

Problem 1(a):

To simplify the $\exp(\xi\hat{a}^\dagger - \xi^*\hat{a})$ in the definition of a coherent state $|\xi\rangle$, we use the product-of-exponentials formula

$$\forall \hat{A}, \hat{B}: \quad e^{\hat{A}}e^{\hat{B}} = \exp\left(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}[(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] + \dots\right). \quad (\text{S.1})$$

In particular, for $\hat{A} = \xi\hat{a}^\dagger$, $\hat{B} = -\xi^*\hat{a}$ and $[\hat{A}, \hat{B}] = \xi\xi^*$ being a c-number, all the multiple commutators vanish and

$$e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}} = \exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a} + \frac{1}{2}\xi\xi^*\right), \quad \text{exactly.} \quad (\text{S.2})$$

Consequently

$$|\xi\rangle \stackrel{\text{def}}{=} e^{\xi\hat{a}^\dagger - \xi^*\hat{a}} |0\rangle = e^{-|\xi|^2/2} e^{\xi\hat{a}^\dagger} e^{-\xi^*\hat{a}} |0\rangle = e^{-|\xi|^2/2} e^{\xi\hat{a}^\dagger} |0\rangle, \quad (\text{S.3})$$

where the last equality follows from $\hat{a}|0\rangle = 0$ and hence $\exp(-\xi^*\hat{a})|0\rangle = |0\rangle$.

Next, let's show that

$$e^{\xi\hat{a}^\dagger}\hat{a}e^{-\xi\hat{a}^\dagger} = \hat{a} - \xi. \quad (\text{S.4})$$

Indeed, the Campbell identity for any operators or matrices X and Y says that

$$\exp(+X)Y\exp(-X) = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{6}[X, [X, [X, Y]]] + \dots \quad (\text{S.5})$$

But for $X = \xi\hat{a}^\dagger$ and $Y = \hat{a}$ we have $[X, Y] = -\xi$, and since this is a c-number all the multiple commutators vanish. Consequently,

$$\exp(\xi\hat{a}^\dagger)\hat{a}\exp(-\xi\hat{a}^\dagger) = \hat{a} - \xi + 0, \quad (\text{S.6})$$

exactly as in eq. (S.4).

In light of this formula,

$$(\hat{a} - \xi) |\xi\rangle = e^{\xi \hat{a}^\dagger} \hat{a} e^{-\xi \hat{a}^\dagger} \times e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} \times \hat{a} |0\rangle = 0 \quad (\text{S.7})$$

where the last equality follows from $\hat{a} |0\rangle = 0$. Thus, the coherent state $|\xi\rangle$ indeed satisfies

$$(\hat{a} - \xi) |\xi\rangle = 0 \implies \hat{a} |\xi\rangle = \xi |\xi\rangle. \quad (\text{S.8})$$

BTW, the coherent states are often *defined* by the condition $\hat{a} |\xi\rangle = \xi |\xi\rangle$. We may formally prove the existence of such a coherent state for any complex number ξ by noting that the operators $\hat{a}' = \hat{a} - \xi$ and $\hat{a}'^\dagger = \hat{a}^\dagger - \xi^*$ satisfy the same commutation relation $[\hat{a}', \hat{a}'^\dagger] = 1$ as the $\hat{a}$ and $\hat{a}^\dagger$ operators. Consequently, the same formal argument that proves the existence of the ground state $|0\rangle$ annihilated by the $\hat{a}$ operator also prove the existence of the state $|\xi\rangle$ annihilated by the $\hat{a}' = \hat{a} - \xi$. Alas, the formal proof of existence does not tell us what that state looks like, so the explicit construction used in this problem provides the missing description.

Problem 1(b):

In the coordinate basis, the annihilation operator $\hat{a}$ acts as

$$\hat{a} = \frac{\omega m \hat{x} + i \hat{p}}{\sqrt{2\hbar\omega m}} = \frac{\omega m \hat{x} + \hbar \partial_x}{\sqrt{2\hbar\omega m}} \quad (\text{S.9})$$

and the condition $\hat{a} |\xi\rangle = \xi |\xi\rangle$ becomes a first-order differential equation

$$\left(\hbar \frac{d}{dx} + \omega m \times x - \sqrt{2\hbar\omega m} \times \xi \right) \psi_\xi(x) = 0 \quad (\text{S.10})$$

for the wave-function $\psi_\xi(x)$ of the coherent state. This equation has a unique solution (up to an overall normalization), namely

$$\psi_\xi(x) = \text{const} \times \exp \left(\xi \sqrt{\frac{2m\omega}{\hbar}} \times x - \frac{m\omega}{2\hbar} \times x^2 \right), \quad (\text{S.11})$$

or equivalently,

$$\psi_\xi(x) = \text{const} \times e^{i\bar{p}x/\hbar} \times e^{-m\omega(x-\bar{x})^2/2\hbar}, \quad (\text{S.12})$$

a Gaussian wave-packet with

$$\bar{x} = \sqrt{\frac{2\hbar}{\omega m}} \times \text{Re} \xi \quad \text{and} \quad \bar{p} = \sqrt{2\hbar\omega m} \times \text{Im} \xi. \quad (\text{S.13})$$

Note that the width of the wave packet (S.12) does not depend on ξ , so all coherent states have the same Δx . In particular, since $|\xi = 0\rangle$ is the oscillator's ground state, all coherent states have the same width as the ground state.

Problem 1(c):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has

$$\langle \xi | (\hat{a}^\dagger)^k (\hat{a})^\ell | \xi \rangle = (\xi^*)^k \xi^\ell, \quad (\text{S.14})$$

simply because $\hat{a} |\xi\rangle = \xi |\xi\rangle \implies (\hat{a})^\ell |\xi\rangle = \xi^\ell |\xi\rangle$ and $\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi | \implies \langle \xi | (\hat{a}^\dagger)^k = (\xi^*)^k \langle \xi |$. In particular, $\langle \xi | (\hat{n} = \hat{a}^\dagger \hat{a}) | \xi \rangle = \xi^* \xi$. On the other hand,

$$\hat{n}^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} \implies \langle \xi | \hat{n}^2 | \xi \rangle = (\xi^*)^2 \xi^2 + \xi^* \xi = \bar{n}^2 + \bar{n} \quad (\text{S.15})$$

hence $\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \bar{n}^2} = \sqrt{\bar{n}}$.

In a similar manner,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{x}^2 = \frac{\hbar}{2m\omega} \left((\hat{a})^2 + (\hat{a}^\dagger)^2 + 2\hat{a}^\dagger \hat{a} + 1 \right), \quad (\text{S.16})$$

hence

$$\langle \xi | \hat{x}^2 | \xi \rangle = \frac{\hbar}{2m\omega} ((\xi + \xi^*)^2 + 1) = \langle \xi | \hat{x} | \xi \rangle^2 + \frac{\hbar}{2m\omega}. \quad (\text{S.17})$$

Likewise,

$$\langle \xi | \hat{p}^2 | \xi \rangle = \frac{m\omega\hbar}{2} ((-i\xi + i\xi^*)^2 + 1) = \langle \xi | \hat{p} | \xi \rangle^2 + \frac{m\omega\hbar}{2}.$$

Altogether, this gives us for any coherent state

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}, \quad \Delta p = \sqrt{\frac{m\omega\hbar}{2}}, \quad \Delta x \times \Delta p = \frac{\hbar}{2}. \quad (\text{S.18})$$

Q.E.D.

Problem 1(d):

In a classical harmonic oscillator, the position $\bar{x}(t)$ and the momentum $\bar{p}(t)$ oscillate as

$$\begin{aligned} \bar{x}(t) &= \bar{x}(0) \times \cos(\omega t) + \frac{\bar{p}(0)}{m\omega} \times \sin(\omega t), \\ \bar{p}(t) &= \bar{p}(0) \times \cos(\omega t) - m\omega \bar{x}(0) \times \sin(\omega t). \end{aligned} \quad (\text{S.19})$$

Consequently,

$$\begin{aligned} \xi(t) &= \frac{m\omega \bar{x}(t) + i\bar{p}(t)}{\sqrt{2\hbar\omega m}} = \frac{m\omega \bar{x}(0) + i\bar{p}(0)}{\sqrt{2\hbar\omega m}} \times \cos(\omega t) + \frac{-im\omega \bar{x}(0) + \bar{p}(0)}{\sqrt{2\hbar\omega m}} \times \sin(\omega t) \\ &= \frac{m\omega \bar{x}(0) + i\bar{p}(0)}{\sqrt{2\hbar\omega m}} \times e^{-i\omega t} = \xi(0) \times e^{-i\omega t}. \end{aligned} \quad (\text{S.20})$$

Now consider the quantum state $|\xi(t)\rangle$ for the classically oscillating $\xi(t) = \xi_0 \times e^{-i\omega t}$. In light of eq. (1),

$$|\xi(t)\rangle = e^{-|\xi|^2/2} e^{\xi(t)\hat{a}^\dagger} |0\rangle, \quad (\text{S.21})$$

and only the second factor here depends on time. Indeed, $|\xi(t)|^2 = |\xi_0|^2 = \text{const} \implies e^{-|\xi|^2/2} = \text{const}$, while $|0\rangle$ is time-independent because we work in the Schrödinger picture. In this picture, the $\hat{a}^\dagger$ operator is also time independent, hence

$$\frac{d}{dt} e^{\xi\hat{a}^\dagger} = \frac{d\xi}{dt} \hat{a}^\dagger \times e^{\xi\hat{a}^\dagger} = -i\omega \xi \hat{a}^\dagger \times e^{\xi\hat{a}^\dagger}, \quad (\text{S.22})$$

and therefore

$$\frac{d}{dt} |\xi\rangle = -i\omega \xi \hat{a}^\dagger |\xi\rangle = -i\omega \hat{a}^\dagger \hat{a} |\xi\rangle \quad (\text{S.23})$$

where the second equality follows from $\xi |\xi\rangle = \hat{a} |\xi\rangle$. Consequently,

$$i\hbar \frac{d}{dt} |\xi(t)\rangle = \hbar\omega \hat{a}^\dagger \hat{a} |\xi(t)\rangle \equiv \hat{H} |\xi(t)\rangle \quad (\text{S.24})$$

— the time-dependent coherent state $|\xi(t)\rangle$ obeys the Schrödinger equation. *Q.E.D.*

Problem 1(e):

For any analytic function $f(\hat{a})$ of the operator $\hat{a}$, we have

$$[\hat{a}^\dagger, f(\hat{a})] = -f'(\hat{a}) \quad (\text{S.25})$$

(which follows from expanding $f(\hat{a})$ into a power series in $\hat{a}$), and consequently

$$[\hat{a}^\dagger, [\hat{a}^\dagger, f(\hat{a})]] = +f''(\hat{a}), \quad [\hat{a}^\dagger, [\hat{a}^\dagger, [\hat{a}^\dagger, f(\hat{a})]]] = -f'''(\hat{a}), \quad \text{etc.} \quad (\text{S.26})$$

Thus, by the Campbell identity (S.5),

$$\begin{aligned} \exp(\xi \hat{a}^\dagger) f(\hat{a}) \exp(-\xi \hat{a}^\dagger) &= f(\hat{a}) + \xi [\hat{a}^\dagger, f(\hat{a})] + \frac{\xi^2}{2} [\hat{a}^\dagger, [\hat{a}^\dagger, f(\hat{a})]] \\ &\quad + \frac{\xi^3}{6} [\hat{a}^\dagger, [\hat{a}^\dagger, [\hat{a}^\dagger, f(\hat{a})]]] + \dots \\ &= f(a) - \xi \times f'(\hat{a}) + \frac{\xi^2}{2} \times f''(\hat{a}) - \frac{\xi^3}{6} \times f'''(\hat{a}) + \dots \\ &= f(\hat{a} - \xi). \end{aligned} \quad (\text{S.27})$$

In particular,

$$\exp(\xi \hat{a}^\dagger) \times \exp(\eta^* \hat{a}) \times \exp(-\xi \hat{a}^\dagger) = \exp(\eta^* \hat{a} - \eta^* \xi), \quad (\text{S.28})$$

or equivalently

$$\exp(\xi \hat{a}^\dagger) \times \exp(\eta^* \hat{a}) = \exp(-\eta^* \xi) \times \exp(\eta^* \hat{a}) \times \exp(\xi \hat{a}^\dagger). \quad (\text{S.29})$$

Consequently, the quantum overlap of the coherent states $|\xi\rangle$ and $\langle\eta|$ is

$$\begin{aligned} \langle\eta|\xi\rangle &= e^{-|\eta|^2/2} e^{-|\xi|^2/2} \times \langle 0 | \exp(\eta^* \hat{a}) \exp(\xi \hat{a}^\dagger) | 0 \rangle \\ &= e^{-|\eta|^2/2} e^{-|\xi|^2/2} e^{+\eta^* \xi} \langle 0 | \exp(\xi \hat{a}^\dagger) \exp(\eta^* \hat{a}) | 0 \rangle \\ &= \exp\left(-\frac{1}{2}|\eta|^2 - \frac{1}{2}|\xi|^2 + \eta^* \xi\right) \times 1 \end{aligned} \quad (\text{S.30})$$

because $e^{\eta^* \hat{a}} | 0 \rangle = | 0 \rangle$, $\langle 0 | e^{\xi \hat{a}^\dagger} = \langle 0 |$, and $\langle 0 | 0 \rangle = 1$. In terms of the probability overlap,

$$|\langle\eta|\xi\rangle|^2 = \exp(-|\eta - \xi|^2). \quad (\text{S.31})$$

Problem 1(f):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \stackrel{\text{def}}{=} \exp(\hat{F}^\dagger - \hat{F}) |0\rangle = e^{-\bar{N}/2} e^{\hat{F}^\dagger} |0\rangle \quad (\text{S.32})$$

where

$$\hat{F}^\dagger = \xi \hat{a}^\dagger \rightarrow \sum_{\alpha} \xi_{\alpha} \hat{a}_{\alpha}^\dagger \rightarrow \int d^3\mathbf{x} \Phi(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}). \quad (\text{S.33})$$

Similar to the single-oscillator theory, $(\hat{\Psi}(\mathbf{x}) - \Phi(\mathbf{x}))e^{\hat{F}^\dagger} = e^{\hat{F}^\dagger} \hat{\Psi}^\dagger(\mathbf{x})$, hence

$$\hat{\Psi}(\mathbf{x}) |\Phi\rangle = \Phi(\mathbf{x}) |\Phi\rangle. \quad (\text{S.34})$$

Problem 1(g):

Using eq. (S.34) and its hermitian conjugate, we have

$$\langle \Phi | \hat{\Psi}^\dagger(\mathbf{x}_1) \cdots \hat{\Psi}^\dagger(\mathbf{x}_k) \hat{\Psi}(\mathbf{y}_1) \cdots \hat{\Psi}(\mathbf{y}_\ell) | \Phi \rangle = \Phi^*(\mathbf{x}_1) \cdots \Phi^*(\mathbf{x}_k) \Phi(\mathbf{y}_1) \cdots \Phi(\mathbf{y}_\ell) \quad (\text{S.35})$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator $\hat{N}$ we have eq. (4), while for its square — whose normal-ordered form

$$\hat{N}^2 = \iint d^3\mathbf{x} d^3\mathbf{y} \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{y}) \hat{\Psi}(\mathbf{x}) \hat{\Psi}(\mathbf{y}) + \int d^3\mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}(\mathbf{x}) \quad (\text{S.36})$$

generalizes eq. (S.15) — we have

$$\langle \Phi | \hat{N}^2 | \Phi \rangle = \iint d^3\mathbf{x} d^3\mathbf{y} \Phi^*(\mathbf{x}) \Phi^*(\mathbf{y}) \Phi(\mathbf{x}) \Phi(\mathbf{y}) + \int d^3\mathbf{x} \Phi^*(\mathbf{x}) \Phi(\mathbf{x}) = \langle \Phi | \hat{N} | \Phi \rangle^2 + \langle \Phi | \hat{N} | \Phi \rangle, \quad (\text{S.37})$$

and hence $\Delta N = \sqrt{\bar{N}}$. *Q.E.D.*

Problem 1(h):

First of all, if $\Phi(\mathbf{x}, t)$ satisfies the classical field equation (6) — which looks exactly like a one-particle Schrödinger equation — then $\bar{N}$ remains constant. (This is undergraduate-level QM.) Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt} e^{\hat{F}^\dagger} = \frac{d\hat{F}^\dagger}{dt} e^{\hat{F}^\dagger} = \left[\int d^3\mathbf{x} \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} \hat{\Psi}^\dagger(\mathbf{x}) \right] e^{\hat{F}^\dagger} \quad (\text{S.38})$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in part (e),

$$\begin{aligned} i\hbar \frac{d}{dt} \left(|\Phi\rangle = e^{-\bar{N}/2} e^{\hat{F}^\dagger} |0\rangle \right) &= \left[\int d^3\mathbf{x} i\hbar \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} \hat{\Psi}^\dagger(\mathbf{x}) \right] |\Phi\rangle \\ &\ll \text{using eq. (6)} \gg \\ &= \left[\int d^3\mathbf{x} \left(\left(\frac{-\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \Phi(\mathbf{x}) \right) \hat{\Psi}^\dagger(\mathbf{x}) \right] |\Phi\rangle \\ &\ll \text{using eq. (3)} \gg \\ &= \left[\int d^3\mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) \left(\frac{-\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \hat{\Psi}(\mathbf{x}) \right] |\Phi\rangle \\ &= \hat{H} |\Phi\rangle. \end{aligned} \quad (\text{S.39})$$

Q.E.D.

Problem 1(i):

Generalizing (e) to multi-oscillatory systems is completely straightforward:

$$|\langle \eta | \xi \rangle|^2 = \prod_{\alpha} e^{-|\xi_{\alpha} - \eta_{\alpha}|^2} = \exp \left(- \sum_{\alpha} |\xi_{\alpha} - \eta_{\alpha}|^2 \right)$$

or for the field theory,

$$|\langle \Phi_1 | \Phi_2 \rangle|^2 = \exp \left(\int d^3\mathbf{x} |\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})|^2 \right), \quad (\text{S.40})$$

which is exponentially small for any macroscopic $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$. Indeed, a *macroscopic* difference between two coherent states means (by definition) that $\delta\Phi$ affects a large number of particles, $\int |\delta\Phi|^2 \gg 1$, which makes for an *exponentially* tiny overlap (S.40).