

Problem 1(a):

In 3D notations (but $\hbar = c = 1$ units), the Lagrangian density (1) for the massive vector field is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2) - J_0A_0 + \mathbf{J} \cdot \mathbf{A} \\ &= \frac{1}{2}(-\dot{\mathbf{A}} - \nabla A_0)^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2 + \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2) - J_0A_0 + \mathbf{J} \cdot \mathbf{A}.\end{aligned}\quad (\text{S.1})$$

Note that only the first term on the last line contains any time derivatives at all, and it does not contain the $\dot{A}_0$ but only the $\dot{\mathbf{A}}$. Consequently, $\partial\mathcal{L}/\partial\dot{A}_0 = 0$ and the scalar potential $A_0(\mathbf{x})$ does not have a canonical conjugate field. On the other hand, the vector potential $\mathbf{A}(\mathbf{x})$ does have a canonical conjugate, namely

$$\frac{\delta L}{\delta \dot{\mathbf{A}}(x)} = \left. \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} \right|_{\mathbf{x}} = -(-\dot{\mathbf{A}}(\mathbf{x}) - \nabla A_0(\mathbf{x})) = -\mathbf{E}(\mathbf{x}). \quad (\text{S.2})$$

Problem 1(b):

In terms of the Hamiltonian and Lagrangian densities, eq. (2) means

$$\mathcal{H} = -\dot{\mathbf{A}} \cdot \mathbf{E} - \mathcal{L}. \quad (2')$$

Expressing all fields in terms $\mathbf{A}$, $\mathbf{E}$, and A_0 , we get

$$\begin{aligned}\dot{\mathbf{A}} &= -\mathbf{E} - \nabla A_0, \\ -\dot{\mathbf{A}} \cdot \mathbf{E} &= \mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0, \\ \mathcal{L} &= \frac{1}{2}(\mathbf{E}^2 - (\nabla \times \mathbf{A})^2) + \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2) - (A_0J_0 - \mathbf{A} \cdot \mathbf{J}),\end{aligned}\quad (\text{S.3})$$

and consequently,

$$\mathcal{H} = \frac{1}{2}\mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0 - \frac{1}{2}m^2A_0^2 + A_0J_0 + \frac{1}{2}(\nabla \times \mathbf{A})^2 + \frac{1}{2}m^2\mathbf{A}^2 - \mathbf{A} \cdot \mathbf{J}. \quad (\text{S.4})$$

Taking the $\int d^3\mathbf{x}$ integral of this density and integrating by parts the $\mathbf{E} \cdot \nabla A_0$ term, we arrive at the Hamiltonian (3). *Quod erat demonstrandum.*

Problem 1(c):

Evaluating the derivatives of $\mathcal{H}$ in eq. (4) gives us

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(A_0)} - \nabla_i \frac{\partial \mathcal{H}}{\partial(\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i. \quad (\text{S.5})$$

Were there a canonical conjugate $\pi_0(\mathbf{x}, t)$ of the $A_0(\mathbf{x}, t)$, its time derivative $\partial\pi_0/\partial t$ would be given by the right hand side of eq. (S.5). But the $A_0(\mathbf{x}, t)$ does not have a canonical conjugate, so instead of a Hamilton equation of motion we have a time-independent *constraint* (4), namely

$$m^2 A_0 = J_0 - \nabla \cdot \mathbf{E}. \quad (\text{S.6})$$

In the massless EM case, a similar constraint gives rise to the Gauss Law $\nabla \cdot \mathbf{E} = J_0$. But the massive vector field does not obey the Gauss Law; instead, eq. (S.6) gives us a formula for the scalar potential A_0 in terms of $\mathbf{E}$ and J_0 .

On the other hand, the Hamilton equations for the vector fields $\mathbf{A}$ and $\mathbf{E}$ are honest equations of motion. Specifically, evaluating the derivatives of $\mathcal{H}$ in eq. (5), we find

$$\frac{\delta H}{\delta E^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(E^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j E^i)} = E^i + \nabla_i A_0, \quad (\text{S.7})$$

which leads to Hamilton equation

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}, t) = -\mathbf{E}(\mathbf{x}, t) - \nabla A_0(\mathbf{x}, t). \quad (\text{S.8})$$

Similarly, in eq. (6) we have

$$\frac{\delta H}{\delta A^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(A^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j (\epsilon^{jik} (\nabla \times \mathbf{A})^k) \quad (\text{S.9})$$

and hence Hamilton equation

$$\frac{\partial}{\partial t} \mathbf{E}(\mathbf{x}, t) = m^2 \mathbf{A} - \mathbf{J} + \nabla \times (\nabla \times \mathbf{A}). \quad (\text{S.10})$$

Problem 1(d):

In 3D notations, the Euler–Lagrange field equations (7) or $\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu$ become

$$\nabla \cdot \mathbf{E} + m^2 A^0 = J^0, \quad (\text{S.11})$$

$$-\dot{\mathbf{E}} + \nabla \times \mathbf{B} + m^2 \mathbf{A} = \mathbf{J}, \quad (\text{S.12})$$

where

$$\mathbf{E} \stackrel{\text{def}}{=} -\dot{\mathbf{A}} - \nabla A^0, \quad (\text{S.13})$$

$$\mathbf{B} \stackrel{\text{def}}{=} \nabla \times \mathbf{A}. \quad (\text{S.14})$$

Clearly, eq. (S.11) is equivalent to eq. (S.6) while eq. (S.12) is equivalent to eq. (S.10) (provided $\mathbf{B}$ is defined as in eq. (S.14)). Finally, eq. (S.13) is equivalent to eq. (S.8), although their origins differ: In the Lagrangian formalism, eq. (S.13) is the definition of the $\mathbf{E}$ field in terms of A_0 , $\mathbf{A}$ and their derivatives, while in the Hamiltonian formalism, $\mathbf{E}$ is an independent conjugate field and eq. (S.8) is the dynamical equation of motion for the $\dot{\mathbf{A}}$. *Quod erat demonstrandum.*

Problem 2:

Let start with the $[\hat{\mathbf{A}}, \hat{H}]$ commutator. In light of eq. (11) for the Hamiltonian, we have

$$[\hat{A}^i(\mathbf{x}), \hat{H}] = \int d^3\mathbf{y} \left[\hat{A}^i(\mathbf{x}), \left(\frac{1}{2} \hat{\mathbf{E}}^2 + \frac{1}{2m^2} (\hat{J}_0 - \nabla \cdot \hat{\mathbf{E}})^2 + \frac{1}{2} (\nabla \times \hat{\mathbf{A}})^2 + \frac{1}{2} m^2 \hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right)_{@ \mathbf{y}} \right] \quad (\text{S.15})$$

where all operators are at the same time t as the $\hat{A}^i(\mathbf{x}, t)$. Since all the $\hat{A}^i(\mathbf{x})$ operators commute with each other at equal times, the last three terms in the Hamiltonian density do not contribute to the commutator (S.15). But for the first term we have

$$\begin{aligned} [\hat{A}^i(\mathbf{x}), \frac{1}{2} \hat{\mathbf{E}}^2(\mathbf{y})] &= \frac{1}{2} \{ \hat{E}^j(\mathbf{y}), [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \} \\ &= \frac{1}{2} \{ \hat{E}^j(\mathbf{y}), -i \delta^{ij} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \} \\ &= -i \delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \hat{E}^i(\mathbf{y}), \end{aligned} \quad (\text{S.16})$$

while for the second term we have

$$\left[\hat{A}^i(\mathbf{x}), \left(\hat{J}_0(\mathbf{y}) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{y}) \right) \right] = 0 - \frac{\partial}{\partial y^j} [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] = +i\delta^{ij} \frac{\partial}{\partial y^j} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (\text{S.17})$$

and hence

$$\begin{aligned} \left[\hat{A}^i(\mathbf{x}), \frac{1}{2m^2} \left(\hat{J}_0(\mathbf{y}) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{y}) \right)^2 \right] &= \frac{1}{m^2} \left(\hat{J}_0(\mathbf{y}) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{y}) \right) \times +i\delta^{ij} \frac{\partial}{\partial y^j} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \hat{A}^0(\mathbf{y}) \times i \frac{\partial}{\partial y^i} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (\text{S.18})$$

where the second equality follows from eq. (10). Plugging these all these commutators into eq. (S.15) and integrating over $\mathbf{y}$, we obtain

$$\begin{aligned} [\hat{A}^i(\mathbf{x}), \hat{H}] &= \int d^3\mathbf{y} \left(-i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \hat{E}^i(\mathbf{y}) + \hat{A}^0(\mathbf{y}) \times i \frac{\partial}{\partial y^i} \delta^{(3)}(\mathbf{x} - \mathbf{y}) + 0 + 0 + 0 \right) \\ &\quad \langle\langle \text{integrating by parts} \rangle\rangle \\ &= \int d^3\mathbf{y} (-i)\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \left(\hat{E}^i(\mathbf{y}) + \frac{\partial}{\partial y^i} \hat{A}^0(\mathbf{y}) \right) \\ &= -i \left(\hat{E}^i(\mathbf{x}) + \frac{\partial}{\partial x^i} \hat{A}^0(\mathbf{x}) \right). \end{aligned} \quad (\text{S.19})$$

In other words, $[\hat{\mathbf{A}}(\mathbf{x}), \hat{H}] = -i\hat{\mathbf{E}}(\mathbf{x}) - i\nabla \hat{A}^0(\mathbf{x})$ and consequently in the Heisenberg picture,

$$\frac{\partial}{\partial t} \hat{\mathbf{A}}(\mathbf{x}, t) = -i[\hat{\mathbf{A}}(\mathbf{x}), \hat{H}] = -\hat{\mathbf{E}}(\mathbf{x}, t) - \nabla \hat{A}^0(\mathbf{x}, t). \quad (\text{S.20})$$

Clearly, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.8).

Now consider the $[\hat{\mathbf{E}}, \hat{H}]$ commutator. Similarly to eq.(S.15), we have

$$[\hat{E}^i(\mathbf{x}, t), \hat{H}] = \int d^3\mathbf{y} \left[\hat{E}^i(\mathbf{x}, t), \left(\frac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2}m^2\hat{A}_0^2 + \frac{1}{2}\hat{\mathbf{B}}^2 + \frac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right)(\mathbf{y}, t) \right] \quad (\text{S.21})$$

where $m^2\hat{A}^0 = \hat{J}^0 - \nabla \cdot \hat{\mathbf{E}}$ according to eq. (12) and $\hat{\mathbf{B}} \stackrel{\text{def}}{=} \nabla \times \hat{\mathbf{A}}$. At equal times, the $\hat{E}^i(\mathbf{x})$ operator commutes with all the $\hat{E}^j(\mathbf{y})$ and hence with the $\hat{\mathbf{E}}^2(\mathbf{y})$, and also with the $\hat{A}_0(\mathbf{y})$

and hence with the $\hat{A}_0^2(\mathbf{y})$; this eliminates the first two terms in the Hamiltonian density from the commutator (S.21). For the remaining three terms we have

$$\begin{aligned}
\left[\hat{E}^i(\mathbf{x}), (-\hat{\mathbf{J}} \cdot \hat{\mathbf{A}})(\mathbf{y}) \right] &= -\hat{J}^j(\mathbf{y}) \times \left[\hat{E}^i(\mathbf{x}), \hat{A}^j(\mathbf{y}) \right] \\
&= -\hat{J}^j(\mathbf{y}) \times +i\delta^{ij}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= -i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \hat{J}^i(\mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \frac{1}{2}m^2\hat{\mathbf{A}}^2(\mathbf{y}) \right] &= +im^2\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \hat{A}^i(\mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \hat{B}^j(\mathbf{y}) \right] &= \epsilon^{jkl} \frac{\partial}{\partial y^k} \left[\hat{E}^i(\mathbf{x}), \hat{A}^l(\mathbf{y}) \right] \\
&= \epsilon^{jkl} \frac{\partial}{\partial y^k} \left(+i\delta^{il}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \right) \\
&= +i\epsilon^{jki} \frac{\partial}{\partial y^k} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \frac{1}{2}\hat{\mathbf{B}}^2(\mathbf{y}) \right] &= \hat{B}^j(\mathbf{y}) \times +i\epsilon^{jki} \frac{\partial}{\partial y^k} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= -i\epsilon^{jki} \frac{\partial}{\partial y^k} \hat{B}^j(\mathbf{y}) \times \delta^{(3)}(\mathbf{x} - \mathbf{y}) + \text{a total derivative} \\
&= +i(\nabla \times \hat{\mathbf{B}})^i(\mathbf{y}) \times \delta^{(3)}(\mathbf{x} - \mathbf{y}) + \text{a total derivative.}
\end{aligned} \tag{S.22}$$

Thus

$$\left[\hat{\mathbf{E}}(\mathbf{x}), \hat{\mathcal{H}}(\mathbf{y}) \right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \left(\nabla \times \hat{\mathbf{B}}(\mathbf{y}) + m^2\hat{\mathbf{A}}(\mathbf{y}) - \hat{\mathbf{J}}(\mathbf{y}) \right) + \text{a total derivative}, \tag{S.23}$$

hence

$$\begin{aligned}
\left[\hat{\mathbf{E}}(\mathbf{x}), \hat{H} \right] &= \int d^3\mathbf{y} \left(i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \left(\nabla \times \hat{\mathbf{B}}(\mathbf{y}) + m^2\hat{\mathbf{A}}(\mathbf{y}) - \hat{\mathbf{J}}(\mathbf{y}) \right) \right. \\
&\quad \left. + \text{a total derivative} \right) \\
&= i \left(\nabla \times \hat{\mathbf{B}}(\mathbf{x}) + m^2\hat{\mathbf{A}}(\mathbf{x}) - \hat{\mathbf{J}}(\mathbf{x}) \right),
\end{aligned} \tag{S.24}$$

and therefore in the Heisenberg picture

$$\frac{\partial}{\partial t} \hat{\mathbf{E}}(\mathbf{x}, t) = -i \left[\hat{\mathbf{E}}(\mathbf{x}, t), \hat{H} \right] = +\nabla \times \hat{\mathbf{B}}(\mathbf{x}) + m^2\hat{\mathbf{A}}(\mathbf{x}) - \hat{\mathbf{J}}(\mathbf{x}). \tag{S.25}$$

Again, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.10).

Problem 3(a):

The relations $\hat{A}_{\mathbf{k},\lambda}^\dagger = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}_{\mathbf{k},\lambda}^\dagger = -\hat{E}_{-\mathbf{k},\lambda}$ follow from the hermiticity of the quantum fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$, and also from the convention

$$\mathbf{e}_\lambda(-\mathbf{k}) = -\mathbf{e}_\lambda^*(+\mathbf{k}) \quad (16.c)$$

for the polarization vectors. Indeed, taking the hermitian conjugate of eq. (18) for the $\hat{A}_{\mathbf{k},\lambda}$ we get

$$\begin{aligned} \hat{A}_{\mathbf{k},\lambda}^\dagger &= \left(\int d^3\mathbf{x} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda^*(\mathbf{k}) \cdot \hat{\mathbf{A}}(\mathbf{x}) \right)^\dagger = \int d^3\mathbf{x} L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \cdot \hat{\mathbf{A}}^\dagger(\mathbf{x}) \\ &= \int d^3\mathbf{x} L^{-3/2} e^{i(-\mathbf{k})\mathbf{x}} (-\mathbf{e}_\lambda^*(-\mathbf{k})) \cdot \hat{\mathbf{A}}(\mathbf{x}) = -\hat{A}_{-\mathbf{k},\lambda} \end{aligned} \quad (S.26)$$

and likewise for the $\hat{E}_{\mathbf{k},\lambda}$:

$$\begin{aligned} \hat{E}_{\mathbf{k},\lambda}^\dagger &= \left(\int d^3\mathbf{x} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda^*(\mathbf{k}) \cdot \hat{\mathbf{E}}(\mathbf{x}) \right)^\dagger = \int d^3\mathbf{x} L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \cdot \hat{\mathbf{E}}^\dagger(\mathbf{x}) \\ &= \int d^3\mathbf{x} L^{-3/2} e^{i(-\mathbf{k})\mathbf{x}} (-\mathbf{e}_\lambda^*(-\mathbf{k})) \cdot \hat{\mathbf{E}}(\mathbf{x}) = -\hat{E}_{-\mathbf{k},\lambda} \end{aligned} \quad (S.27)$$

Next, the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ modes, they follow directly from eqs. (9) and (16). First, the $\hat{\mathbf{A}}(\mathbf{x})$ fields commute with each other, so obviously all the $\hat{A}_{\mathbf{k},\lambda}$ modes commute with each other,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = 0. \quad (S.28)$$

Likewise, the $\hat{\mathbf{E}}(\mathbf{x})$ fields commute with each other hence

$$[\hat{E}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = 0. \quad (S.29)$$

Less obviously,

$$\begin{aligned}
[\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}^\dagger] &= L^{-3} \int d^3\mathbf{x} \int d^3\mathbf{y} e^{-i\mathbf{k}\mathbf{x}} (\mathbf{e}_\lambda^*(\mathbf{k}))^i \times e^{+i\mathbf{k}'\mathbf{y}} (\mathbf{e}_{\lambda'}(\mathbf{k}'))^j \times [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \\
&= L^{-3} \int d^3\mathbf{x} \int d^3\mathbf{y} e^{-i\mathbf{k}\mathbf{x}} (\mathbf{e}_\lambda^*(\mathbf{k}))^i \times e^{+i\mathbf{k}'\mathbf{y}} (\mathbf{e}_{\lambda'}(\mathbf{k}'))^j \times (-i)\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta^{ij} \\
&= -iL^{-3} \int_{\text{box}} d^3\mathbf{x} e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}} \times (\mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}')) \\
&= -i\delta_{\mathbf{k},\mathbf{k}'} \times (\mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k})) \\
&= -i\delta_{\mathbf{k},\mathbf{k}'} \times \delta_{\lambda,\lambda'},
\end{aligned} \tag{S.30}$$

or equivalently,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = +i\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \times \delta_{\lambda,\lambda'}. \tag{S.31}$$

Problem 3(b):

In the absence of the current $\hat{J}^\mu$, the Hamiltonian (11) reduces to 4 terms,

$$\hat{\mathcal{H}} = \int d^3\mathbf{x} \left(\frac{1}{2} \hat{\mathbf{E}}^2 + \frac{1}{2m^2} (\nabla \cdot \hat{\mathbf{E}})^2 + \frac{1}{2} (\nabla \times \hat{\mathbf{A}})^2 + \frac{m^2}{2} \hat{\mathbf{A}}^2 \right). \tag{S.32}$$

Let's re-express each of the 4 terms here in terms of the field-mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$.

For the first term, we have

$$\begin{aligned}
\int d^3\mathbf{x} \hat{\mathbf{E}}^2(\mathbf{x}) &= \int d^3\mathbf{x} \hat{\mathbf{E}}^\dagger(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}) \\
&= \int d^3\mathbf{x} \left(\sum_{\mathbf{k},\lambda} L^{-3/2} e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda^*(\mathbf{k}) \hat{E}_{\mathbf{k},\lambda}^\dagger \right) \cdot \left(\sum_{\mathbf{k}',\lambda'} L^{-3/2} e^{+i\mathbf{k}'\cdot\mathbf{x}} \mathbf{e}_{\lambda'}(\mathbf{k}') \hat{E}_{\mathbf{k}',\lambda'} \right) \\
&= \sum_{\mathbf{k},b\mathbf{k}'} \sum_{\lambda,\lambda'} \hat{E}_{\mathbf{k},\lambda}^\dagger \hat{E}_{\mathbf{k}',\lambda'} \times L^{-3} \int_{\text{box}} d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} (\mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}'))
\end{aligned} \tag{S.33}$$

where

$$L^{-3} \int_{\text{box}} d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} = \delta_{\mathbf{k},\mathbf{k}'} \tag{S.34}$$

and then for $\mathbf{k}' = \mathbf{k}$

$$\mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}' = \mathbf{k}) = \delta_{\lambda, \lambda'} . \quad (\text{S.35})$$

Plugging this into the bottom line of eq. (S.33), we get

$$\int d^3\mathbf{x} \hat{\mathbf{E}}^2(\mathbf{x}) = \sum_{\mathbf{k}, \mathbf{k}'} \sum_{\lambda, \lambda'} \hat{E}_{\mathbf{k}, \lambda}^\dagger \hat{E}_{\mathbf{k}', \lambda'} \times \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\lambda, \lambda'} = \sum_{\mathbf{k}, \lambda} \hat{E}_{\mathbf{k}, \lambda}^\dagger \hat{E}_{\mathbf{k}, \lambda} . \quad (\text{S.36})$$

In exactly the same way, for the fourth term in the Hamiltonian (S.32) we have

$$\int d^3\mathbf{x} \hat{\mathbf{A}}^2(\mathbf{x}) = \sum_{\mathbf{k}, \lambda} \hat{A}_{\mathbf{k}, \lambda}^\dagger \hat{A}_{\mathbf{k}, \lambda} . \quad (\text{S.37})$$

Next, consider the second term in the Hamiltonian (S.32). It involves

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}) = \nabla \cdot \left(\sum_{\mathbf{k}, \lambda} L^{-3/2} e^{i\mathbf{x} \cdot \mathbf{k}} \mathbf{e}_\lambda(\mathbf{k}) \hat{E}_{\mathbf{k}, \lambda} \right) = \sum_{\mathbf{k}, \lambda} L^{-3/2} \left(\nabla e^{i\mathbf{x} \cdot \mathbf{k}} \cdot \mathbf{e}_\lambda(\mathbf{k}) \right) \hat{E}_{\mathbf{k}, \lambda} \quad (\text{S.38})$$

where

$$\nabla e^{i\mathbf{x} \cdot \mathbf{k}} = e^{i\mathbf{x} \cdot \mathbf{k}} i\mathbf{k} \quad (\text{S.39})$$

and

$$\mathbf{k} \cdot \mathbf{e}_\lambda(\mathbf{k}) = |\mathbf{k}| \delta_{\lambda, 0} \quad (\text{S.40})$$

because in the helicity basis $\mathbf{e}_0(\mathbf{k})$ points in the direction of the $\mathbf{k}$ while $\mathbf{e}_{\pm 1} \perp \mathbf{k}$. Consequently

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}) = \sum_{\mathbf{k}} i|\mathbf{k}| L^{-3/2} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{E}_{\mathbf{k}, 0} , \quad (\text{S.41})$$

which involves only the longitudinal modes with $\lambda = 0$. Therefore,

$$\begin{aligned}
\int d^3\mathbf{x} (\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}))^2 &= \int d^3\mathbf{x} (\nabla \cdot \hat{\mathbf{E}}^\dagger(\mathbf{x})) \cdot (\nabla \cdot \hat{\mathbf{E}}(\mathbf{x})) \\
&= \int d^3\mathbf{x} \left(\sum_{\mathbf{k}} (-i)|\mathbf{k}| L^{-3/2} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{E}_{\mathbf{k},0}^\dagger \right) \cdot \left(\sum_{\mathbf{k}'} (+i)|\mathbf{k}'| L^{-3/2} e^{i\mathbf{k}'\cdot\mathbf{x}} \hat{E}_{\mathbf{k}',0} \right) \\
&= \sum_{\mathbf{k},\mathbf{k}'} |\mathbf{k}| \times |\mathbf{k}'| \times \hat{E}_{\mathbf{k},0}^\dagger \hat{E}_{\mathbf{k}',0} \times \left(L^{-3} \int_{\text{box}} d^3\mathbf{x} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} = \delta_{\mathbf{k},\mathbf{k}'} \right) \\
&= \sum_{\mathbf{k}} |\mathbf{k}|^2 \times \hat{E}_{\mathbf{k},0}^\dagger \hat{E}_{\mathbf{k},0}.
\end{aligned} \tag{S.42}$$

Finally, in the magnetic third term in the Hamiltonian (S.32), we have

$$\hat{\mathbf{B}}(\mathbf{x}) = \nabla \times \hat{\mathbf{A}}(\mathbf{x}) = \sum_{\mathbf{k},\lambda} \nabla \times \left(L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \right) \hat{A}_{\mathbf{k},\lambda}$$

where

$$\nabla \times \left(e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \right) = e^{i\mathbf{k}\cdot\mathbf{x}} (i\mathbf{k}) \times \mathbf{e}_\lambda(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}} \lambda |\mathbf{k}| \mathbf{e}_\lambda(\mathbf{k}). \tag{S.43}$$

Note that due the factor of λ , only the transverse modes $\hat{A}_{\mathbf{k},\lambda}$ with $\lambda = \pm 1$ contribute to the magnetic field

$$\hat{\mathbf{B}}(\mathbf{x}) = \sum_{\mathbf{k},\lambda} L^{-3/2} \lambda |\mathbf{k}| e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \hat{A}_{\mathbf{k},\lambda} \tag{S.44}$$

but the longitudinal modes with $\lambda = 0$ do not contribute. Consequently, the magnetic term

in the Hamiltonian (S.32) becomes

$$\begin{aligned}
\int d^3\mathbf{x} \hat{\mathbf{B}}^2(\mathbf{x}) &= \int d^3\mathbf{x} \hat{\mathbf{B}}^\dagger(\mathbf{x}) \cdot \hat{\mathbf{B}}(\mathbf{x}) \\
&= \int d^3\mathbf{x} \left(\sum_{\mathbf{k},\lambda} L^{-3/2} \lambda |\mathbf{k}| e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda^*(\mathbf{k}) \hat{A}_{\mathbf{k},\lambda}^\dagger \right) \cdot \left(\sum_{\mathbf{k}',\lambda'} L^{-3/2} \lambda' |\mathbf{k}'| e^{+i\mathbf{k}'\cdot\mathbf{x}} \mathbf{e}_{\lambda'}(\mathbf{k}') \hat{A}_{\mathbf{k}',\lambda'} \right) \\
&= \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda,\lambda'} \lambda |\mathbf{k}| \lambda' |\mathbf{k}'| \hat{A}_{\mathbf{k}',\lambda'}^\dagger \hat{A}_{\mathbf{k},\lambda} \times L^{-3} \int_{\text{box}} d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} \times (\mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}')) \\
&\quad \langle\langle \text{where } L^{-3} \int d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} = \delta_{\mathbf{k},\mathbf{k}'} \rangle\rangle \\
&\quad \langle\langle \text{and } \mathbf{e}_\lambda^*(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}') = \delta_{\lambda,\lambda'} \text{ for } \mathbf{k} = \mathbf{k}' \rangle\rangle \\
&= \sum_{\mathbf{k},\lambda} \lambda^2 |\mathbf{k}|^2 \hat{A}_{\mathbf{k},\lambda}^\dagger \hat{A}_{\mathbf{k},\lambda}
\end{aligned} \tag{S.45}$$

Altogether, the 4 terms in the Hamiltonian (S.32) add up to

$$\hat{H} = \sum_{\mathbf{k},\lambda} \left(\frac{1}{2} \left(1 + \frac{\mathbf{k}^2}{m^2} \delta_{\lambda,0} \right) \times \hat{E}_{\mathbf{k},\lambda}^\dagger \hat{E}_{\mathbf{k},\lambda} + \frac{m^2 + \lambda^2 \mathbf{k}^2}{2} \times \hat{A}_{\mathbf{k},\lambda}^\dagger \hat{A}_{\mathbf{k},\lambda} \right). \tag{S.46}$$

Taking a closer look at the coefficients here and comparing them to the $\omega_{\mathbf{k}}$ and $C_{\mathbf{k},\lambda}$ in eqs. (17.b-c), we see that

$$1 + \frac{\mathbf{k}^2}{m^2} \times \delta_{\lambda,0} = \left\{ \begin{array}{ll} 1 & \text{for } \lambda = \pm 1 \\ \omega_{\mathbf{k}}^2/m^2 & \text{for } \lambda = 0 \end{array} \right\} = C_{\mathbf{k},\lambda}$$

while

$$m^2 + \lambda^2 \mathbf{k}^2 = \left\{ \begin{array}{ll} \omega_{\mathbf{k}}^2 & \text{for } \lambda = \pm 1 \\ m^2 & \text{for } \lambda = 0 \end{array} \right\} = \frac{\omega_{\mathbf{k}}^2}{C_{\mathbf{k},\lambda}}, \tag{S.47}$$

thus the Hamiltonian (S.46) amounts to

$$\hat{H} = \sum_{\mathbf{k},\lambda} \left(\frac{C_{\mathbf{k},\lambda}}{2} \times \hat{E}_{\mathbf{k},\lambda}^\dagger \hat{E}_{\mathbf{k},\lambda} + \frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}} \times \hat{A}_{\mathbf{k},\lambda}^\dagger \hat{A}_{\mathbf{k},\lambda} \right), \tag{S.48}$$

precisely as in eq. (17.a).

Problem 3(c):

Given the definitions (18) of the creation and the annihilation operators and the commutation relations (S.28), (S.29), and (S.31) of the field modes $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$, we obtain

$$\begin{aligned}
[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}] &= -i\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} \left([\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = (+i)\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \right) \\
&\quad - i\sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \left([\hat{E}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = (-i)\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \right) \\
&= \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \times \left(\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} - \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \right) \\
&= 0
\end{aligned} \tag{S.49}$$

because for $\mathbf{k}' + \mathbf{k} = \mathbf{0}$ and $\lambda' = \lambda$ we have $\omega_{\mathbf{k}'} = \omega_{\mathbf{k}}$ and $C_{\mathbf{k}',\lambda'} = C_{\mathbf{k},\lambda}$.

Likewise, $[\hat{a}_{\mathbf{k},\lambda}^\dagger, \hat{a}_{\mathbf{k}',\lambda'}^\dagger] = 0$.

On the other hand,

$$\begin{aligned}
[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^\dagger] &= -i\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} \left([\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = (+i)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \right) \\
&\quad + i\sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \left([\hat{E}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = (-i)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \right) \\
&= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \times \left(\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} + \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \right) \\
&= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \times 1
\end{aligned} \tag{S.50}$$

because for $\mathbf{k}' = \mathbf{k}$ and $\lambda' = \lambda$

$$\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} + \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} = 1. \tag{S.51}$$

Quod erat demonstrandum.

Problem 3(d):

Lets compare the first eq. (18) for the annihilation operator and the second eq. (18) for the creation operators for the two modes with opposite momenta $\pm \mathbf{k}$. For the annihilation operator we have

$$\sqrt{2\omega_{\mathbf{k}}C_{\mathbf{k},\lambda}}\hat{a}_{\mathbf{k},\lambda} = \omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \quad (\text{S.52})$$

while for the creation operator

$$\sqrt{2\omega_{-\mathbf{k}}C_{-\mathbf{k},\lambda}}\hat{a}_{-\mathbf{k},\lambda}^{\dagger} = \omega_{-\mathbf{k}}\hat{A}_{-\mathbf{k},\lambda}^{\dagger} + iC_{-\mathbf{k},\lambda}\hat{E}_{-\mathbf{k},\lambda}^{\dagger} = -\omega_{-\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{-\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \quad (\text{S.53})$$

and hence

$$\sqrt{2\omega_{\mathbf{k}}C_{\mathbf{k},\lambda}}\hat{a}_{-\mathbf{k},\lambda}^{\dagger} = -\omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \quad (\text{S.54})$$

because $\omega_{-\mathbf{k}} = \omega_{\mathbf{k}}$ and $C_{-\mathbf{k},\lambda} = C_{\mathbf{k},\lambda}$. Eqs. (S.52) and (S.54) involve the same field-mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$, so adding and subtracting them from each other gives us

$$\begin{aligned} \hat{E}_{\mathbf{k},\lambda} &= \frac{i\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2C_{\mathbf{k},\lambda}}}(\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^{\dagger}), \\ \hat{A}_{\mathbf{k},\lambda} &= \frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}^{\dagger}). \end{aligned} \quad (\text{S.55})$$

Now let's plug eqs. (S.55) into the Hamiltonian (17). For the first term inside the integral/sum, we have

$$\begin{aligned} \frac{C_{\mathbf{k},\lambda}}{2} \hat{E}_{\mathbf{k},\lambda}^{\dagger} \hat{E}_{\mathbf{k},\lambda} &= \frac{\omega_{\mathbf{k}}}{4} (\hat{a}_{\mathbf{k},\lambda}^{\dagger} + \hat{a}_{-\mathbf{k},\lambda}) (\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^{\dagger}) \\ &= \frac{\omega_{\mathbf{k}}}{4} (\hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{-\mathbf{k},\lambda}^{\dagger} + \hat{a}_{-\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda}^{\dagger}), \end{aligned} \quad (\text{S.56})$$

while for the second term we have

$$\begin{aligned} \frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}} \hat{A}_{\mathbf{k},\lambda}^{\dagger} \hat{A}_{\mathbf{k},\lambda} &= \frac{\omega_{\mathbf{k}}}{4} (\hat{a}_{\mathbf{k},\lambda}^{\dagger} - \hat{a}_{-\mathbf{k},\lambda}) (\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}^{\dagger}) \\ &= \frac{\omega_{\mathbf{k}}}{4} (\hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{\mathbf{k},\lambda} - \hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{-\mathbf{k},\lambda}^{\dagger} - \hat{a}_{-\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda}^{\dagger}), \end{aligned} \quad (\text{S.57})$$

so altogether

$$\begin{aligned}
\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}_{\mathbf{k},\lambda}^\dagger \hat{E}_{\mathbf{k},\lambda} + \frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}} \hat{A}_{\mathbf{k},\lambda}^\dagger \hat{A}_{\mathbf{k},\lambda} &= \frac{\omega_{\mathbf{k}}}{4} (2\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + 2\hat{a}_{-\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda}^\dagger) \\
&= \frac{\omega_{\mathbf{k}}}{2} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^\dagger \hat{a}_{-\mathbf{k},\lambda} + 1).
\end{aligned} \tag{S.58}$$

Consequently, the net Hamiltonian (17) amounts to

$$\begin{aligned}
\hat{H} &= \sum_{\mathbf{k},\lambda} \left(\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}_{\mathbf{k},\lambda}^\dagger \hat{E}_{\mathbf{k},\lambda} + \frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}} \hat{A}_{\mathbf{k},\lambda}^\dagger \hat{A}_{\mathbf{k},\lambda} \right) \\
&= \sum_{\mathbf{k},\lambda} \frac{\omega_{\mathbf{k}}}{2} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^\dagger \hat{a}_{-\mathbf{k},\lambda} + 1) \\
&= \frac{1}{2} \sum_{\mathbf{k},\lambda} \omega_{\mathbf{k}} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}) + \frac{1}{2} \sum_{\mathbf{k},\lambda} \omega_{\mathbf{k}} (\hat{a}_{-\mathbf{k},\lambda}^\dagger \hat{a}_{-\mathbf{k},\lambda} + \frac{1}{2}) \\
&\quad \langle\langle \text{renaming the summation variable in the second term only from } \mathbf{k} \text{ to } -\mathbf{k} \rangle\rangle \\
&= \frac{1}{2} \sum_{\mathbf{k},\lambda} \omega_{\mathbf{k}} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}) + \frac{1}{2} \sum_{-\mathbf{k},\lambda} \omega_{-\mathbf{k}} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}) \\
&\quad \langle\langle \text{by } \mathbf{k} \leftrightarrow -\mathbf{k} \text{ symmetry of the summation range and thanks to } \omega_{-\mathbf{k}} = \omega_{\mathbf{k}} \rangle\rangle \\
&\quad \langle\langle \text{the two terms here are equal} \rangle\rangle \\
&= \sum_{\mathbf{k},\lambda} \omega_{\mathbf{k}} (\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}).
\end{aligned} \tag{S.59}$$

Problem 3(e):

As explained in class — *cf.* [my notes on identical bosons](#), — Hamiltonians of the form

$$\hat{H} = \sum_{\alpha} \omega_{\alpha} \hat{a}_{\alpha}^\dagger \hat{a}_{\alpha} + \text{const} \tag{S.60}$$

— where $\hat{a}_{\alpha}^\dagger$ and $\hat{a}_{\alpha}$ are some kind of creation and annihilation operators obeying the bosonic commutation relations — describe theories of arbitrary numbers of identical bosons. Specifically, each boson has an independent 1-particle Hamiltonian with eigenstates $|\alpha\rangle$ and eigenvalues ω_{α} labeled by the same index α as in the sum (S.60).

In our case of the Hamiltonian (19), the ‘index’ α becomes $(\mathbf{k}, \lambda)$ where the momentum $\mathbf{k}$ has a discrete but dense spectrum corresponding to a particle living in a large L^3 box with periodic boundary conditions, while the helicity λ takes 3 discrete values $-1, 0, +1$. Physically, this means that the Hamiltonian (19) describes the Fock space of identical bosons, where each boson has a basis of 1-particle states $|\mathbf{k}, \lambda\rangle$ parametrized by the in-box momentum $\mathbf{k}$ and the helicity λ , with single-particle energies

$$\omega_{\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2} \quad (\text{S.61})$$

(in the $\hbar = c = 1$ units). Thus, each boson is a free relativistic particle of mass m . Moreover, for each particular momentum $\mathbf{k}$, the boson has 3 degenerate states with different helicities $\lambda = -1, 0, +1$. *Quod erat demonstrandum.*

PS: For the purpose of this problem, λ is just a label distinguishing the 3 distinct polarization states of the relativistic boson. Later in class we shall see that λ happens to be the boson’s *helicity*, that is the component of its spin along the direction of its motion.

Problem 4(a):

By the Leibniz rule for the commutators

$$[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] = \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\gamma^\dagger] + [\hat{a}_\alpha^\dagger, \hat{a}_\gamma^\dagger] \hat{a}_\beta = \hat{a}_\alpha^\dagger \times \delta_{\beta\gamma} + 0 \times \hat{a}_\beta = \delta_{\beta\gamma} \hat{a}_\alpha^\dagger \quad (\text{S.62})$$

and likewise

$$[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] = \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\delta] + [\hat{a}_\alpha^\dagger, \hat{a}_\delta] \hat{a}_\beta = \hat{a}_\alpha^\dagger \times 0 + (-\delta_{\alpha\delta}) \times \hat{a}_\beta = -\delta_{\alpha\delta} \hat{a}_\beta. \quad (\text{S.63})$$

Consequently, applying the Leibniz rule once again we get

$$\begin{aligned} [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] &= \hat{a}_\gamma^\dagger [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] + [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] \hat{a}_\delta \\ &= \hat{a}_\gamma^\dagger \times (-\delta_{\alpha\delta} \hat{a}_\beta) + (+\delta_{\beta\gamma} \hat{a}_\alpha^\dagger) \hat{a}_\delta \\ &= \delta_{\beta\gamma} \hat{a}_\alpha^\dagger \hat{a}_\delta - \delta_{\alpha\delta} \hat{a}_\gamma^\dagger \hat{a}_\beta. \end{aligned} \quad (\text{S.64})$$

Finally,

$$\begin{aligned}
[\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu] &= \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \times [\hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu] + \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \times [\hat{a}_\gamma, \hat{a}_\mu^\dagger \hat{a}_\nu] \times \hat{a}_\delta \\
&\quad + \hat{a}_\alpha^\dagger \times [\hat{a}_\beta^\dagger, \hat{a}_\mu^\dagger \hat{a}_\nu] \times \hat{a}_\gamma \hat{a}_\delta + [\hat{a}_\alpha^\dagger, \hat{a}_\mu^\dagger \hat{a}_\nu] \times \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \\
&= \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \times (+\delta_{\delta\mu} \hat{a}_\nu) + \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \times (+\delta_{\gamma\mu} \hat{a}_\nu) \times \hat{a}_\delta \\
&\quad + \hat{a}_\alpha^\dagger \times (-\delta_{\beta\nu} \hat{a}_\mu^\dagger) \times \hat{a}_\gamma \hat{a}_\delta + (-\delta_{\alpha\nu} \hat{a}_\mu^\dagger) \times \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \\
&= +\delta_{\delta\mu} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\nu + \delta_{\gamma\mu} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\delta - \delta_{\beta\nu} \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\gamma \hat{a}_\delta - \delta_{\alpha\nu} \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta.
\end{aligned}
\tag{S.65}$$