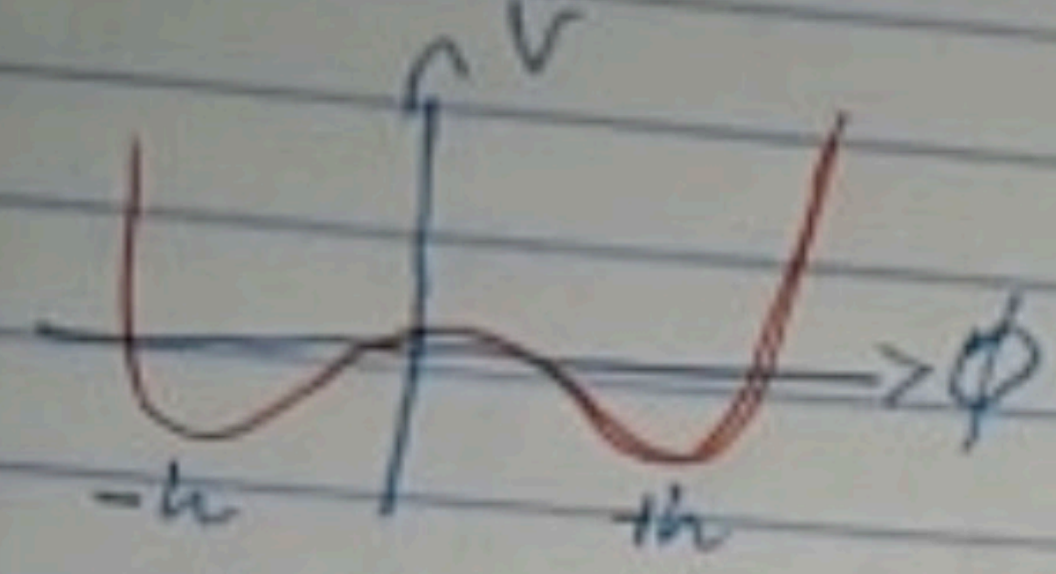
E2/1

Domain wall between diff ground states

For ex $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \underbrace{\frac{\lambda}{8}(\phi^2-h^2)^2}_{V}$



2 vacua

$\langle v1|\hat\phi|v1\rangle = +h$

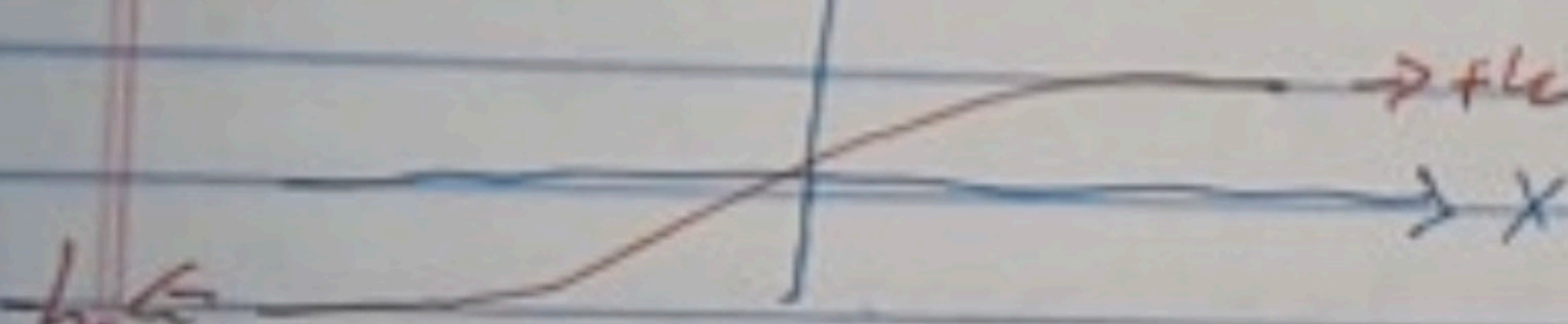
$\langle v2|\hat\phi|v2\rangle = -h$

De-excitation → not quite gr. state

domains $\langle\hat\phi(x)\rangle = +h$ or $-h$, dep on $\vec{x}$

walls between domains.

For ex $\langle\phi(x)\rangle$ ~ depends on x but not y, z



mid-wall ϕ vanishes

topology: $\phi(x,y,z)$ must vanish somewhere between 2 domains.

eq: $\phi(x,y,z)=0$: 1 real eq for 3 real vars.

→ 2d family of solutions

domain wall is a 2D surface in 3D.

E2/2

Co-dimension

codim[object] = dim[space] - dim[object].

$\forall$ dim[space], codim[domain wall] = 1

Vortices: codim[vortex] = 2

in 3D, a vortex = line

in 2D, a vortex = particle

in 4D, a vortex = membrane

in 5D, a vortex = 3D ~~objet~~ object 3-brane

Vortices in a superfluid

Landau-Ginzburg theory of $\varphi(x)$:

$$\varphi(x) = \langle SF | \hat{\varphi}(x) | SF \rangle.$$

For the SF component

$$N_{\text{atoms in SF}} = \int d^3x\, \varphi^* \varphi$$

$$\vec{P}_{net} = \int d^3x\, \varphi^* (-i\hbar \vec{\nabla} \varphi) = \hbar \int d^3x\, \mathrm{Im}(\varphi^* \vec{\nabla} \varphi).$$

atom # density $n_s = \varphi^* \varphi$

momentum density $= \hbar\, \mathrm{Im}(\varphi^* \nabla \varphi) = \hbar |\varphi|^2 \vec{\nabla} \mathrm{phase}(\varphi)$

$$\overset{\parallel}{n_s M \times \vec{v}_s} = M |\varphi|^2 \vec{v}_s$$

$$\boxed{\vec{v}_s = \frac{\hbar}{M} \nabla \mathrm{phase}(\varphi)}$$

E2(3)

$\vec{\nabla}\times\vec{v} = \nabla\times\nabla(\ldots) = 0$

naively, irrotational flow

$\forall$ loop L, circulation $\oint_L \vec{v}\cdot d\vec{x} = 0$

Real life: not true

~~SF Helium does rotate in rotating dewar.~~

How: vortices.

phase(ψ) ill-defined when $\psi = 0$.

∇phase(ψ) is singular where $\psi = 0$

$\nabla\times\nabla$phase(ψ) has a δ-spike where $\psi = 0$.

$\oint_L d\text{phase}(\psi) = 2\pi \times \text{integer}$

but if $\neq 0$, then ψ must vanish somewhere inside the loop L.

For ex, if along a circle phase(ψ) $= \theta$

then $\oint d\text{phase}(\psi) = 2\pi \neq 0$.

but $\psi(r=0)$ ~~is~~ $= 0$ ~~or else phase is~~

or else $\psi(r,\theta)$ is singular.

---

vortices $\oint_L d\text{phase}(\psi) = 2\pi N$

$\Rightarrow$ circulation $= \frac{\hbar}{m}\oint \nabla\text{phase}(\psi)\cdot d\vec{x}$

$= \frac{\hbar}{m}\oint d\text{phase}(\psi) = \frac{2\pi\hbar}{m}\times N$

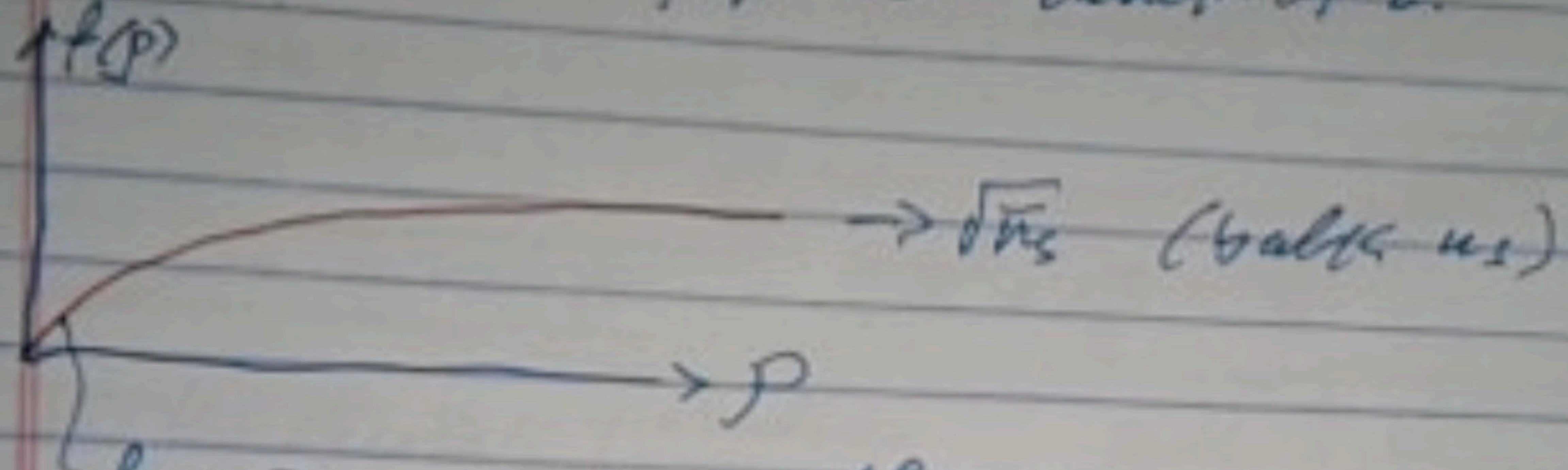
$\rightarrow$ quantized circulation $= \frac{2\pi\hbar}{m}\times$ integer.

E2/4)

Vortex along z axis.

Cylindrical coord. (ρ, θ, z)

$\varphi(\rho,\theta,z) = f(\rho) \times e^{i\theta}$ indep. of z.



$f \sim \rho \;\to\; \varphi \sim \rho e^{i\theta} = (x+iy)$

↳ analytic in (x,y,z).

circulation $= \dfrac{2\pi\hbar}{m}$.

the profile $f(\rho)$ obtained by minimizing

$$\frac{\text{Energy}}{\text{Length}} = \int d^2x_\perp \left(\frac{\hbar^2}{2m}|\nabla\varphi|^2 + \frac{\lambda}{8}\left(|\varphi|^2 - n_s\right)^2\right)$$

$$= \int d\rho \times 2\pi\rho \left(\frac{\hbar^2}{2m} \times \left(\frac{df}{d\rho}\right)^2 + \frac{\hbar^2}{2m} \times \frac{f^2}{\rho^2} + \frac{\lambda}{8}\left(f^2 - n_s\right)^2\right)$$

bec. of potential term

$f \to n = \sqrt{n_s}$ @ $\rho \to \infty$.

E2/5

Rotating Liq He II (in a round dewar, rad R)

macroscopically, circulation $= \Omega \times 2\pi R$

$= \Omega R \times 2\pi R$

$\rightarrow \frac{2\pi\hbar}{m} \times N_{vortices}$

$\rightarrow N_{vortices} = \Omega R^2 \times \frac{m}{\hbar}$

vortex density $\boxed{\frac{N_{vortices}}{Area} = \frac{\Omega m}{\pi\hbar}}$

Energy of a standalone vortex

$$\frac{E}{L} = \int_0^\infty d\rho\, 2\pi\rho \left(\frac{\hbar^2}{2m}\left(f'^2 + \frac{f^2}{\rho^2}\right) + \frac{\lambda}{8}\left(f^2 - h^2\right)^2 \right)$$

large ρ $f \sim h$ $\rightarrow$ 2^{nd} term $\rightarrow \underbrace{\int d\rho \frac{\rho}{\rho^2}}_{\text{divergent as } \rho\to\infty} \times \frac{2\pi\hbar^2 n_s}{2m}$

$\Rightarrow$ need long-distance cutoff.

due to dewar size or other vortices:

$$\frac{E}{L} = \frac{2\pi\hbar^2 \bar{n}_s}{2m} \times \log R_{max} + \text{finite constant}.$$

vortex in superconductor has finite $\frac{E}{L}$ even for $R_{max} = \infty$

vortex ring: moves @ $v \sim \frac{\log R}{R}$

$\rightarrow$ critical speed of SF flow.

E2/6

Topology: @ $r \to \infty$, $|\varphi|^2 \to \overline{n_s}$
as in the ground state.
but $\langle gr | \hat{\varphi} | gr \rangle$ may have any phase
→ may have phase(φ) slowly varying under ϱ.
if the phase (ϱ, θ, z) @ large ϱ
is phase $= \kappa\theta$
then there should be κ vortices
where $\varphi = 0$.
↳ topological defects.

Vortex is axis where $\varphi(x,y,z) = 0$
1 complex eqn ≡ 2 real eqs for (x,y,z)
→ $\boxed{\text{co-dimension} = 2}$

A vortex is surrounded by a large loop
where phase(φ) changes by 2π.

Loop → 1 dimension
Radius: vortex to the loop → 1 dim.
} co-dim = 2

all other D-2 dimensions are
spanned by the vortex.

Defect of co-dimension = 3.
Magnetic monopole.

in 3, e.g. monopole = particle.

Surround a monopole with a sphere S^2

Mag. Flux through that sphere

$$\mathcal{F} = \oiint_{S^2} \vec{B}\cdot d^2\text{area} \neq 0.$$

↳ top. defect.

in $D > 3$

mag field vec. a 2-form $F_{ij}\, dx^i \wedge dx^j$
$F_{ij} = -F_{ji}$

$$\text{Flux } \mathcal{F} = \oiint F_{ij}\, dx^i \wedge dx^j$$

$\oiint$ is over a closed 2D surface.
like S^2.

in any D, a monopole is surrounded
by a 2-sphere S^2.

$$\left.\begin{array}{l} S^2 \to 2\text{ dim.} \\ \text{radius} \to 1\text{ dim} \end{array}\right\}\ \text{codim} = 3.$$

remaining $D-3$ dim are spanned
by the monopole.

E2/8

in 3+1 dim monopole = particle
in 4+1 dim monopole = string
in 5+1 dim monopole = membrane
etc.

what about $d = 2+1$?
no monopoles
but analytically continuing
$d = 2+1 \to d = 3+0$ (Euclidean 3D)

in $d = 3$ Eucl, monopole is an
instanton: exists only in
a small volume of spacetime
very short time duration

in $d = 2+1$, monopoles contribute
to non-pert. effects.

Yang-Mills instantons
have codimension = 4.

in Eucl. 4D: instantons
→ contribute to non-pert. effects
in 3+1 dim.

in 4+1 dim, a YM instanton = a particle
in 5+1 dim, " " " = a string
etc.