

Problem 1(a):

In the Noether's formula (1) for the stress-energy tensor, the ϕ_a stand for independent fields or field components, however they might be labeled. In the electromagnetic case, the independent fields are components of the 4-vector $A_\lambda(x)$, hence

$$\begin{aligned} T_{\text{Noether}}^{\mu\nu}(\text{EM}) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L} \\ &= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \end{aligned} \quad (\text{S.1})$$

While the second term here is clearly both gauge invariant and symmetric in $\mu \leftrightarrow \nu$, the first term is neither.

Problem 1(b):

Let $\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu} A^\nu$ as in eq. (5). Then

$$\partial_\lambda \mathcal{K}^{\lambda\mu,\nu} = -(\partial_\lambda F^{\lambda\mu}) A^\nu - F^{\lambda\mu} (\partial_\lambda A^\nu) = -J^\mu A^\nu + F^{\mu\lambda} (\partial_\lambda A^\nu), \quad (\text{S.2})$$

where the first term on the RHS vanishes for the free EM fields (*i.e.*, when $J^\mu = 0$). Consequently, for the free EM fields

$$\begin{aligned} T_{\text{phys}}^{\mu\nu} &= T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu,\nu} \\ &= -F^{\mu\lambda} (\partial^\nu A_\lambda) + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} + F^{\mu\lambda} (\partial_\lambda A^\nu) \\ &= -F^{\mu\lambda} (\partial^\nu A_\lambda - \partial_\lambda A^\nu) + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \\ &= -F^{\mu\lambda} F_\lambda^\nu + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}, \end{aligned} \quad (\text{S.3})$$

exactly as in eq. (4).

Problem 1(c):

Let's start with the Lagrangian (3). In component form,

$$F^{i0} = -F^{0i} = E^i, \quad F^{ij} = -\epsilon^{ijk} B^k. \quad (\text{S.4})$$

Therefore, $F^{i0}F_{i0} = F^{0i}F_{0i} = -E^i E^i$ where the minus sign comes from raising one space index. Likewise, $F^{ij}F_{ij} = +\epsilon^{ijk} B^k \epsilon^{ij\ell} B^\ell = +2B^k B^k$ where the plus sign comes from raising two space indices at once. Thus,

$$\mathcal{L} = -\frac{1}{4} \left(F^{\mu\nu} F_{\mu\nu} = F^{i0}F_{i0} + F^{0i}F_{0i} + F^{ij}F_{ij} \right) = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2). \quad (\text{S.5})$$

Now let's work out the stress-energy tensor (4) in components. For the energy density component T^{00} , we have

$$\mathcal{U} \equiv T^{00} = -F^{0i}F_{i0} - \mathcal{L} = +\mathbf{E}^2 - \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \quad (\text{S.6})$$

in agreement with the standard electromagnetic formulæ (in the *rationalized* $c = 1$ units).

Likewise, the energy flux and the momentum density are

$$S^i \equiv T^{i0} = T^{0i} = -F^{0j}F_{j0}^i = -(-E^j)(+\epsilon^{ijk} B^k) = +\epsilon^{ijk} E^j B^k = (\mathbf{E} \times \mathbf{B})^i, \quad (\text{S.7})$$

in agreement with the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ (again, in the *rationalized* $c = 1$ units).

Finally, the (3-dimensional) stress tensor is

$$\begin{aligned} T_{\text{EM}}^{ij} &= -F^{i\lambda}F_{\lambda}^j - g^{ij}\mathcal{L} = -F^{i0}F_0^j - F^{ik}F_k^j + \delta^{ij}\mathcal{L} \\ &= -E^i E^j + \epsilon^{ik\ell} B^\ell \epsilon^{jkm} B^m + \frac{1}{2}\delta^{ij} (\mathbf{E}^2 - \mathbf{B}^2) \\ &= -E^i E^j + (\delta^{ij} B^\ell B^\ell - B^i B^j) + \frac{1}{2}\delta^{ij} (\mathbf{E}^2 - \mathbf{B}^2) \\ &= -E^i E^j - B^i B^j + \frac{1}{2}\delta^{ij} (\mathbf{E}^2 + \mathbf{B}^2). \end{aligned} \quad (\text{S.8})$$

Again, this is the well-known Maxwell stress tensor for the EM fields (in the rationalized $c = 1$ units), except for the overall sign convention: In our convention, positive EM stress is

compression rather than tension, thus EM pressure

$$P = +\frac{1}{3}T^{ii} = +\frac{1}{6}(\mathbf{E}^2 + \mathbf{B}^2), \quad (\text{S.9})$$

while most EM books define positive stress to be tension rather than compression, hence

$$T_{\text{EM}}^{ij} = +E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(\mathbf{E}^2 + \mathbf{B}^2) \quad (\text{S.10})$$

$$\text{but } P = -\frac{1}{3}T^{ii} = +\frac{1}{6}(\mathbf{E}^2 + \mathbf{B}^2). \quad (\text{S.11})$$

Problem 1(d):

In a sense, eq. (7) follows from part (b), but it is just as easy to derive it directly from the Maxwell equations. Starting with eq. (4), we immediately have

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -(\partial_\mu F^{\mu\lambda})F_\lambda^\nu - F^{\mu\lambda}(\partial_\mu F_\lambda^\nu) + \frac{1}{2}F_{\kappa\lambda}(\partial^\nu F^{\kappa\lambda}), \quad (\text{S.12})$$

where the last two terms cancel each other. Indeed, we may rewrite the second term here as

$$\begin{aligned} \text{second term} &= -F^{\mu\lambda} \times \partial_\mu F_\lambda^\nu = -F_{\mu\lambda} \times \partial^\mu F^{\nu\lambda} \\ &\quad \langle\langle \text{exchanging summation indices } \mu \leftrightarrow \lambda \rangle\rangle \\ &= -F_{\lambda\mu} \times \partial^\lambda F^{\nu\mu} = +F_{\mu\lambda} \times \partial^\lambda F^{\nu\mu} \\ &= \text{average_of} \left(+F_{\mu\lambda} \times \partial^\lambda F^{\nu\mu}, -F_{\mu\lambda} \times \partial^\mu F^{\nu\lambda} \right) \\ &= \frac{1}{2}F_{\mu\lambda} \left(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} \right), \end{aligned} \quad (\text{S.13})$$

hence combining the second and the third term on the RHS of eq. (S.12) gives us

$$\begin{aligned} \text{second term} + \text{third term} &= \frac{1}{2}F_{\mu\lambda} \left(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} \right) + \frac{1}{2}F_{\kappa\lambda}(\partial^\nu F^{\kappa\lambda}) \\ &\quad \langle\langle \text{renaming the summation index } \kappa \rightarrow \mu \text{ in the third term} \rangle\rangle \\ &= \frac{1}{2}F_{\mu\lambda} \left(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} \right) \\ &= 0 \end{aligned} \quad (\text{S.14})$$

thanks to the homogeneous Maxwell equation

$$\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0. \quad (\text{S.15})$$

Thus, on the RHS of eq. (S.12) the second and the the third term cancel each other and we

are left with the first term only:

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -(\partial_\mu F^{\mu\lambda})F^\nu_\lambda = -J^\lambda F^\nu_\lambda \quad (\text{S.16})$$

where the second equality comes from the inhomogeneous Maxwell equation $\partial_\mu F^{\mu\lambda} = J^\lambda$. This proves eq. (7). Finally, eq. (8) follows from eq. (7) and the net stress-energy conservation (6). *Quod erat demonstrandum.*

Problem 1(e):

For $\nu = 0$, on the LHS of eq. (7) we have

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = \frac{\partial \mathcal{U}}{\partial t} + \nabla \cdot \mathbf{S}, \quad (\text{S.17})$$

while on the RHS of eq. (7) we have

$$-J_\lambda F^{0\lambda} = +J^i F^{0i} = -\mathbf{J} \cdot \mathbf{E}. \quad (\text{S.18})$$

Thus, eq. (7) for $\nu = 0$ is the *Poynting theorem*

$$\frac{\partial \mathcal{U}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \quad (\text{S.19})$$

which is the local form of the work-energy theorem for the EM fields: The rate of the EM energy's non-conservation is equal to the power expended by the EM forces on the electric current $\mathbf{J}$.

Now consider eq. (7) for $\nu = i = 1, 2, 3$. On the LHS, we have

$$\partial_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_j T^{ji} = \frac{\partial S^i}{\partial t} + \nabla_j T^{ji}, \quad (\text{S.20})$$

which is the local non-conservation of the i^{th} component of the EM momentum. Indeed, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ gives not only the flux of the EM energy but also the density of the EM momentum, while the stress-tensor T^{ij} (in our sign convention) gives the flux of

the EM momentum. Physically, this non-conservation is due to mechanical forces between the EM fields and the currents, so we should have

$$\frac{\partial S^i}{\partial t} + \nabla^j T^{ji} = -f^i, \quad (\text{S.21})$$

where $\mathbf{f}$ is the density of the net EM forces on the charges and the currents,

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (\text{S.22})$$

And indeed, on the RHS of eq. (7) for $\nu = i$ we have

$$-J_\lambda F^{i\lambda} = -J^0 F^{i0} + J^j F^{ij} = -\rho E^i - \epsilon^{ijk} J^j B^k = -(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})^i = -f^i, \quad (\text{S.23})$$

in perfect agreement with eq. (S.21). In other words, the $\nu = 1, 2, 3$ components of eq. (7) give the local form of the momentum-impulse theorem for the EM field.

Problem 2(a):

As discussed in class — see [my notes on gauge symmetries](#) (pages 5–6), — for the EM field coupled to any kinds of charged scalars with a Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_a (D_\mu \Phi_a^*)(D^\mu \Phi_a) - V(\text{scalars}), \quad (\text{S.24})$$

the Euler–Lagrange equations for the charged fields are

$$D^\mu D_\mu \Phi_a + \frac{\partial V}{\partial \Phi_a^*} = 0, \quad D^\mu D_\mu \Phi_a^* + \frac{\partial V}{\partial \Phi_a} = 0, \quad (\text{S.25})$$

while the electric current is

$$J^\mu = \sum_a \left(-iq_a \Phi_a D^\mu \Phi_a^* + iq_a \Phi_a^* D^\mu \Phi_a \right). \quad (\text{S.26})$$

In particular, for the theory at hand there is only one charged scalar field Φ (and its conjugate Φ^*), the Lagrangian density is as in eq. (9), so the Euler–Lagrange equations for the charged

fields are

$$D_\mu D^\mu \Phi + m^2 \Phi = 0 = D_\mu D^\mu \Phi^* + m^2 \Phi^*, \quad (\text{S.27})$$

while the electric current is simply

$$J^\mu = -iq\Phi D^\mu \Phi^* + iq\Phi^* D^\mu \Phi. \quad (\text{S.28})$$

This current is manifestly gauge invariant, and is conserved, $\partial_\mu J^\mu = 0$ when the scalar fields obey their equation of motion (S.27). Indeed, by the Leibniz rule for the covariant derivatives

$$\begin{aligned} \partial_\mu (\Phi D^\mu \Phi^*) &= (D_\mu \Phi)(D^\mu \Phi^*) + \Phi(D_\mu D^\mu \Phi^*) \\ \langle\langle \text{by eq. (S.27)} \rangle\rangle &= (D_\mu \Phi)(D^\mu \Phi^*) + \Phi(-m^2 \Phi^*), \end{aligned} \quad (\text{S.29})$$

$$\begin{aligned} \partial_\mu (\Phi^* D^\mu \Phi) &= (D_\mu \Phi^*)(D^\mu \Phi) + \Phi^*(D_\mu D^\mu \Phi) \\ \langle\langle \text{by eq. (S.27)} \rangle\rangle &= (D_\mu \Phi^*)(D^\mu \Phi) + \Phi^*(-m^2 \Phi), \end{aligned} \quad (\text{S.30})$$

hence

$$\partial_\mu J^\mu = -\cancel{iq(D_\mu \Phi)(D^\mu \Phi^*)} + \cancel{iqm^2 \Phi \Phi^*} + \cancel{iq(D_\mu \Phi^*)(D^\mu \Phi)} - \cancel{iqm^2 \Phi^* \Phi} = 0. \quad (\text{S.31})$$

Problem 2(b):

According to the Noether theorem,

$$\begin{aligned} T_{\text{Noether}}^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial^\nu \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \partial^\nu \Phi^* - g^{\mu\nu} \mathcal{L} \\ &= T_{\text{Noether}}^{\mu\nu}(\text{EM}) + T_{\text{Noether}}^{\mu\nu}(\text{matter}) \end{aligned} \quad (\text{S.32})$$

where

$$T_{\text{Noether}}^{\mu\nu}(\text{EM}) = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \quad (\text{S.33})$$

similar to the free EM fields, and

$$T_{\text{Noether}}^{\mu\nu}(\text{matter}) = D^\mu \Phi^* \partial^\nu \Phi + D^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} (D^\lambda \Phi^* D_\lambda \Phi - m^2 \Phi^* \Phi). \quad (\text{S.34})$$

Both terms on the second line of eq. (S.32) lack $\mu \leftrightarrow \nu$ symmetry and gauge invariance and thus need $\partial_\lambda \mathcal{K}^{\lambda\mu,\nu}$ corrections for some $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$. We would like to show that the

same $\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu}A^\nu$ we used to improve the free electromagnetic stress-energy tensor will now symmetrize both the $T_{\text{EM}}^{\mu\nu}$ and $T_{\text{mat}}^{\mu\nu}$ at the same time!

Indeed, we saw in problem 1(b) that

$$\partial_\lambda \mathcal{K}^{\lambda\mu,\nu} = -J^\mu A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \quad (\text{S.2})$$

where the second term symmetrizes the EM stress-energy tensor,

$$T_{\text{Noether}}^{\mu\nu}(\text{EM}) + (\text{second term in eq. (S.2)}) = T_{\text{phys}}^{\mu\nu}(\text{EM}) = T_{\text{phys}}^{\nu\mu}(\text{EM}). \quad (\text{S.3})$$

At the same time, the first term in eq. (S.2) — namely the $-J^\mu A^\nu$ term — happens to symmetrize the matter stress-energy tensor. Indeed, consider the difference

$$\begin{aligned} \Delta T_{\text{matter}}^{\mu\nu} &\equiv T_{\text{phys}}^{\mu\nu}(\text{matter}) - T_{\text{Noether}}^{\mu\nu}(\text{matter}) \\ &= D^\mu \Phi^* (D^\nu \Phi - \partial^\nu \Phi) + D^\mu \Phi (D^\nu \Phi^* - \partial^\nu \Phi^*) \\ &= D^\mu \Phi^* (iqA^\nu \Phi) + D^\mu \Phi (-iqA^\nu \Phi^*) \\ &= -A^\nu \times (iq\Phi^* D^\mu \Phi - iq\Phi D^\mu \Phi^*) \\ &= -A^\nu J^\mu = \text{first term in eq. (S.2)}, \end{aligned} \quad (\text{S.35})$$

which immediately tells us that

$$T_{\text{Noether}}^{\mu\nu}(\text{matter}) + (\text{first term in eq. (S.2)}) = T_{\text{phys}}^{\mu\nu}(\text{matter}) = T_{\text{phys}}^{\nu\mu}(\text{matter}). \quad (\text{S.36})$$

Altogether, the divergence of the $\mathcal{K}^{\lambda\mu,\nu}$ as in eq. (2) symmetrizes the net stress-energy tensor (12), *quod erat demonstrandum*.

Problem 2(c):

Since the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and

therefore

$$\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi). \quad (\text{S.37})$$

In a similar manner

$$\begin{aligned} \partial_\mu((D^\mu\Phi^*)(D^\nu\Phi)) &= (D_\mu D^\mu\Phi^*)(D^\nu\Phi) + (D^\mu\Phi^*)(D_\mu D^\nu\Phi) \\ &= (D^2\Phi^*)(D^\nu\Phi) + (D_\mu\Phi^*)(D^\mu D^\nu\Phi = D^\nu D^\mu\Phi + [D^\mu, D^\nu]\Phi) \\ &\quad \langle\langle \text{using equation of motion } D^2\Phi^* = -m^2\Phi^* \text{ for the first term} \rangle\rangle \\ &\quad \langle\langle \text{and } [D^\mu, D^\nu]\Phi = iqF^{\mu\nu}\Phi \text{ for the second term} \rangle\rangle \\ &= (-m^2\Phi^*)(D^\nu\Phi) + (D_\mu\Phi^*)(D^\nu D^\mu\Phi) + (D_\mu\Phi^*)(iqF^{\mu\nu}\Phi), \end{aligned} \quad (\text{S.38})$$

and likewise

$$\begin{aligned} \partial_\mu((D^\mu\Phi)(D^\nu\Phi^*)) &= (D_\mu D^\mu\Phi)(D^\nu\Phi^*) + (D^\mu\Phi)(D_\mu D^\nu\Phi^*) \\ &= (-m^2\Phi)(D^\nu\Phi^*) + (D_\mu\Phi)(D^\nu D^\mu\Phi^*) + (D_\mu\Phi)(-iqF^{\mu\nu}\Phi^*). \end{aligned} \quad (\text{S.39})$$

Finally,

$$\begin{aligned} \partial_\mu \left[-g^{\mu\nu} (D_\lambda\Phi^* D^\lambda\Phi - m^2\Phi^*\Phi) \right] &= -\partial^\nu (D_\lambda\Phi^* D^\lambda\Phi) + m^2\partial^\nu(\Phi^*\Phi) \\ &= -(D^\nu D^\mu\Phi^*)(D_\mu\Phi) - (D_\mu\Phi^*)(D^\nu D^\mu\Phi) \\ &\quad + m^2\Phi(D^\nu\Phi^*) + m^2\Phi^*(D^\nu\Phi). \end{aligned} \quad (\text{S.40})$$

Together, the left hand sides of eqs. (S.38), (S.39) and (S.40) comprise $\partial_\mu T_{\text{mat}}^{\mu\nu}$ — *cf.* eq. (13). On the other hand, totaling up the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Therefore,

$$\begin{aligned} \partial_\mu T_{\text{mat}}^{\mu\nu} &= (D_\mu\Phi^*)(iqF^{\mu\nu}\Phi) + (D_\mu\Phi)(-iqF^{\mu\nu}\Phi^*) + \text{massive cancellation} \\ &= F^{\mu\nu}(iq\Phi D_\mu\Phi^* - iq\Phi^* D_\mu\Phi) \\ &= F^{\mu\nu} \times (-J_\mu) = +F^{\nu\lambda}J_\lambda \end{aligned} \quad (\text{S.41})$$

in accordance with eq. (15).

Finally, combining this formula with eq. (7) immediately shows us that the net stress-energy tensor (13) is conserved,

$$\partial_\mu T_{\text{tot}}^{\mu\nu} = \partial_\mu T_{\text{tot}}^{\mu\nu} + \partial_\mu T_{\text{EM}}^{\mu\nu} = 0. \quad (\text{S.42})$$

Quod erat demonstrandum.

Problem 3(a):

In $N \times N$ matrix form, the local $SU(N)$ symmetry acts on the adjoint matter field $\Phi(x)$ and the gauge field $\mathcal{A}_\mu(x)$ according to

$$\Phi'(x) = U(x)\Phi(x)U^\dagger(x), \quad \mathcal{A}'_\mu(x) = U(x)\mathcal{A}_\mu(x)U^\dagger(x) + i\partial_\mu U(x)U^\dagger(x). \quad (\text{S.43})$$

Consequently, the covariant derivatives (20) become

$$D_\mu \Phi(x) \rightarrow D'_\mu \Phi'(x) = \partial_\mu \Phi(x) + i[\mathcal{A}'_\mu(x), \Phi'(x)] \quad (\text{S.44})$$

where the first term on the RHS expands to

$$\begin{aligned} \partial_\mu \Phi' &= \partial_\mu (U\Phi U^\dagger) \\ &= (\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger + U\Phi(\partial_\mu U^\dagger) \\ &= UU^\dagger(\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger - U\Phi U^\dagger(\partial_\mu U)U^\dagger \\ &= U\left((U^\dagger \partial_\mu U)\Phi + \partial_\mu \Phi - \Phi(U^\dagger \partial_\mu U)\right)U^\dagger \\ &= U\left(\partial_\mu \Phi + [U^\dagger \partial_\mu U, \Phi]\right)U^\dagger \end{aligned} \quad (\text{S.45})$$

while the second term expands to

$$\begin{aligned} [\mathcal{A}'_\mu, \Phi'] &= [U\mathcal{A}_\mu U^\dagger, U\Phi U^\dagger] + [i(\partial_\mu U)U^\dagger, U\Phi U^\dagger] \\ &= U\left([\mathcal{A}_\mu, \Phi] + i[U^\dagger \partial_\mu U, \Phi]\right)U^\dagger \end{aligned} \quad (\text{S.46})$$

Combining the two expansions, we arrive at

$$D'_\mu \Phi' = U\left(\partial_\mu \Phi + \cancel{[U^\dagger \partial_\mu U, \Phi]} + i[\mathcal{A}_\mu, \Phi] - \cancel{[U^\dagger \partial_\mu U, \Phi]}\right)U^\dagger = U(D_\mu \Phi)U^\dagger. \quad (\text{S.47})$$

Thus, the $D_\mu \Phi(x)$ matrix transforms exactly like the $\Phi(x)$ matrix itself, which makes the D_μ derivative (20) covariant. *Quod erat demonstrandum.*

Problem 3(b):

Let's start with the second line of eq. (22). In the matrix form, the adjoint multiplet Φ is a matrix, the fundamental multiplet Ψ is a column vector, and their matrix product $\Phi\Psi$ is also a column vector. The covariant derivatives acts on these matrices and vectors as

$$D_\mu\Phi = \partial_\mu\Phi + i[\mathcal{A}_\mu, \Phi], \quad D_\mu\Psi = \partial_\mu\Psi + i\mathcal{A}_\mu\Psi, \quad (\text{S.48})$$

while

$$\begin{aligned} D_\mu(\Phi\Psi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Psi) + i\mathcal{A}_\mu(\Phi\Psi) \\ &= (\partial_\mu\Phi)\Psi + \Phi(\partial_\mu\Psi) + i[\mathcal{A}_\mu, \Phi]\Psi + i\Phi\mathcal{A}_\mu\Psi \\ &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi). \end{aligned} \quad (\text{S.49})$$

Likewise, on the third line of eq. (22), Ξ is a matrix, $\Psi^\dagger$ is a row vector, and their matrix product $\Psi^\dagger\Xi$ is also a row vector. Therefore

$$D_\mu\Psi^\dagger = \partial_\mu\Psi^\dagger - i\Psi^\dagger\mathcal{A}_\mu, \quad D_\mu\Xi = \partial_\mu\Xi + i[\mathcal{A}_\mu, \Xi], \quad (\text{S.50})$$

while

$$\begin{aligned} D_\mu(\Psi^\dagger\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Psi^\dagger\Xi) - i(\Psi^\dagger\Xi)\mathcal{A}_\mu \\ &= (\partial_\mu\Psi^\dagger)\Xi + \Psi^\dagger(\partial_\mu\Xi) - i\Psi^\dagger\mathcal{A}_\mu\Xi - i\Psi^\dagger[\Xi, \mathcal{A}_\mu] \\ &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \quad (\text{S.51})$$

Finally, on the first line of eq. (22), Φ and Ξ are both $N \times N$ matrices and their product $\Phi\Xi$ is also a matrix. Consequently,

$$\begin{aligned} D_\mu(\Phi\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Xi) + i[\mathcal{A}_\mu, \Phi\Xi] \\ &= (\partial_\mu\Phi)\Xi + \Phi(\partial_\mu\Xi) + i[\mathcal{A}_\mu, \Phi]\Xi + i\Phi[\mathcal{A}_\mu, \Xi] \\ &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi). \end{aligned} \quad (\text{S.52})$$

Quod erat demonstrandum.

Problem 3(c):

For the adjoint multiplet of fields $\Phi(x)$,

$$\begin{aligned}
D_\mu D_\nu \Phi &= \partial_\mu (D_\nu \Phi) + i[\mathcal{A}_\mu, D_\nu \Phi] \\
&= \partial_\mu \left(\partial_\nu \Phi + i[\mathcal{A}_\nu, \Phi] \right) + i[\mathcal{A}_\mu, \left(\partial_\nu \Phi + i[\mathcal{A}_\nu, \Phi] \right)] \\
&= \partial_\mu \partial_\nu \Phi + i[\partial_\mu \mathcal{A}_\nu, \Phi] + i[\mathcal{A}_\nu, \partial_\mu \Phi] + i[\mathcal{A}_\mu, \partial_\nu \Phi] - [\mathcal{A}_\mu, [\mathcal{A}_\nu, \Phi]]
\end{aligned} \tag{S.53}$$

where the blue marks terms that are symmetric WRT $\mu \leftrightarrow \nu$ while the asymmetric terms are marked in red. Only the red terms contribute to the difference

$$\begin{aligned}
[D_\mu, D_\nu] \Phi &= D_\mu D_\nu \Phi - D_\nu D_\mu \Phi \\
&= i[\partial_\mu \mathcal{A}_\nu, \Phi] - [\mathcal{A}_\mu, [\mathcal{A}_\nu, \Phi]] - i[\partial_\nu \mathcal{A}_\mu, \Phi] + [\mathcal{A}_\nu, [\mathcal{A}_\mu, \Phi]] \\
&= i[\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \Phi] - [[\mathcal{A}_\mu, \mathcal{A}_\nu], \Phi]
\end{aligned} \tag{S.54}$$

where the second term on the bottom line follows from the Jacobi identity for the matrix commutator:

$$\begin{aligned}
-[\mathcal{A}_\mu, [\mathcal{A}_\nu, \Phi]] + [\mathcal{A}_\nu, [\mathcal{A}_\mu, \Phi]] &= +[\mathcal{A}_\mu, [\Phi, \mathcal{A}_\nu]] + [\mathcal{A}_\nu, [\mathcal{A}_\mu, \Phi]] \\
&= -[\Phi, [\mathcal{A}_\nu, \mathcal{A}_\mu]] = -[[\mathcal{A}_\mu, \mathcal{A}_\nu], \Phi].
\end{aligned} \tag{S.55}$$

Altogether, we have

$$[D_\mu, D_\nu] \Phi = i[(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]), \Phi] \equiv i[\mathcal{F}_{\mu\nu}, \Phi] = ig[F_{\mu\nu}, \Phi] \tag{S.56}$$

where the second equality follows from the definition of the non-abelian $\mathcal{F}_{\mu\nu}$ and the third equality from $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$. Finally, in components

$$ig[F_{\mu\nu}, \Phi] = igF_{\mu\nu}^b \Phi^c \times \left[\frac{\lambda^b}{2}, \frac{\lambda^c}{2} \right] = igF_{\mu\nu}^b \Phi^c \times if^{bca} \frac{\lambda^a}{2} \tag{S.57}$$

and hence $[D_\mu, D_\nu] \Phi^a = -gf^{bca} F_{\mu\nu}^b \Phi^c = -gf^{abc} F_{\mu\nu}^b \Phi^c$.

Problem 3(d):

In matrix notations, the non-abelian gauge symmetries act on vector potentials $\mathcal{A}_\mu(x)$ according to

$$\mathcal{A}'_\mu(x) = U(x)\mathcal{A}_\mu(x)U^\dagger(x) + i\partial_\mu U(x)U^\dagger(x). \quad (\text{S.58})$$

Taking

$$\mathcal{F}_{\mu\nu}(x) \stackrel{\text{def}}{=} \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) + i[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)] \quad (\text{S.59})$$

as the definition of the tension fields $\mathcal{F}_{\mu\nu}(x)$, we then have

$$\mathcal{F}'_{\mu\nu}(x) = \partial_\mu \mathcal{A}'_\nu(x) - \partial_\nu \mathcal{A}'_\mu(x) + i[\mathcal{A}'_\mu(x), \mathcal{A}'_\nu(x)], \quad (\text{S.60})$$

whatever that evaluates to. Specifically, the first term here evaluates to

$$\begin{aligned} \partial_\mu \mathcal{A}'_\nu &= \partial_\mu \left(U\mathcal{A}_\nu U^\dagger + i(\partial_\nu U)U^\dagger \right) \\ &= U(\partial_\mu \mathcal{A}_\nu)U^\dagger + \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger \right] + i(\partial_\mu \partial_\nu U)U^\dagger - i(\partial_\nu U)U^\dagger \times (\partial_\mu U)U^\dagger \end{aligned} \quad (\text{S.61})$$

where the second equality follows from

$$\partial_\mu \left(U\mathcal{A}_\nu U^\dagger \right) = U(\partial_\mu \mathcal{A}_\nu)U^\dagger + \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger \right] \quad (\text{S.62})$$

— *cf.* similar formula (S.45) — and

$$\partial_\mu \left((\partial_\nu U)U^\dagger \right) = (\partial_\mu \partial_\nu U)U^\dagger + (\partial_\nu U)(\partial_\mu U^\dagger) = (\partial_\mu \partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger (\partial_\mu U)U^\dagger. \quad (\text{S.63})$$

Likewise

$$\partial_\nu \mathcal{A}'_\mu = U(\partial_\nu \mathcal{A}_\mu)U^\dagger + \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger \right] + i(\partial_\nu \partial_\mu U)U^\dagger - i(\partial_\mu U)U^\dagger \times (\partial_\nu U)U^\dagger \quad (\text{S.64})$$

and hence

$$\begin{aligned} \partial_\mu \mathcal{A}'_\nu - \partial_\nu \mathcal{A}'_\mu &= U(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu)U^\dagger + \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger \right] - \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger \right] \\ &\quad + 0 + i \left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger \right]. \end{aligned} \quad (\text{S.65})$$

At the same time, the commutator part of the tension field transforms into

$$\begin{aligned}
i [\mathcal{A}'_\mu, \mathcal{A}'_\nu] &= i \left[\left(U \mathcal{A}_\mu U^\dagger + i(\partial_\mu U) U^\dagger \right), \left(U \mathcal{A}_\nu U^\dagger + i(\partial_\nu U) U^\dagger \right) \right] \\
&= i \left[U \mathcal{A}_\mu U^\dagger, U \mathcal{A}_\nu U^\dagger \right] - \left[(\partial_\mu U) U^\dagger, U \mathcal{A}_\nu U^\dagger \right] \\
&\quad - \left[U \mathcal{A}_\mu U^\dagger, (\partial_\nu U) U^\dagger \right] - i \left[(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right],
\end{aligned} \tag{S.66}$$

Combining eqs. (S.65) and (S.66) gives us 8 terms on the RHS, but fortunately 6 of them cancel each other. Only the first terms on right hand sides of (S.65) and (S.66) survive the cancellation, thus

$$\begin{aligned}
\partial_\mu \mathcal{A}'_\nu - \partial_\nu \mathcal{A}'_\mu + i [\mathcal{A}'_\mu, \mathcal{A}'_\nu] &= U (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu) U^\dagger + i \left[U \mathcal{A}_\mu U^\dagger, U \mathcal{A}_\nu U^\dagger \right] \\
&= U \left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i [\mathcal{A}_\mu, \mathcal{A}_\nu] \right) U^\dagger,
\end{aligned} \tag{S.67}$$

or in other words,

$$\mathcal{F}'_{\mu\nu}(x) = U(x) \mathcal{F}_{\mu\nu}(x) U^\dagger(x). \tag{S.68}$$

Quod erat demonstrandum.

Problem 4(a):

There are two ways to prove the non-abelian differential identity (24): (i) using parts (b,c) of problem 3, or (I) a hard calculation based directly on eq. (S.59). Either way we shall need the Jacobi identity for the commutators.

Let me start with the easier proof. In problem (3.b) we have proved the Leibniz rule for the covariant derivatives of a matrix product of two adjoint fields $\Phi(x)$ and $\Xi(x)$. Clearly, the same Leibniz rule also applies to the commutator $[\Phi, \Xi]$:

$$D_\mu [\Phi, \Xi] = [D_\mu \Phi, \Xi] + [\Phi, D_\mu \Xi]. \tag{S.69}$$

In particular, for $\Phi = \mathcal{F}_{\mu\nu}$ and arbitrary Ξ , we have

$$D_\lambda ([\mathcal{F}_{\mu\nu}, \Xi]) = [D_\lambda \mathcal{F}_{\mu\nu}, \Xi] + [\mathcal{F}_{\mu\nu}, D_\lambda \Xi]. \tag{S.70}$$

On the other hand, in problem (3.c) we saw that for any adjoint field $\Xi(x)$ we have $[\mathcal{F}_{\mu\nu}, \Xi] = -i[D_\mu, D_\nu]\Xi$. Likewise, for the $D_\lambda\Xi(x)$ we also have $[\mathcal{F}_{\mu\nu}, D_\lambda\Xi] = -i[D_\mu, D_\nu]D_\lambda\Xi$. Consequently, eq. (S.70) becomes

$$-iD_\lambda[D_\mu, D_\nu]\Xi = [D_\lambda\mathcal{F}_{\mu\nu}, \Xi] - i[D_\mu, D_\nu]D_\lambda\Xi \quad (\text{S.71})$$

and hence

$$i[D_\lambda\mathcal{F}_{\mu\nu}, \Xi] = [D_\lambda, [D_\mu, D_\nu]]\Xi. \quad (\text{S.72})$$

Now, let's sum 3 such formulae, one for each cyclic permutations of the indices λ, μ, ν . On the left hand side, this gives us

$$i[(D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu}), \Xi] = \dots$$

while on the right hand side we obtain

$$\dots = ([D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]])\Xi = 0 \quad (\text{S.73})$$

due to Jacobi identity for the double commutators of the three covariant derivatives D_λ, D_μ , and D_ν . Consequently

$$[(D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu}), \Xi] = 0,$$

and this must be true for any adjoint field $\Xi(x)$. Moreover, for any x, λ, μ, ν , the $N \times N$ matrix

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu}$$

is traceless, and the only way it may commute with all traceless hermitian matrices $\Xi(x)$ is by being zero, thus

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \quad (\text{S.74})$$

This is my first proof of the non-abelian differential identity (24).

The second proof of the differential identity (24) follows directly from the definition (S.59) of the non-abelian tension fields and the covariant derivatives (20). Let's spell out $D_\lambda \mathcal{F}_{\mu\nu}$ in detail:

$$\begin{aligned}
D_\lambda \mathcal{F}_{\mu\nu} &= \partial_\lambda \mathcal{F}_{\mu\nu} + i[\mathcal{A}_\lambda, \mathcal{F}_{\mu\nu}] \\
&= \partial_\lambda (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]) + i[\mathcal{A}_\lambda, (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu])] \\
&= \partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu + i[\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] + i[\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] \\
&\quad + i[\mathcal{A}_\lambda, \partial_\mu \mathcal{A}_\nu] - i[\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu] - [\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]] \\
&= (\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu) + i([\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] - [\partial_\mu \mathcal{A}_\nu, \mathcal{A}_\lambda]) \\
&\quad + i([\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] - [\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu]) - ([\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]]).
\end{aligned} \tag{S.75}$$

On the bottom two lines here I have grouped terms in $()$ so that after summing over cyclic permutations of the indices λ, μ, ν , we get a zero sum separately for each group. Indeed,

$$\begin{aligned}
(\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu) + \text{cyclic} &= (\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\nu \partial_\lambda \mathcal{A}_\mu) + \text{cyclic} \\
&= 0 \quad \langle\langle \text{by inspection} \rangle\rangle, \\
([\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] - [\partial_\mu \mathcal{A}_\nu, \mathcal{A}_\lambda]) + \text{cyclic} &= 0 \quad \langle\langle \text{by inspection} \rangle\rangle, \\
([\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] - [\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu]) + \text{cyclic} &= 0 \quad \langle\langle \text{by inspection} \rangle\rangle, \text{ and} \\
[\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]] + \text{cyclic} &= 0 \quad \langle\langle \text{by Jacobi identity} \rangle\rangle.
\end{aligned} \tag{S.76}$$

Therefore,

$$D_\lambda \mathcal{F}_{\mu\nu} + \text{cyclic} \equiv D_\lambda \mathcal{F}_{\mu\nu} + D_\mu \mathcal{F}_{\nu\lambda} + D_\nu \mathcal{F}_{\lambda\mu} = 0. \tag{24}$$

Problem 4(b):

The Euler–Lagrange field equations follow from requiring zero first variation of the action $S = \int \mathcal{L}$ under infinitesimal variation of the independent fields $\mathcal{A}_\mu(x)$. Let's start by calculating the variation of the tension fields $\mathcal{F}_{\mu\nu}$:

$$\begin{aligned}
\delta \mathcal{F}_{\mu\nu} &\equiv \delta (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]) \\
&= \partial_\mu \delta \mathcal{A}_\nu - \partial_\nu \delta \mathcal{A}_\mu + i[\delta \mathcal{A}_\mu, \mathcal{A}_\nu] + i[\mathcal{A}_\mu, \delta \mathcal{A}_\nu] \\
&= (\partial_\mu \delta \mathcal{A}_\nu + i[\mathcal{A}_\mu, \delta \mathcal{A}_\nu]) - (\partial_\nu \delta \mathcal{A}_\mu + i[\mathcal{A}_\nu, \delta \mathcal{A}_\mu]) \\
&= D_\mu \delta \mathcal{A}_\nu - D_\nu \delta \mathcal{A}_\mu
\end{aligned} \tag{S.77}$$

where we treat the matrix-valued variations $\delta\mathcal{A}_\nu(x)$ as adjoint fields so their covariant derivative work according to eq. (20), $D_\mu\delta\mathcal{A}_\nu \equiv \partial_\mu\delta\mathcal{A}_\nu + i[\mathcal{A}_\mu, \delta\mathcal{A}_\nu]$ and likewise for the $D_\nu\delta\mathcal{A}_\mu$.

In light of eq. (S.77), the trace in the Yang–Mills Lagrangian (25) varies by

$$\begin{aligned}\delta \operatorname{tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) &= 2 \operatorname{tr}(\mathcal{F}^{\mu\nu}\delta\mathcal{F}_{\mu\nu}) = 2 \operatorname{tr}(\mathcal{F}^{\mu\nu}(D_\mu\delta\mathcal{A}_\nu - D_\nu\delta\mathcal{A}_\mu)) \\ &= 4 \operatorname{tr}(\mathcal{F}^{\mu\nu}D_\mu\delta\mathcal{A}_\nu) \quad \langle\langle \text{since } \mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu} \rangle\rangle \\ &= -4 \operatorname{tr}((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu) + 4\partial_\mu \operatorname{tr}(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)\end{aligned}\tag{S.78}$$

where the last equality follows from the Leibniz rule for the two adjoint fields $\mathcal{F}^{\mu\nu}$ and $\delta\mathcal{A}_\nu$:

$$\begin{aligned}\operatorname{tr}((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu) + \operatorname{tr}(\mathcal{F}^{\mu\nu}(D_\mu\delta\mathcal{A}_\nu)) &= \\ &= \operatorname{tr}(D_\mu(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)) \\ &= \operatorname{tr}(\partial_\mu(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)) + i \operatorname{tr}([\mathcal{A}_\mu, (\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)]) \\ &= \partial_\mu \operatorname{tr}(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu) + 0 \quad \langle\langle \text{since trace of a commutator is zero} \rangle\rangle.\end{aligned}\tag{S.79}$$

Thus,

$$\delta\mathcal{L}_{\text{YM}} = \frac{2}{g^2} \operatorname{tr}((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu) - \text{a total divergence}\tag{S.80}$$

so the net Yang–Mills action varies by

$$\delta S = \frac{2}{g^2} \int d^4x \operatorname{tr}(D_\mu\mathcal{F}^{\mu\nu}(x)\delta\mathcal{A}_\nu(x)) = \frac{1}{g^2} \int d^4x \sum_a D_\mu\mathcal{F}^{a\mu\nu}(x) \times \delta\mathcal{A}_\nu^a(x).\tag{S.81}$$

To make this variation vanish for any infinitesimal $\delta\mathcal{A}_\nu^a(x)$ we need $D_\mu\mathcal{F}^{a\mu\nu}(x) \equiv 0$, and this becomes the Euler–Lagrange equation for the Yang–Mills theory,

$$D_\mu\mathcal{F}^{a\mu\nu} = 0.\tag{S.82}$$

Problem 4(c):

In problem part (b) we saw that under infinitesimal variations of the gauge fields, the YM Lagrangian varies by

$$\delta\mathcal{L}_{\text{YM}} = \frac{1}{g^2} \sum_a D_\mu \mathcal{F}^{a\mu\nu} \times \delta\mathcal{A}_\nu^a + \partial_\mu(\cdots) = \sum_a D_\mu F^{a\mu\nu} \times \delta A_\nu^a + \partial_\mu(\cdots). \quad (\text{S.83})$$

Now let's add the matter Lagrangian $\mathcal{L}_{\text{mat}}(\phi, D\phi)$ for some matter fields in a non-trivial multiplet (or multiplets) of the gauge symmetry. When we vary the gauge fields $A_\nu^a(x)$ while keeping the matter fields $\phi(x)$ fixed, the covariant derivatives $D\phi$ vary due to $igA_\nu^a t^a \phi$ terms in $D_\nu \phi$, which leads to non-trivial variation

$$\delta\mathcal{L}_{\text{mat}} = \sum_a \frac{\partial\mathcal{L}_{\text{mat}}}{\partial A_\nu^a} \times \delta A_\nu^a \equiv - \sum_a J^{a\nu} \times \delta A_\nu^a. \quad (\text{S.84})$$

Altogether, the net action of the theory varies by

$$\delta S = \int d^4x \sum_a \left(D_\mu F^{a\mu\nu}(x) - J^{a\nu}(x) \right) \times \delta A_\nu^a(x). \quad (\text{S.85})$$

Requiring this variation to vanish for any $\delta A_\nu^a(x)$ leads to the field equations

$$D_\mu F^{a\mu\nu} = J^{a\nu}, \quad (\text{S.86})$$

or in matrix notations $D_\mu F^{\mu\nu} = J^\nu$. This is the non-abelian version of the Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$.

In the abelian EM theory, the equations $\partial_\mu F^{\mu\nu} = J^\nu$ require the electric current to be conserved, $\partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$ since $F^{\mu\nu} = -F^{\nu\mu}$ and the derivatives commute with each other. The non-abelian tension fields $F^{\mu\nu}$ are also antisymmetric in $\mu \leftrightarrow \nu$, but the covariant derivatives do not commute, $D_\mu D_\nu \neq D_\nu D_\mu$. Therefore,

$$D_\nu J^\nu = D_\nu D_\mu F^{\mu\nu} = \frac{1}{2} [D_\mu, D_\nu] \mathcal{F}^{\mu\nu} = \frac{i}{2} [\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}] \quad (\text{S.87})$$

where the last equality works exactly as in problem 3(c) — the $\mathcal{F}^{a\mu\nu}$ fields form an adjoint multiplet of fields, and for any such multiplet packed into an hermitian $N \times N$ matrix Φ ,

$[D_\mu, D_\nu]\Phi = i[\mathcal{F}_{\mu\nu}, \Phi]$. However, unlike a generic matrix Φ which may commute or not commute with the $\mathcal{F}_{\mu\nu}$, for any μ and ν the $\mathcal{F}^{\mu\nu}$ matrix always commutes with *itself*. Thus,

$$[\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}] = 0 \quad \text{even before summing over } \mu \text{ and } \nu. \quad (\text{S.88})$$

Of course, after the summing over μ and ν we still have a zero, thus $D_\nu D_\mu \mathcal{F}^{\mu\nu}(x) \equiv 0$.

Thus, consistency of the field equations (S.86) for the gauge fields requires the non-abelian currents $J^{a\mu}$ to be *covariantly conserved*:

$$D_\nu J^\nu = D_\mu D_\nu F^{\mu\nu} = 0, \quad (\text{S.89})$$

or in components

$$\partial_\nu J^{a\nu} - f^{abc} \mathcal{A}_\nu^b J^{c\nu} = 0. \quad (\text{S.90})$$

Note: because of the covariantizing term here, we do not have conserved net charges; alas,

$$\frac{d}{dt} \int d^3\mathbf{x} J^{a0}(\mathbf{x}, t) \neq 0. \quad (\text{S.91})$$

Problem 4(d):

The currents J_μ^a come from the covariant derivatives in the Lagrangian for the scalar fields

$$\mathcal{L}_{\text{mat}} = D_\mu \Psi^\dagger D^\mu \Psi - V(\Psi^\dagger \Psi). \quad (\text{S.92})$$

Expanding the covariant derivatives $D_\mu \Psi^\dagger$ and $D^\mu \Psi$ in components of Ψ^i , Ψ_i^* , and A_μ^a , we obtain

$$D_\mu \Psi_i^* = \partial_\mu \Psi_i^* - \frac{ig}{2} A_\mu^a \Psi_i^* (\lambda^a)^i_j, \quad D^\mu \Psi^i = \partial^\mu \Psi^i + \frac{ig}{2} A^{a\mu} (\lambda^a)^i_j \Psi^j, \quad (\text{S.93})$$

and hence

$$\begin{aligned} J_\mu^a &= -\frac{\partial \mathcal{L}_{\text{mat}}}{\partial A_\mu^a} = -D^\nu \Psi_i^* \times \frac{\partial D_\nu \Psi^i}{\partial A_\mu^a} - \frac{\partial D_\nu \Psi_i^*}{\partial A_\mu^a} \times D^\nu \Psi^i \\ &= \frac{g}{2} \left(-i D^\nu \Psi_i^* \times \delta_\nu^\mu (\lambda^a)^i_j \Psi^j + i \delta_\nu^\mu \Psi_j^* (\lambda^a)^j_i \times D^\nu \Psi^i \right) \\ &= \frac{g}{2} \left(-i (D^\mu \Psi^\dagger) \lambda^a \Psi + i \Psi^\dagger \lambda^a (D^\mu \Psi) \right) \\ &= -g \text{Im} \left(\Psi^\dagger \lambda^a D^\mu \Psi \right), \end{aligned} \quad (\text{S.94})$$

exactly as in eq. (32).

Now consider a gauge transform of these currents. In matrix notations, the fields $\Psi(x)$, $\Psi^\dagger(x)$, and their covariant derivatives transform to

$$\begin{aligned}\Psi'(x) &= U(x)\Psi(x), & (D_\mu\Psi)'(x) &= U(x)D_\mu\Psi(x), \\ \Psi'^\dagger(x) &= \Psi^\dagger(x)U^\dagger(x), & (D_\mu\Psi')^\dagger(x) &= D_\mu\Psi^\dagger(x)U^\dagger(x).\end{aligned}\tag{S.95}$$

Consequently,

$$\begin{aligned}\left(\Psi^\dagger\lambda^a D_\mu\Psi\right)' &= (\Psi^\dagger U^\dagger)\lambda^a (U D_\mu\Psi) \\ &= \Psi^\dagger (U^\dagger\lambda^a U) D_\mu\Psi \\ \ll \text{by eq. (33)} \gg &= \Psi^\dagger \times (R_{\text{adj}}^{ab}(U)\lambda^b) \times D_\mu\Psi \\ &= R_{\text{adj}}^{ab}(U) \times \Psi^\dagger\lambda^b D_\mu\Psi,\end{aligned}\tag{S.96}$$

and likewise

$$\left((D_\mu\Psi^\dagger)\lambda^a\Psi\right)' = R_{\text{adj}}^{ab}(U) \times (D_\mu\Psi^\dagger)\lambda^b\Psi.\tag{S.97}$$

Thus, both terms in eq. (32) for the currents transforms like members of an adjoint multiplet, so the currents themselves transform as

$$J_\mu'^a(x) = R_{\text{adj}}^{ab}(U(x)) \times J_\mu^b(x).\tag{S.98}$$

Quod erat demonstrandum.

Problem 4(e):

First, let's derive the Leibniz rule for the adjoint multiplets of the form $Q^a = \Psi^\dagger\lambda^a\Psi'$:

$$D_\mu(\Psi^\dagger\lambda^a\Psi') = D_\mu\Psi^\dagger\lambda^a\Psi' + \Psi^\dagger\lambda^a D_\mu\Psi'.\tag{S.99}$$

Proof:

$$\begin{aligned}D_\mu(\Psi^\dagger\lambda^a\Psi') &\equiv \partial_\mu(\Psi^\dagger\lambda^a\Psi') - gf^{abc}A_\mu^b(\Psi^\dagger\lambda^c\Psi') \\ &= (\partial_\mu\Psi^\dagger)\lambda^a\Psi' + \Psi^\dagger\lambda^a(\partial_\mu\Psi') - gA_\mu^b\Psi^\dagger(f^{abc}\lambda^c = -\frac{i}{2}[\lambda^a, \lambda^b])\Psi' \\ &= \left(\partial_\mu\Psi^\dagger - \frac{ig}{2}A_\mu^b\Psi^\dagger\lambda^b\right)\lambda^a\Psi' + \Psi^\dagger\lambda^a\left(\partial_\mu\Psi' + \frac{ig}{2}A_\mu^b\lambda^b\Psi'\right) \\ &= D_\mu\Psi^\dagger\lambda^a\Psi' + \Psi^\dagger\lambda^a D_\mu\Psi'.\end{aligned}\tag{S.100}$$

Thanks to this lemma, the non-abelian currents (S.94) satisfy

$$\begin{aligned}
D_\mu J^{a\mu} &= -\frac{ig}{2} D_\mu \left(\Psi^\dagger \lambda^a D^\mu \Psi - D^\mu \Psi^\dagger \lambda^a \Psi \right) \\
&= -\frac{ig}{2} \left(\cancel{D_\mu \Psi^\dagger \lambda^a D^\mu \Psi} + \Psi^\dagger \lambda^a D_\mu D^\mu \Psi - D_\mu D^\mu \Psi^\dagger \lambda^a \Psi - \cancel{D^\mu \Psi^\dagger \lambda^a D_\mu \Psi} \right) \\
&= g \operatorname{Im} \left(\Psi^\dagger \lambda^a D_\mu D^\mu \Psi \right).
\end{aligned} \tag{S.101}$$

Now let the scalar fields satisfy their covariant equations of motion

$$D_\mu \frac{\partial \mathcal{L}}{\partial (D_\mu \Psi_i)} = \frac{\partial \mathcal{L}}{\partial \Psi_i}, \quad D_\mu \frac{\partial \mathcal{L}}{\partial (D_\mu \Psi^{*i})} = \frac{\partial \mathcal{L}}{\partial \Psi^{*i}}. \tag{S.102}$$

For the Lagrangian (31) these equations read

$$\begin{aligned}
D_\mu D^\mu \Psi^{*i} &= -\frac{\partial V}{\partial \Psi_i} = -\Psi^{i*} \times \left(m^2 + \frac{\lambda}{2} \Psi^\dagger \Psi \right), \\
D_\mu D^\mu \Psi_i &= -\frac{\partial V}{\partial \Psi^{*i}} = -\left(m^2 + \frac{\lambda}{2} \Psi^\dagger \Psi \right) \times \Psi_i,
\end{aligned} \tag{S.103}$$

so for the fields obeying these equations

$$\Psi^\dagger \lambda^a D_\mu D^\mu \Psi = -\left(m^2 + \frac{\lambda}{2} \Psi^\dagger \Psi \right) \times \Psi^\dagger \lambda^a \Psi = (\text{real}) \times (\text{real}) \tag{S.104}$$

for any hermitian matrix λ^a , and therefore

$$D_\mu J^{a\mu} = g \operatorname{Im} \left(\Psi^\dagger \lambda^a D_\mu D^\mu \Psi \right) = 0. \tag{S.105}$$