SPINOR FIELDS IN DIFFERENT DIMENSIONS

In class, we have focused on Dirac, Majorana, and Weyl spinor fields in $3+1$ dimensions we happen to live in. But in string theory — as well as other hypothetical unified theories — one often deals with spacetimes of other dimensions, from 2 to 11 and beyond that. So in these notes I shall briefly review the Dirac, Majorana, and Weyl spinor fields in different dimensions.

Dirac Spinor Fields

Dirac spinors and Dirac spinor fields exist in all space dimension. Similar to 4d, in other dimensions we start with d Dirac matrices γ^μ obeying anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2g^{\mu\nu}, \tag{1}$$

define the spin matrices $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, and use them to construct the Dirac spinor representation of the d-dimensional Lorentz group $SO^+(d-1,1)$. And then the Dirac spinor fields are multiplets of fermionic component fields in the Dirac spinor representation of this group, with the free Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \;=\; \Psi^\dagger(i\gamma^0\gamma^\mu\partial_\mu - m\gamma^0)\Psi. \tag{2}$$

Everything work exactly as in $d=4$, except for the size of the γ^μ matrices and hence the size of the Dirac spinor multiplet:

d	size
2,3	2
4,5	4
6,7	8
8,9	16
10,11	32
$(2n), (2n+1)$	2^n

$$\begin{aligned} &\text{in an even spacetime dimension } d, \quad \text{size} = 2^{d/2},\\ &\text{in an odd spacetime dimension } d, \quad \text{size} = 2^{(d-1)/2}.\end{aligned}$$

These formulae for the Dirac spinor size come from counting the independent product of Dirac matrices γ^μ. Since these matrices anticommute with each other and square to ± 1, any product of these matrices can be simplified to $\pm(1\,\text{or}\,\gamma^0)\times(1\,\text{or}\,\gamma^1)\times\cdots\times(1\,\text{or}\,\gamma^{d-1})$. Altogether, this gives 2^d different product, and in even dimensions d all these product are linearly independent. Consequently, the Dirac matrices must have 2^d independent matrix elements to allow for 2^d independent products, which calls for the matrix size $2^{d/2}\times 2^{d/2}$ and hence $2^{d/2}$ components of the Dirac spinor multiplet.

In odd d, only a half of the 2^d matrix products are independent due to a different nature of the product of *all* Dirac matrices

$$\Gamma \stackrel{\text{def}}{=} \gamma^0\gamma^1\cdots\gamma^{d-1}\times(\pm 1\ \text{or}\ \pm i) \tag{3}$$

where the last factor provides for $\Gamma^\dagger=+\Gamma$ and $\Gamma^2=+1$. In even spacetime dimensions d, this Γ matrix anti-commutes with all the γ^μ, so it acts as a d-dimensional analogue of the 4D γ^5. Consequently, it will be instrumental in constructing the Weyl spinors, as we shall see in the next section of these notes.

But in odd dimensions d, the Γ matrix commutes rather than anticommutes with all the γ^μ and hence also commutes with the entire Clifford algebra of the γ-matrix products. Consequently, we may just as well restrict the whole algebra to an eigenblock of Γ, which has an effect of simply identifying $\Gamma=+1$ or $\Gamma=-1$. This identification reduces the number of independent γ-matrix products by half, from 2^d down to 2^{d-1}, so we only need 2^{d-1} independent matrix elements rather than 2^d. And this is why in odd d, the Dirac matrices have size $2^{(d-1)/2}\times 2^{(d-1)/2}$ and the Dirac spinors have $2^{(d-1)/2}$ independent components.

Parity

The *parity* — the space reflection symmetry — works differently in spacetimes of even vs. odd dimensions. For an even d — and hence an odd space dimension $d-1$ — the parity combines an a mirror reflection $x^1\to -x^1$ with 180° rotations in pairs of other space dimensions $(x^2,x^3),\ldots$ to a space reversal

$$\mathbf{x}\to\mathbf{x}' = -\mathbf{x} \qquad (\text{but}\quad t\to +t). \tag{4}$$

Under this symmetry, a Dirac spinor field transforms as

$$\Psi'(x') = \pm\gamma^0\Psi(x), \tag{5}$$

which leaves the Dirac Lagrangian invariant, or rather transforms as a true scalar, $\mathcal{L}'(x') = \mathcal{L}(x)$. Indeed,

$$\begin{aligned} \mathcal{L}'(x') &= \Psi'^{\dagger}(x')\big(i\partial'_0 + i\gamma^0\vec{\gamma}\cdot\nabla' - m\gamma^0\big)\Psi'(x') \\ &\qquad \langle\!\langle \text{ where } \nabla' = -\nabla \text{ but } \partial'_0 = +\partial_0 \,\rangle\!\rangle \\ &= \Psi^{\dagger}(x)\gamma^0\big(i\partial_0 - i\gamma^0\vec{\gamma}\cdot\nabla - m\gamma^0\big)\gamma^0\Psi(x) \\ &= \Psi^{\dagger}(x)\big(i\partial_0 + i\gamma^0\vec{\gamma}\cdot\nabla - m\gamma^0\big)\Psi(x) \\ &= \mathcal{L}(x). \end{aligned} \tag{6}$$

But for an odd d — and hence an even space dimension $d-1$, — the space reversal (4) is not a reflection but a rotation (by 180° in $(d-1/2)$ planes). Instead, the parity acts as a mirror reflection in just one space dimensions, for example

$$x^0 \to +x^0, \quad x^1 \to -x^1, \quad \text{other } x^i \to +\, x^i. \tag{7}$$

Consequently, the parity transformation rule for a Dirac fermion has a more general form

$$\Psi'(x') = P\Psi(x) \tag{8}$$

for some unitary matrix $P \neq \gamma^0$. In order for this transformation to preserve the *massless* Dirac Lagrangian

$$\mathcal{L}_0 = \Psi^{\dagger}(i\partial_0 + i\gamma^0\vec{\gamma}\cdot\nabla)\Psi, \tag{9}$$

where the mirror reflection flips the sign of the ∇^1 but not of other space derivatives ∇^i, we need

$$P^{\dagger}\gamma^0\gamma^i P = \begin{cases} -\gamma^0\gamma^i & \text{for } i=1, \\ +\gamma^0\gamma^i & \text{for } i\neq 1. \end{cases} \tag{10}$$

and hence $P=\gamma^1$ (or rather $P=i\gamma^1$ since we want $P^2=1$). But then the parity transform ends up reversing the sign of the Lagrangian mass term

$$\mathcal{L}_m = -m\Psi^{\dagger}\gamma^0\Psi \xrightarrow[\text{parity}]{} -m\Psi^{\dagger}(i\gamma^1)\gamma^0(+i\gamma^1)\Psi = -m\Psi^{\dagger}(-\gamma^0)\Psi = -\mathcal{L}_m\,. \tag{11}$$

This *in odd spacetime dimensions, a non-zero mass of a Dirac fermion breaks its parity symmetry!*

However, a pair of massive Dirac fermion fields with $m_1 = -m_2$ (and hence the same particle mass $|m|$), do have an unbroken combination of parity with an internal symmetry exchanging the two fields.

Weyl Spinor Fields

In 4D, the γ^5 matrix commutes with all the spin matrices $S^{\mu\nu}$ and hence with all the *continuous* Lorentz transforms. Consequently, the Dirac spinor is a reducible representation $\mathbf{2}+\bar{\mathbf{2}}$ of the Spin(3, 1) symmetry group, with the irreducible components — the LH and the RH Weyl spinors $\mathbf{2}$ and $\bar{\mathbf{2}}$ — being the eigenblock of the γ^5. In particular, in the Weyl basis where γ^5 is diagonal,

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \tag{12}$$

the 4-component Dirac spinor field splits into 2-component Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$. Or in the basis-independent way, the two Weyl spinors obtain by projecting

$$\psi_L(x) = \frac{1-\gamma^5}{2}\Psi(x), \qquad \psi_R(x) = \frac{1+\gamma^5}{2}\Psi(x). \tag{13}$$

Likewise, in any *even* spacetime dimension d, there is a matrix Γ which anticommutes with all the γ^μ and hence commutes with all the spin matrices $S^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Consequently, all the *continuous* d-dimensional Lorentz symmetries Spin$(d-1,1)$ commute with the Γ, and since Γ has two distinct eigenvalues ± 1 (with equal numbers of eigenspinors for each eigenvalue), the Dirac spinor representation is reducible, with one irreducible component for each eigenblock of Γ. Similar to 4D, these irreducible spinor representations are respectively the LH and the RH Weyl spinors.

In terms of d-dimensional spinor fields, the $2^{d/2}$ component Dirac spinor field splits into two $2^{(d-2)/2}$ components Weyl spinor fields,

$$\psi_L(x) = \frac{1-\Gamma}{2}\Psi(x), \qquad \psi_R(x) = \frac{1+\Gamma}{2}\Psi(x). \tag{14}$$

In particular, in the basis where

$$\Gamma = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \quad \text{in } 2^{(d-2)/2} \times 2^{(d-2)/2} \text{ blocks,} \tag{15}$$

$$\Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}. \tag{16}$$

But note that all of the above analysis depends on the Γ matrix which anticommutes with all the Dirac matrices γ^μ. Such a Γ matrix exists for any even spacetime dimension d, but it does not exist for an odd d. Consequently, *the Weyl spinor fields exist in even spacetime dimensions but not in odd dimensions!*

Finally, a few words about complex-conjugate Weyl spinor fields. In 4D, the conjugate of a LH Weyl spinor transforms equivalently to a RH Weyl spinor and vice verse, the conjugate of a RH Weyl spinor transforms equivalently to a LH Weyl spinor,

$$\psi_L^* \cong \psi_R\,, \quad \psi_R^* \cong \psi_L\,. \tag{17}$$

The same is true in other spacetime dimensions divisible by 4, $d = 4n = 4, 8, 12, \ldots$. Consequently, in all such dimensions a LH Weyl field (together with its conjugate) is physically equivalent to a RH Weyl field (together with its conjugate), so the same physical particle species (and the corresponding antiparticles) can be described in terms of either LH Weyl fields $\psi_L(x), \psi_L^\dagger(x)$ or in terms of RH Weyl fields $\psi_R(x), \psi_R^\dagger(x)$ of opposite charge(s), whichever is convenient. Or in terms of particle-antiparticle pairings, a LH fermions comes with a RH antifermion while a RH fermion comes with a LH antifermion.

However, in even spacetime dimensions not divisible by 4, — in $d = 4n - 2 = 2, 6, 10, \ldots$, — the conjugate of a LH spinor transform equivalently to a LH spinor, while the conjugate of a RH spinor transform equivalently to a RH spinor,

$$\psi_L^* \cong \psi_L\,, \quad \psi_R^* \cong \psi_R\,. \tag{18}$$

Consequently, in these dimensions, you cannot trade LH Weyl fields for RH Weyl fields of opposite charge(s), but you are stuck with a particular chirality for each fermion species.

Likewise, the particles and their antiparticles have similar 'helicity' states: either both left-handed or both right-handed.

In particular, in $d = 2$ — one space dimension and one time — the massless Weyl fermions move at the speed of light in a particular direction, depending on the Weyl spinor's chirality: the $\psi_L(x+t)$ move to the left while the $\psi_R(x-t)$ move to the right. And of course, a particle and its antiparticle always move in the same direction: both to the left, or both to the right,

Majorana Spinor Fields

A Majorana spinor has a s many components as a Dirac spinor, but these components are linearly related to each other's complex conjugates,

$$\Psi \;=\; C\Psi^* \tag{19}$$

for some matrix C such that

$$C^*C \;=\; 1 \quad \text{and} \quad C\gamma^\mu \;=\; -\gamma^{\mu*}C. \tag{20}$$

Consequently, a Majorana spinor field $\Psi(x) = C\Psi^*(x)$ has half as many degrees of freedom as a Dirac field, and its quanta are inherently neutral particles (same as their antiparticles).

The Majorana spinor fields exists in any spacetime dimension d for which we can find a matrix C obeying the criteria (20). Although the specific form of such a C matrix depends on a particular convention of the γ^μ matrices — in particular, on which γ^μ are real and which are imaginary — the existence or non-existence of the matrix C is convention independent. Instead, it depends only on the dimension d, or rather on d modulo 8! Specifically:

⋆ The C matrix and hence Majorana spinor fields exist for $d \equiv 0, 1, 2, 3, 4 \pmod 8$ but do not exist for $d \equiv 5, 6, 7 \pmod 8$. In other words, they exist for

$$d \;=\; 2, 3, 4;\; 8, 9, 10, 11, 12;\; 16, 17, 18, 19, 20;\; \ldots. \tag{21}$$

Some other features of Majorana spinor fields also depend on d modulo 8:

- In $d \equiv 0, 4 \pmod 8$ dimensions — *i.e.*, in $d = 4, 8, 12, 16, \ldots$ dimensions, — A massive Majorana field is physically equivalent to a neutral Weyl field — left-handed or right-handed — with a Majorana mass term. And a massless Majorana field with axial charge Q is equivalent to a massless LH Weyl field with charge $+Q$ or a massless RH Weyl field with charge $-Q$.
- In $d \equiv 1, 3 \pmod 8$ dimensions — *i.e.*, in $d = 3, 9, 11, 17, 19, \ldots$ dimensions — There are Dirac or Majorana spinor fields but there are no Weyl spinor fields, hence no Majorana/Weyl equivalence.
- Finally, in $d \equiv 2 \pmod 8$ dimensions — *i.e.*, in $d = 2, 10, 18, \ldots$ dimensions, there are both Majorana and Weyl spinor fields, and they are nor equivalent to each other. Instead, the Majorana condition $\Psi(x) = C\Psi^*(x)$ is independent from but compatible with the Weyl condition $\Gamma\Psi(x) = \pm\Psi(x)$. Consequently, both conditions can be imposed at the same time, which leads to the Majorana–Weyl spinor fields.

  Altogether, in $d \equiv 2 \pmod 8$ dimensions there are several kinds of spinors and spinor fields: Dirac, Majorana, Weyl (LH or RH), and Majorana–Weyl (LH or RH). So let me complete this section by counting the components (and the degrees of freedom) of all such spinor fields in $d = 2$ and $d = 10$ dimensions.

  * In $d = 1 + 1$ dimensions,

$$\begin{array}{lll} \Psi_{\text{Dirac}} & \text{has 2 complex components,} & 2\ \text{DoF} \\ \Psi_{\text{Majorana}} & \text{has 2 real components,} & 1\ \text{DoF} \\ \Psi_{\text{Weyl}} & \text{has 1 complex components,} & 1\ \text{DoF} \\ \Psi_{\text{MW}} & \text{has 1 real components,} & \tfrac{1}{2}\ \text{DoF} \end{array} \tag{22}$$

  * In $d = 9 + 1$ dimensions,

$$\begin{array}{lll} \Psi_{\text{Dirac}} & \text{has 32 complex components,} & 32\ \text{DoF} \\ \Psi_{\text{Majorana}} & \text{has 32 real components,} & 16\ \text{DoF} \\ \Psi_{\text{Weyl}} & \text{has 16 complex components,} & 16\ \text{DoF} \\ \Psi_{\text{MW}} & \text{has 16 real components,} & 8\ \text{DoF} \end{array} \tag{23}$$

Summary

Altogether, the existence of different spinor types — and the relations between them — depends on the spacetime dimension d modulo 8. This periodicity is related to the Bott periodicity of the homotopy groups of $SO(N)$ group manifolds, but the math of this relation is way beyond the scope of our QFT class. So instead of this math, let me simply give you the periodic table of spinor types in different dimensions:

$d \pmod 8$	spinor fields
0	Dirac, Majorana, Weyl (LH and RH), but $\mathrm{M} \cong \mathrm{W_L} \cong \mathrm{W_R}$.
1	Dirac and Majorana, but no Weyl.
2	Dirac, Majorana, Weyl (LH and RH), Majorana–Weyl (LH and RH), all inequivalent.
3	Dirac and Majorana, but no Weyl.
4	Dirac, Majorana, Weyl (LH and RH), but $\mathrm{M} \cong \mathrm{W_L} \cong \mathrm{W_R}$.
5	Dirac only — no Majorana, no Weyl.
6	Dirac and Weyl (LH and RH) but no Majorana; $W_L \ncong W_R$.
7	Dirac only — no Majorana, no Weyl.

Math: Real, Pseudoreal, and Complex Spinors

From the mathematical point of view, the existence of different types of spinors and the relations between them depends on the irreducible spinor representation of the d-dimensional Lorentz group $SO^+(d-1,1)$, or rather its double cover $\mathrm{Spin}(d-1,1)$. Specifically, is there just one irreducible spinor representation or are there two inequivalent irreducible spinors? Also, is this spinor representation (or representations) real, pseudo-real, or complex?

Before I answer these questions for the Lorentz groups, let me start with a few definitions.

- Two matrix representations (a) and (b) of the same group G,

$$g \mapsto M_a(g) \quad \text{and} \quad g \mapsto M_b(g) \tag{24}$$

  are called *equivalent* iff they has the same matrix size and there is a constant matrix

V such that for any $g \in G$

$$M_b(g) \;=\; V^{-1} M_a(g) V. \tag{25}$$

Note: same V for all group elements g.

In multiplet terms, equivalent representations describe symmetry transformations of the same multiplet but in different bases for the multiplet's components, and the matrix V relates the two bases to each other. For example, in a triplet multiplet of the $SO(3)$ rotation symmetry, we may use a basis of $|\ell, m\rangle$ states with $\ell = 1$ and $m = -1, 0, +1$, or we may use a basis of vector components x, y, z. The same rotation $R(\phi, \mathbf{n})$ is describes in one basis by the complex $\mathcal{D}^{(1)}_{m,m'}(R)$ matrices and in the other by the real R_{ij} matrices, but these two representations are equivalent to each other.

- A representation (r) is called *real* iff it's equivalent to a representation by real matrices. For example, all integer-j representations of the $SO(3)$ rotation group are real, since they are all equivalent to tensor representations. For example, members of a $j = 2$ multiplet transform into each other equivalently to the independent components of a real traceless symmetric 2-index tensor,

$$T_{ik} \;\mapsto\; T'_{ik} \;=\; R_{im} R_{kn} T_{kn} \,. \tag{26}$$

- A representation (s) is called *self-conjugate* iff it's equivalent to its complex conjugate, thus

$$M^*_s(g) \;=\; V^{-1} M_s(g) V, \qquad \text{same } V \text{ for all } M_s(g). \tag{27}$$

All real representations are self-conjugate, but there are self-conjugate representations that are not real; such self-conjugate but not real representations are called *pseudo-real.*

For example, all the half-integer-j representations of the Spin(3) rotation group are pseudo-real. In particular, the doublet representation of Spin(3) $= SU(2)$ is pseudo-real. Indeed, for any $SU(2)$ matrix U,

$$U^* \;=\; \sigma_2 U \sigma_2, \tag{28}$$

so the doublet **2** is a self-conjugate representation, but it's not equivalent to a representation by any real 2×2 matrices, so **2** is a pseudo-real rather than a real representation.

- Finally, a representation (c) is called *complex* iff it's not equivalent to its complex-conjugate representation $(\bar{c})$. For example, the fundamental **2** representation of the $SL(2, \mathbf{C})$ group is complex — it is not equivalent to the conjugate doublet $\bar{\mathbf{2}}$. Likewise, for all $SU(N)$ groups with $N \geq 3$, the fundamental representation **N** (by the column vector of N complex components) is complex: For $N \geq 3$, there is no matrix V such that

$$U^* \;=\; V^{-1} U V \quad \forall U \in SU(N). \tag{29}$$

(Unlike the $SU(2)$ group for which $V = \sigma_2$ does the job.)

Now with all these definitions in mind, let's classify the irreducible spinor representations of the Lorentz-like groups $SO^+(a,b)$ in a space and b time dimensions. For Physics purposes, we are primarily interested in the orthogonal $SO(N)$ groups of internal symmetries (rigid or gauged) and in the Lorentz groups $SO^+(d-1,1)$, but from the Mathematical point of view it's easier to consider the general $SO^+(a,b)$ case. While the size of an irreducible spinor representation of such a group depends only on the net dimension $d = a + b$,

$$\text{size} \;=\; \begin{cases} 2^{(d-2)/2} & \text{for even } d, \\ 2^{(d-1)/2} & \text{for odd } d, \end{cases} \tag{30}$$

the type of the irreducible spinor depends only on the *difference* $a - b$ *modulo 8.* Specifically,

$a - b \pmod 8$	irreducible spinor representations
0	two different real spinors R_1 and R_2, $\quad R_1 \not\cong R_2$
± 1	one real spinor R
± 2	two different complex spinors, C and its conjugate $\overline{C}$, $\quad C \not\cong \overline{C}$
± 3	one pseudoreal spinor P
4	two different pseudoreal spinors P_1 and P_2, $\quad P_1 \not\cong P_2$

Now let's apply this general table to the spinor fields in $d-1$ space dimensions and 1 time, thus $a - b = d - 2$. In general, a Majorana spinor field has real components, so it must be in a real representation of the Lorentz group. And the two Weyl spinors — the left-handed and the right-handed — must be in different representations of the Lorentz group. Therefore:

- For $d-2 \equiv 0 \pmod 8$ — *i.e.*, $d = 2, 10, 18, 26, \ldots,$ — two distinct and real spinor representations allow for the Majorana–Weyl spinor fields, left-handed Ψ_L^{MW} in the R_1 representation and right-handed Ψ_R^{MW} in the R_2 representation. The Majorana spinor

$$\Psi^M(x) \;=\; \begin{pmatrix} \Psi_L^{MW}(x) \\ \Psi_R^{MW}(x) \end{pmatrix} \tag{31}$$

  is in the reducible real $R_1 \oplus R_2$ representation. A Weyl spinor has complex components in a real representation of the Lorentz group, so it's physically equivalent to two real Majorana–Weyl spinors of the same chirality,

$$\Psi_L^W(x) \;=\; \Psi_{L1}^{MW}(x) \;+\; i\Psi_{L2}^{MW}(x), \qquad \Psi_R^W(x) \;=\; \Psi_{R1}^{MW}(x) \;+\; i\Psi_{R2}^{MW}(x). \tag{32}$$

  Finally, a Dirac spinor

$$\Psi^D(x) \;=\; \begin{pmatrix} \Psi_L^W(x) \\ \Psi_R^W(x) \end{pmatrix} \;=\; \Psi_1^M(x) \;+\; i\Psi_2^M(x) \tag{33}$$

  has complex components in the reducible real $R_1 \oplus R_2$ representation. Physically, 1 Dirac spinor field is equivalent to 2 Majorana fields, or two Weyl fields (1LH + 1RH), or two 4 Majorana–Weyl fields (2LH + 2RH).

- For $d-2 \equiv \pm 1 \pmod 8$ — *i.e.*, $d = 3, 9, 11, 17, 19, \ldots,$ — the only irreducible spinor representation allows for the Majorana spinor field $\Psi^M(x)$, but there are no Weyl or Majorana-Weyl fields. As to the Dirac spinor field, it has complex components in a real representation R, so it's equivalent to two Majorana fields,

$$\Psi^D(x) \;=\; \Psi_1^M(x) \;+\; i\Psi_2^M(x). \tag{34}$$

- For $d-2 \equiv \pm 2 \pmod 8$ — *i.e.*, $d = 4, 8, 12, 16, \ldots$ — two distinct spinor representations C and $\overline{C}$ allow for the Weyl spinor fields $\Psi_L^W(x)$ and $\Psi_R^W(x)$, and since the two representations are complex conjugate to each other, the LH Weyl spinor is physically equivalent to the RH Weyl spinor with opposite charges, and vice verse. Also, the reducible spinor representation $C \oplus \overline{C}$ is real, which allows for Majorana spinor fields, but any such field is physically equivalent to a Weyl field (LH or RH) together with its conjugate, thus

$$\Psi^M \;\cong\; \Psi_L^W \;+\; \left(\Psi_L^W\right)^\dagger \;\cong\; \Psi_R^W \;+\; \left(\Psi_R^W\right)^\dagger . \tag{35}$$

Lastly, a Dirac spinor field is a combination of two Weyl spinor fields of opposite chiralities,

$$\Psi^D(x) \;=\; \begin{pmatrix} \Psi_L^W(x) \\ \Psi_R^W(x) \end{pmatrix} . \tag{36}$$

- For $d-2 \equiv \pm 3 \pmod 8$ — *i.e.*, $d = 5, 7, 13, 15, \ldots$ — there is only one irreducible spinor representation P, hence no Weyl spinor fields, and since P is pseudoreal rather than real, there are no Majorana fields either. Thus, in these dimensions there are Dirac spinor fields but no other kinds of spinors.

- Finally, for $d-2 \equiv 4 \pmod 8$ — *i.e.*, $d = 6, 14, 22, \ldots$ — the two distinct irreducible spinor representations P_1 and P_2 give rise to the Weyl spinor fields $\Psi_L^W(x)$ and $\Psi_R^W(x)$, but since P_1 and P_2 are pseudoreal rather than real (and so is $P_1 \oplus P_2$), there are no Majorana or Majorana–Weyl spinor fields. Also, since P_1 and P_2 are not complex conjugates of each other, the LH and the RH Weyl spinor fields are *not* equivalent to each other conjugates. Instead

$$\left(\Psi_L^W\right)^\dagger \;\cong\; \Psi_L^W \;\ncong\; \Psi_R^W \quad \text{and} \quad \left(\Psi_R^W\right)^\dagger \;\cong\; \Psi_R^W \;\ncong\; \Psi_L^W . \tag{37}$$

Lastly, a Dirac spinor field is a combination of two Weyl spinor fields of opposite chiralities,

$$\Psi^D(x) \;=\; \begin{pmatrix} \Psi_L^W(x) \\ \Psi_R^W(x) \end{pmatrix} . \tag{38}$$