the same  $\ell$  but different  $n' \neq n$ ) are of the same type — the  $(\ell)$  representation of the  $SO(3)$  — but they are different multiplets because they comprise different states.

In gauge theory, the matter fields form multiplets of the gauge theory group  $G$ , and there can be several distinct multiplets of the same type. For example, in QCD (quantum

chromodynamics) the gauge group is  $SU(3)$  while the matter fields are the quarks — Dirac spinor fields  $\Psi_\alpha^{if}(x)$  which come in 3 colors  $i = 1, 2, 3$  and 6 flavors  $f = u, d, s, c, b, t$ . The  $SU(3)$  gauge symmetry acts only on the colors,

$$\Psi_\alpha^{if}(x) = \sum_j U_j^i(x) \Psi_\alpha^{jf} \quad \langle\langle \text{for the same } f \rangle\rangle, \quad (3)$$

so the quarks form 6 distinct multiplets according to their flavors: the 3 quark fields of different colors  $i = 1, 2, 3$  but of the same flavor form a multiplet. But all 6 of these multiplet are of the same type — the fundamental  $\mathbf{3}$  representation of the  $SU(3)$  group.