

Problem 1(a):

In the previous homework set #2, problem 3(d), we saw that (in the Schrödinger picture and in the box normalization)

$$\begin{aligned}\hat{E}_{\mathbf{k},\lambda} &= \frac{i\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2C_{\mathbf{k},\lambda}}}(\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^\dagger), \\ \hat{A}_{\mathbf{k},\lambda} &= \frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}^\dagger),\end{aligned}\tag{S2.55}$$

cf. eq. (55) on page 12 of the [solutions to set #2](#). Consequently, the vector field $\hat{\mathbf{A}}(\mathbf{x})$ expands into the creation and annihilation operators as

$$\begin{aligned}\hat{\mathbf{A}}(\mathbf{x}) &= \sum_{\mathbf{k},\lambda} L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \hat{A}_{\mathbf{k},\lambda} \\ &= \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} (\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}^\dagger) \\ &= \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda}^\dagger \\ &\quad \langle\langle \text{change } \mathbf{k} \rightarrow -\mathbf{k} \text{ in the second sum but not the first sum} \rangle\rangle \\ &= \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{-\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{-\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{-\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}^\dagger \\ &\quad \langle\langle \text{using } \omega_{-\mathbf{k}} = \omega_{\mathbf{k}} \text{ and } \mathbf{e}_{-\mathbf{k},\lambda} = -\mathbf{e}_{\mathbf{k},\lambda}^* \rangle\rangle \\ &= \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^\dagger \\ &= \sum_{\mathbf{k},\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \left(e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^\dagger \right).\end{aligned}\tag{S.1}$$

Next, let's go to the infinite volume limit and hence continuous momenta. In the infinite-space but non-relativistic normalization, eq. (S.1) becomes

$$\hat{\mathbf{A}}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda} \frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^\dagger \right)^{\text{non rel.}}.\tag{S.2}$$

Finally, in the relativistic normalization

$$(\hat{a}_{\mathbf{k},\lambda})^{\text{rel}} = \sqrt{2\omega_{\mathbf{k}}}(\hat{a}_{\mathbf{k},\lambda})^{\text{non rel}} \implies \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k},\lambda})^{\text{non rel}} = \frac{1}{2\omega_{\mathbf{k}}}(\hat{a}_{\mathbf{k},\lambda})^{\text{rel}} \quad (\text{S.3})$$

and likewise for the creation operators $\hat{a}_{\mathbf{k},\lambda}^\dagger$. Consequently, eq. (S.2) becomes

$$\hat{\mathbf{A}}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \sqrt{C_{\mathbf{k},\lambda}} \left(e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^\dagger \right)^{\text{rel}}. \quad (1)$$

Eq. (1) applies in the Schrödinger picture where all the operators — including the creation and annihilation operators for all the models as well as the quantum fields — are time-independent. In the Heisenberg picture, all operators become time-dependent, but the algebraic relation between different operators *at equal times* remain the same as in the Schrödinger picture, thus

$$\hat{\mathbf{A}}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \sqrt{C_{\mathbf{k},\lambda}} \left(e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}(t) + e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^\dagger(t) \right). \quad (2)$$

Quod erat demonstrandum

Problem 1(b):

In the Heisenberg picture, the time-dependence of the quantum field $\hat{\mathbf{A}}(\mathbf{x}, t)$ follows from the time-dependence of the creation and annihilation operators,

$$i \frac{d}{dt} \hat{a}_{\mathbf{k},\lambda}(t) = [\hat{a}_{\mathbf{k},\lambda}(t), \hat{H}], \quad i \frac{d}{dt} \hat{a}_{\mathbf{k},\lambda}^\dagger(t) = [\hat{a}_{\mathbf{k},\lambda}^\dagger(t), \hat{H}]. \quad (\text{S.4})$$

In the previous homework (problem 4(d)) we wrote the Hamiltonian of the free vector field as

$$\hat{H} = \sum_{\mathbf{k},\lambda} \omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \text{const} \quad (\text{S.5})$$

for the box normalization of creation and annihilation operators. In the infinite-space rela-

tivistic normalization of the operators, this Hamiltonian becomes

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{\mathbf{k},\lambda} + \text{const} = \int \frac{d^3\mathbf{k}}{2(2\pi)^3} \sum_{\lambda} \hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{\mathbf{k},\lambda} + \text{const}. \quad (\text{S.6})$$

Consequently,

$$\begin{aligned} [\hat{a}_{\mathbf{k},\lambda}, \hat{H}] &= \int \frac{d^3\mathbf{k}'}{2(2\pi)^3} \sum_{\lambda'} \left([\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^{\dagger} \hat{a}_{\mathbf{k}',\lambda'}] = [\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^{\dagger}] \hat{a}_{\mathbf{k}',\lambda'} + 0 \right) \\ &= \int \frac{d^3\mathbf{k}'}{2(2\pi)^3} \sum_{\lambda'} \left(+2\omega_{\mathbf{k}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \right) \hat{a}_{\mathbf{k}',\lambda'} \\ &= +\omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}, \\ [\hat{a}_{\mathbf{k},\lambda}^{\dagger}, \hat{H}] &= \int \frac{d^3\mathbf{k}'}{2(2\pi)^3} \sum_{\lambda'} \left([\hat{a}_{\mathbf{k},\lambda}^{\dagger}, \hat{a}_{\mathbf{k}',\lambda'}^{\dagger} \hat{a}_{\mathbf{k}',\lambda'}] = 0 + \hat{a}_{\mathbf{k}',\lambda'}^{\dagger} [\hat{a}_{\mathbf{k},\lambda}^{\dagger}, \hat{a}_{\mathbf{k}',\lambda'}] \right) \\ &= \int \frac{d^3\mathbf{k}'}{2(2\pi)^3} \sum_{\lambda'} \hat{a}_{\mathbf{k}',\lambda'}^{\dagger} \left(-2\omega_{\mathbf{k}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \right) \\ &= -\omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}^{\dagger}, \end{aligned} \quad (\text{S.7})$$

similar to the scalar creation and annihilation operators we have studied in class. Plugging these relations into the Heisenberg equations (S.4) for the creation and annihilation operators, we get

$$\frac{d}{dt} \hat{a}_{\mathbf{k},\lambda}(t) = -i\omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}(t), \quad \frac{d}{dt} \hat{a}_{\mathbf{k},\lambda}^{\dagger}(t) = +i\omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}^{\dagger}(t), \quad (\text{S.8})$$

which give us the time dependence of the creation/annihilation operators:

$$\hat{a}_{\mathbf{k},\lambda}(t) = \exp(-i\omega_{\mathbf{k}}t) \times \hat{a}_{\mathbf{k},\lambda}(0), \quad \hat{a}_{\mathbf{k},\lambda}^{\dagger}(t) = \exp(+i\omega_{\mathbf{k}}t) \times \hat{a}_{\mathbf{k},\lambda}^{\dagger}(0). \quad (\text{S.9})$$

Consequently, substituting this time dependence into eq. (2) for the vector field, we arrive at

$$\begin{aligned} \hat{\mathbf{A}}(\mathbf{x}, t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \sqrt{C_{\mathbf{k},\lambda}} \left(e^{+i\mathbf{k}\mathbf{x} - i\omega_{\mathbf{k}}t} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}(0) + e^{-i\mathbf{k}\mathbf{x} + i\omega_{\mathbf{k}}t} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger}(0) \right) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \sqrt{C_{\mathbf{k},\lambda}} \left(e^{-ik_{\mu}x^{\mu}} \mathbf{e}_{\lambda}(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda}(0) + e^{+ik_{\mu}x^{\mu}} \mathbf{e}_{\lambda}^*(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda}^{\dagger}(0) \right)^{k^0 = +\omega_{\mathbf{k}}} \end{aligned} \quad (\text{S.10})$$

precisely as in eq. (3).

Problem 1(c):

Proceeding exactly as in part (a), we find that in the Schrödinger picture and the box normalization of the creation and annihilation operators, the electric field becomes

$$\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x}) &= \sum_{\mathbf{k},\lambda} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\lambda}(\mathbf{k}) \hat{E}_{\mathbf{k},\lambda} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} (\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}^{\dagger}) \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda}^{\dagger} \\
&\quad \ll \text{change } \mathbf{k} \rightarrow -\mathbf{k} \text{ in the second sum but not the first sum} \gg \tag{S.11} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{-\mathbf{k}}}}{L^{3/2}\sqrt{2C_{-\mathbf{k},\lambda}}} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_{-\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}^{\dagger} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \\
&= \sum_{\mathbf{k},\lambda} \frac{\sqrt{\omega_{\mathbf{k}}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - i e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right).
\end{aligned}$$

Going to the infinite space limit and then to the relativistic normalization of the operators, we turn this formula to

$$\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda} \frac{\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - i e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{\text{non rel}} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda} \frac{1}{2\sqrt{C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - i e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{\text{rel}} \tag{S.12} \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \frac{\omega_{\mathbf{k}}}{2\sqrt{C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - i e^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{\text{rel}}.
\end{aligned}$$

In the Heisenberg picture, this formula becomes

$$\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x}, t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}(t) - i e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger}(t) \right) \\
&\quad \langle\langle \text{using part (b)} \rangle\rangle \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}} \left(i e^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda} e^{-i\omega_{\mathbf{k}}t} \hat{a}_{\mathbf{k},\lambda} - i e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}^* e^{+i\omega_{\mathbf{k}}t} \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right) \quad (\text{S.13}) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}} \left(i e^{-ik_{\mu}x^{\mu}} \mathbf{e}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} - i e^{+ik_{\mu}x^{\mu}} \mathbf{e}_{\mathbf{k},\lambda}^* \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{k^0=+\omega_{\mathbf{k}}}.
\end{aligned}$$

Finally, the $\hat{A}^0(x)$ field obtains from the electric field and eq. (4). Applying this identity to eq. (S.13), we obtain

$$\begin{aligned}
\hat{A}^0(\mathbf{x}, t) &= \frac{-1}{m^2} \nabla \cdot \hat{\mathbf{E}}(\mathbf{x}, t) \\
&= \frac{-1}{m^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}} \left(e^{-i\omega_{\mathbf{k}}t} \times \left(\nabla e^{+i\mathbf{k}\mathbf{x}} \cdot \mathbf{e}_{\mathbf{k},\lambda} \right) \times \hat{a}_{\mathbf{k},\lambda} \right. \\
&\quad \left. - e^{+i\omega_{\mathbf{k}}t} \times \left(\nabla e^{-i\mathbf{k}\mathbf{x}} \cdot \mathbf{e}_{\mathbf{k},\lambda}^* \right) \times \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right). \quad (\text{S.14})
\end{aligned}$$

In this formula

$$\nabla e^{+i\mathbf{k}\mathbf{x}} \cdot \mathbf{e}_{\mathbf{k},\lambda} = e^{+i\mathbf{k}\mathbf{x}} (i\mathbf{k} \cdot \mathbf{e}_{\mathbf{k},\lambda}) = e^{+i\mathbf{k}\mathbf{x}} \times i|\mathbf{k}| \delta_{\lambda,0} \quad (\text{S.15})$$

in the helicity basis, and likewise

$$\nabla e^{-i\mathbf{k}\mathbf{x}} \cdot \mathbf{e}_{\mathbf{k},\lambda}^* = e^{-i\mathbf{k}\mathbf{x}} \times (-i)|\mathbf{k}| \delta_{\lambda,0}. \quad (\text{S.16})$$

Thus, only the $\lambda = 0$ modes contribute to the scalar potential (S.14). Specifically,

$$\hat{A}^0(\mathbf{x}, t) = \frac{-1}{m^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \times \frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},0}}} \times i|\mathbf{k}| \left(e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\mathbf{x}} \times \hat{a}_{\mathbf{k},0} + e^{+i\omega_{\mathbf{k}}t-i\mathbf{k}\mathbf{x}} \times \hat{a}_{\mathbf{k},0}^{\dagger} \right) \quad (\text{S.17})$$

where

$$\sqrt{C_{\mathbf{k},0}} = \frac{\omega_{\mathbf{k}}}{m} \implies \frac{-1}{m^2} \times \frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},0}}} \times i|\mathbf{k}| = +\frac{|\mathbf{k}|}{m}.$$

Thus altogether,

$$\hat{A}^0(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \times \frac{|\mathbf{k}|}{m} \times \left(e^{-ik_{\mu}x^{\mu}} \times \hat{a}_{\mathbf{k},0} + e^{+ik_{\mu}x^{\mu}} \times \hat{a}_{\mathbf{k},0}^{\dagger} \right)^{k^0=+\omega_{\mathbf{k}}}. \quad (\text{S.18}).$$

Problem 1(d):

In light of obvious similarity between eqs. (2) and (S.18), we may immediately combine them into

$$\hat{A}^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \left(e^{-ikx} f^\mu(\mathbf{k}, \lambda) \times \hat{a}_{\mathbf{k},0}(0) + e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda) \times \hat{a}_{\mathbf{k},0}^\dagger(0) \right)_{k^0=+\omega_{\mathbf{k}}} \quad (\text{S.19})$$

— precisely as in eq. (5) — for

$$\mathbf{f}(\mathbf{k}, \lambda) = \sqrt{C_{\mathbf{k},\lambda}} \mathbf{e}_{\mathbf{k},\lambda} \quad \text{and} \quad f^0(\mathbf{k}, \lambda) = \frac{|\mathbf{k}|}{m} \delta_{\lambda,0}. \quad (\text{S.20})$$

Specifically,

$$\text{for } \lambda = \pm 1, \quad \mathbf{f} = \mathbf{e}_\lambda(\mathbf{k}), \quad f^0 = 0, \quad (\text{S.21})$$

while

$$\text{for } \lambda = 0, \quad \mathbf{f} = \frac{\omega_{\mathbf{k}}}{m} \frac{\mathbf{k}}{|\mathbf{k}|}, \quad f^0 = \frac{|\mathbf{k}|}{m}. \quad (\text{S.22})$$

Problem 1(e):

It is easy to see that the polarization 4-vectors $f^\mu(\mathbf{k}, \lambda)$ obey $k_\mu f^\mu(\mathbf{k}, \lambda) = 0$. Indeed, for the transverse polarizations $\lambda = \pm 1$, eq. (S.21) tells us that $f^0 = 0$ while the space part $\mathbf{f} = \mathbf{e}(\lambda = \pm 1)$ is transverse to the 3-vector $\mathbf{k}$, hence $k_\mu f^\mu = 0$. As to the longitudinal polarization $\lambda = 0$, in light of eq. (S.22)

$$k_\mu f^\mu = \omega_{\mathbf{k}} f^0 - \mathbf{k} \cdot \mathbf{f} = \omega_{\mathbf{k}} \frac{|\mathbf{k}|}{m} - \mathbf{k} \cdot \left(\frac{\omega_{\mathbf{k}}}{m} \frac{\mathbf{k}}{|\mathbf{k}|} \right) = \frac{\omega_{\mathbf{k}} |\mathbf{k}|}{m} - \frac{\omega_{\mathbf{k}} \mathbf{k}^2}{m |\mathbf{k}|} = 0. \quad (\text{S.23})$$

Consequently, the quantum field $\hat{A}^\mu(x)$ obeys the classical equation $\partial_\mu \hat{A}^\mu = 0$. Indeed, each plane wave factor in the expansion (24) obeys this equation:

$$\partial_\mu (e^{-ikx} f^\mu(\mathbf{k}, \lambda)) = e^{-ikx} \times (-ik_\mu) f^\mu(\mathbf{k}, \lambda) = 0 \quad \langle\langle \text{because } k_\mu f^\mu(\mathbf{k}, \lambda) = 0 \rangle\rangle, \quad (\text{S.24})$$

and likewise

$$\partial_\mu (e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda)) = e^{+ikx} \times (+ik_\mu) f^{\mu*}(\mathbf{k}, \lambda) = 0 \quad (\text{S.25})$$

because $k_\mu f^{\mu*} = (k_\mu f^\mu)^* = 0$. And therefore, by linearity, the whole quantum field obeys

this equation,

$$\partial_\mu \hat{A}^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \left(\partial_\mu (e^{-ikx} f^\mu(\mathbf{k}, \lambda)) \hat{a}_{\mathbf{k},\lambda} + \partial_\mu (e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda)) \hat{a}_{\mathbf{k},\lambda}^\dagger \right) = 0. \quad (\text{S.26})$$

Likewise, thanks to $k_0 = +\omega_{\mathbf{k}}$ — and hence $k_\mu k^\mu = \omega_{\mathbf{k}}^2 - \mathbf{k}^2 = m^2$ — in every plane wave factor in the decomposition (24) of the quantum field, every plane wave factor obeys the Klein–Gordon equation

$$\begin{aligned} (\partial^2 + m^2)(e^{-ikx} f^\mu(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}) &= (-k^2 + m^2)(e^{-ikx} f^\mu(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}) = 0, \\ (\partial^2 + m^2)(e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}^\dagger) &= (-k^2 + m^2)(e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}^\dagger) = 0. \end{aligned} \quad (\text{S.27})$$

Hence, by linearity, the whole quantum field obeys this equation,

$$\begin{aligned} (\partial^2 + m^2) \hat{A}^\mu(x) &= \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \left((\partial^2 + m^2)(e^{-ikx} f^\mu(\mathbf{k}, \lambda)) \hat{a}_{\mathbf{k},\lambda} + (\partial^2 + m^2)(e^{+ikx} f^{\mu*}(\mathbf{k}, \lambda)) \hat{a}_{\mathbf{k},\lambda}^\dagger \right) \\ &= 0. \end{aligned} \quad (\text{S.28})$$

Problem 2(a):

The Hamiltonian (7) of a free relativistic particle — and hence the evolution operator $\exp(-it\hat{H})$ — are functions of the momentum operator $\hat{\mathbf{p}}$, so they diagonalize in the momentum basis. In the non-relativistic normalization of $|\mathbf{k}\rangle$ states,

$$\begin{aligned} \hat{H} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}\rangle \omega(\mathbf{k}) \langle \mathbf{k}|, \\ \exp(-i\hat{H}t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}\rangle \exp(-it\omega(\mathbf{k})) \langle \mathbf{k}|, \end{aligned} \quad (\text{S.29})$$

where $t = y^0 - x^0$ and $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + M^2}$. In the same non-relativistic normalization

$\langle \mathbf{x} | \mathbf{k} \rangle = \exp(i\mathbf{k} \cdot \mathbf{x})$, therefore

$$\begin{aligned} U(x \rightarrow y) &\equiv \langle \mathbf{y} | \exp(-i\hat{H}t) | \mathbf{x} \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle \mathbf{y} | \mathbf{k} \rangle \exp(-it\omega(\mathbf{k})) \langle \mathbf{k} | \mathbf{x} \rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \exp(i(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} - it\omega(\mathbf{k})). \end{aligned} \quad (\text{S.30})$$

precisely as in eq. (8).

Moreover, this non-relativistically normalized evolution kernel $U(x \rightarrow y)$ is related to the function

$$D(x - y) \stackrel{\text{def}}{=} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \exp(i(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} - i(y^0 - x^0)\omega_{\mathbf{k}}) \quad (\text{S.31})$$

we have used in class. (In the contexts of relativistic causality and also of the Feynman propagators.) Specifically,

$$U(x \rightarrow y) = 2i \frac{\partial}{\partial t} D(y - x) \quad \langle\langle \text{where } t = y^0 - x^0 \rangle\rangle. \quad (11)$$

Indeed,

$$2i \frac{\partial}{\partial t} \exp(i(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} - it\omega_{\mathbf{k}}) = 2\omega_{\mathbf{k}} \times \exp(i(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} - it\omega_{\mathbf{k}}) \quad (\text{S.32})$$

where the $2\omega_{\mathbf{k}}$ factor cancels the similar denominator of the relativistic measure, thus

$$2i \frac{\partial}{\partial t} D(y - x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \exp(i(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} - it\omega(\mathbf{k})) = U(x \rightarrow y). \quad (\text{S.33})$$

Problem 2(b):

Let's simplify the 3D integral on the RHS of eq. (8) by integrating over the directions of the $\mathbf{k}$ vector. In spherical coordinates (k, θ, ϕ) where $k = |\mathbf{k}|$ and θ is the angle between $\mathbf{k}$ and

$\mathbf{y} - \mathbf{x}$,

$$d^3\mathbf{k} = dk k^2 d\cos\theta d\phi, \quad \mathbf{k} \cdot (\mathbf{y} - \mathbf{x}) = rk \cos\theta, \quad (\text{S.34})$$

hence

$$\iint d\cos\theta d\phi e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{x})} = 2\pi \int_{-1}^{+1} d\cos\theta e^{ikr \cos\theta} = \frac{2\pi}{irk} (e^{+irk} - e^{-irk}). \quad (\text{S.35})$$

Consequently, for any symmetric function $f(k) = f(-k)$ we have

$$\begin{aligned} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{x})} \times f(|\mathbf{k}|) &= \frac{1}{4\pi^2} \int_0^\infty dk k^2 \frac{1}{irk} (e^{+irk} - e^{-irk}) \times f(k) \\ &= \frac{1}{4\pi^2 i r} \int_0^\infty dk k (e^{+irk} \times f(k) - e^{-irk} \times (f(k) = f(-k))) \\ &= \frac{1}{4\pi^2 i r} \int_{-\infty}^{+\infty} dk k e^{+irk} \times f(k). \end{aligned} \quad (\text{S.36})$$

In particular, for the $f(k) = e^{-it\omega(k)} = f(-k)$ we have

$$U(x \rightarrow y) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{x})} \times e^{-it\omega(\mathbf{k})} = \frac{1}{4\pi^2 i r} \int_{-\infty}^{+\infty} dk k e^{+irk} \times e^{-it\omega(\mathbf{k})}. \quad (\text{S.37})$$

precisely as in eq. (12). *Quod erat demonstrandum*

Problem 2(d):

As explained in [my notes on the saddle point method](#), integrals of the form

$$I = \int_{\Gamma} dz f(z) e^{Ag(z)} \quad (\text{S.38})$$

in the large A limit become

$$I = e^{Ag(z_0)} \times \frac{\sqrt{2\pi\eta} f(z_0)}{\sqrt{-\eta^2 Ag''(z_0)}} \times (1 + O(A^{-1})). \quad (\text{S.39})$$

In general, the f and g here are complex analytic functions of a complex variable z which is integrated over some contour Γ ; quite often Γ is the real axis, but one should allow for

its deformation in the complex plane. In eq. (S.39), z_0 is a saddle point of $g(z)$ where the derivative $g'(z_0) = 0$; this saddle point does not have to lie on the original integration contour Γ — if it does not, we deform the contour $\Gamma \rightarrow \Gamma'$ so that Γ' does go through the z_0 . If several saddle points are present near the contour Γ , the point with the largest $\text{Re } g$ dominates the integral. Finally, η is the direction dz of the Γ' at the saddle point z_0 ; it should be chosen such that $\text{Re}(-\eta^2 g''(z_0)) \geq 0$, which assures that Γ' crosses z_0 as a mountain path, from a valley to the lowest crossing point to another valley.

For the integral (12) at hand, we identify

$$A = t, \quad g(k) = i\frac{r}{t}k - i\omega(k), \quad f(k) = \frac{k}{4\pi^2 i r}. \quad (\text{S.40})$$

The saddle point in the k plane follows from

$$\frac{dg}{dk} \equiv i\frac{r}{t} - i\frac{d\omega}{dk} \equiv i\frac{r}{t} - i\frac{k}{\omega} = 0. \quad (\text{S.41})$$

For $r < t$ this equation has a real solution, namely

$$k_0 = M \times \frac{r}{\sqrt{t^2 - r^2}}, \quad \omega(k_0) = M \times \frac{t}{\sqrt{t^2 - r^2}}. \quad (\text{S.42})$$

At this point

$$\begin{aligned} Ag(k_0) &= irk_0 - it\omega(k_0) = iM \frac{r^2 - t^2}{\sqrt{t^2 - r^2}} = -iM \times \sqrt{t^2 - r^2}, \\ f(k_0) &= \frac{-iM}{4\pi^2} \frac{1}{\sqrt{t^2 - r^2}}, \\ Ag''(k_0) &\equiv \frac{-it M^2}{\omega^3(k_0)} = \frac{(t^2 - r^2)^{3/2}}{iMt^2}, \end{aligned} \quad (\text{S.43})$$

and the direction of the integration contour at k_0 should be in the fourth quadrant of the complex plane, $\arg(\eta)$ between 0 and $-\pi/2$; the real-axis contour is marginally OK. Substi-

tuting all these data into eq. (S.39) gives us

$$\frac{\sqrt{2\pi}\eta f(z_0)}{\sqrt{-\eta^2 Ag''(z_0)}} = \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \quad (\text{S.44})$$

and therefore

$$U(x \rightarrow y) = \exp\left(-iM\sqrt{t^2 - r^2}\right) \times \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \times \left(1 + O\left(\frac{1}{M\sqrt{t^2 - r^2}}\right)\right) \quad (\text{S.45})$$

in accordance with eq. (14).

Problem 2(d):

Again, we use the saddle point method to evaluate the integral (12). That is, we identify A , $g(k)$, and $f(k)$ according to eq. (S.40), and solve eq. (S.41) to find the saddle point. But this time, for $r > t$ the saddle point is imaginary

$$k_0 = \frac{iMr}{\sqrt{r^2 - t^2}}, \quad \omega(k_0) = \frac{iMt}{\sqrt{r^2 - t^2}}, \quad (\text{S.46})$$

so the integration contour must be deformed away from the real axis. At the saddle point (S.46),

$$\begin{aligned} Ag(k_0) &= irk_0 - it\omega(k_0) = -M \times \sqrt{r^2 - t^2}, \\ f(k_0) &= \frac{M}{4\pi^2} \frac{1}{\sqrt{r^2 - t^2}}, \\ Ag''(k_0) &\equiv \frac{-itM^2}{\omega^3(k_0)} = +\frac{(r^2 - t^2)^{3/2}}{Mt^2}, \end{aligned} \quad (\text{S.47})$$

all being real, and the deformed contour should cross k_0 in the imaginary direction, $\arg(\eta) = \frac{\pi}{2} \pm \frac{\pi}{4}$. Consequently, in eq. (S.39)

$$\frac{\sqrt{\pi}\eta f(z_0)}{\sqrt{-\eta^2 Ag''(z_0)}} = \frac{+iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \quad (\text{S.48})$$

and therefore

$$U(\mathbf{x} - \mathbf{y}; t) = \exp\left(-M\sqrt{r^2 - t^2}\right) \times \frac{iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \times \left(1 + O\left(\frac{1}{M\sqrt{r^2 - t^2}}\right)\right) \quad (\text{S.49})$$

in accordance with eq. (15).

Note that the exponential factor here decays as one goes further outside the future light cone. In other words, the probability of a relativistic particle moving faster than light is exponentially small. But tiny as it is, this probability does not vanish, and this violates the relativistic causality.