

Problem 1(a):

$$\begin{aligned}
 [\hat{J}^i, \hat{J}^j] &\equiv \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn} [\hat{J}^{kl}, \hat{J}^{mn}] = \langle\langle \text{by eq. (1)} \rangle\rangle \\
 &= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn} \left(-ig^{km}\hat{J}^{\ell n} + ig^{kn}\hat{J}^{\ell m} + ig^{\ell m}\hat{J}^{kn} - g^{\ell n}\hat{J}^{km} \right) \\
 &\quad \langle\langle \text{by antisymmetry of the } \epsilon\text{'s} \rangle\rangle \\
 &= \epsilon^{ikl}\epsilon^{jmn} \times -ig^{km}\hat{J}^{\ell n} \\
 &= i\hat{J}^{\ell n} \times \left(-g^{km}\epsilon^{ikl}\epsilon^{jmn} = +\delta^{km}\epsilon^{ikl}\epsilon^{jmn} = \delta^{ij}\delta^{\ell n} - \delta^{in}\delta^{\ell j} \right) \\
 &= 0 - i\hat{J}^{\hat{j}i} = +i\hat{J}^{ij} \\
 &\equiv +i\epsilon^{ijk}\hat{J}^k.
 \end{aligned} \tag{S.1}$$

$$\begin{aligned}
 [\hat{J}^i, \hat{K}^j] &\equiv \frac{1}{2}\epsilon^{ikl} [\hat{J}^{kl}, \hat{J}^{0j}] = \langle\langle \text{by eq. (1)} \rangle\rangle \\
 &= \frac{1}{2}\epsilon^{ikl} \left(-ig^{k0}\hat{J}^{\ell j} + ig^{kj}\hat{J}^{\ell 0} + ig^{\ell 0}\hat{J}^{kj} - ig^{\ell j}\hat{J}^{k0} \right) \\
 &= \frac{1}{2}\epsilon^{ikl} \left(0 - i\delta^{kj}\hat{J}^{\ell 0} + 0 + i\delta^{\ell j}\hat{J}^{k0} \right) \\
 &\equiv \frac{1}{2}\epsilon^{ikl} \left(+i\delta^{kj}\hat{K}^{\ell} - i\delta^{\ell j}\hat{K}^k \right) \\
 &= \frac{1}{2}\epsilon^{ij\ell}\hat{K}^{\ell} - \frac{1}{2}\epsilon^{ikj}\hat{K}^j \\
 &= \epsilon^{ijk}\hat{K}^k,
 \end{aligned} \tag{S.2}$$

$$\begin{aligned}
 [\hat{K}^i, \hat{K}^j] &\equiv [\hat{J}^{0i}, \hat{J}^{0j}] = \langle\langle \text{by eq. (1)} \rangle\rangle \\
 &= -ig^{00}\hat{J}^{ij} + ig^{0j}\hat{J}^{i0} + ig^{i0}\hat{J}^{0j} - ig^{ij}\hat{J}^{00} \\
 &= -i\hat{J}^{ij} + 0 + 0 + 0, \\
 &\equiv -i\epsilon^{ijk}\hat{J}^k.
 \end{aligned} \tag{S.3}$$

Problem 1(b):

$$\begin{aligned}
 [\hat{V}^i, \hat{J}^j] &\equiv \frac{1}{2}\epsilon^{jkl} [\hat{V}^i, \hat{J}^{kl}] = \frac{1}{2}\epsilon^{jkl} \left(ig^{ik}\hat{V}^{\ell} - ig^{i\ell}\hat{V}^k \right) \\
 &= \frac{1}{2}\epsilon^{jkl} \left(-i\delta^{ik}\hat{V}^{\ell} + \delta^{i\ell}\hat{V}^k \right) = -\frac{i}{2}\epsilon^{jil}\hat{V}^{\ell} + \frac{i}{2}\epsilon^{jki}\hat{V}^k \\
 &= i\epsilon^{ijk}\hat{V}^k,
 \end{aligned} \tag{S.4}$$

$$\begin{aligned}
\left[\hat{V}^0, \hat{J}^j \right] &= \frac{1}{2} \epsilon^{jkl} \left[\hat{V}^0, \hat{J}^{kl} \right] = \frac{1}{2} \epsilon^{jkl} \left(\cancel{ig^{0k} \hat{V}^l} - \cancel{ig^{0l} \hat{V}^k} \right) \\
&= 0,
\end{aligned} \tag{S.5}$$

$$\begin{aligned}
\left[\hat{V}^i, \hat{K}^j \right] &= \left[\hat{V}^i, \hat{J}^{0j} \right] = \cancel{ig^{i0} \hat{V}^j} - ig^{ij} \hat{V}^0 \\
&= +i\delta^{ij} \hat{V}^0,
\end{aligned} \tag{S.6}$$

$$\begin{aligned}
\left[\hat{V}^0, \hat{K}^j \right] &= \left[\hat{V}^0, \hat{J}^{0j} \right] = ig^{00} \hat{V}^j - \cancel{ig^{0j} \hat{V}^0} \\
&= +i\hat{V}^j.
\end{aligned} \tag{S.7}$$

Note that the Hamiltonian of a relativistic theory is a member of a 4-vector multiplet $\hat{P}^\mu = (\hat{H}, \hat{\mathbf{P}})$ where $\hat{\mathbf{P}}$ is the net momentum operator. Applying the above equations to the $\hat{P}^\mu$ vectors, we obtain

$$\begin{aligned}
\left[\hat{P}^i, \hat{J}^j \right] &= i\epsilon^{ijk} \hat{P}^k, \\
\left[\hat{H}, \hat{J}^j \right] &= 0, \\
\left[\hat{P}^i, \hat{K}^j \right] &= +i\delta^{ij} \hat{H}, \\
\left[\hat{H}, \hat{K}^j \right] &= +i\hat{P}^j.
\end{aligned} \tag{S.8}$$

In particular, the Hamiltonian $\hat{H}$ commutes with the three angular momenta $\hat{J}^j$ but it does not commute with the three generators $\hat{K}^k$ of the Lorentz boosts.

Problem 1(c):

In the ordinary quantum mechanics, it is often said that generators of continuous symmetries must commute with the Hamiltonian operator. However, this is true only for the symmetries that act in a time independent manner — for example, rotating the 3D space by the same angle at all times t . But when the transformation rules of a symmetry depend on time, the Hamiltonian must change to account for this time dependence.

In a Lorentz boost, the transform $\mathbf{x} \rightarrow \mathbf{x}'$ obviously depends on time, which changes the way the transformed quantum fields such as $\hat{\Phi}'(\mathbf{x}, t)$ depend on t . Consequently, the Hamiltonian $\hat{H}$ of the theory must change so that the new Heisenberg equations would match the new time dependence. In terms of the generators, this means that the boost generators $\hat{K}^i$ should **not** commute with the Hamiltonian.

Note that this non-commutativity is not caused by the Lorentz boosts affecting the time itself, $t' = L^0_\mu x^\mu \neq t$. Even in non-relativistic theories — where the time is absolute — the generators of symmetries which affect the other variables in a time-dependent matter do not commute with the Hamiltonian $\hat{H}$.

Indeed, consider a Galilean transform from one non-relativistic moving frame into another, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ but $t' = t$. This is a good symmetry of non-relativistic particles interacting with each other but not subject to any external potential,

$$\hat{H} = \sum_a \frac{1}{2M_a} \hat{\mathbf{p}}_a^2 + \frac{1}{2} \sum_{a \neq b} V(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b). \quad (\text{S.9})$$

A unitary operator $\hat{\mathcal{G}}$ realizing a Galilean symmetry acts on coordinate and momentum operators as

$$\hat{\mathcal{G}}\hat{\mathbf{x}}_a\hat{\mathcal{G}}^\dagger = \hat{\mathbf{x}}_a + \mathbf{v}t, \quad \hat{\mathcal{G}}\hat{\mathbf{p}}_a\hat{\mathcal{G}}^\dagger = \hat{\mathbf{p}}_a + M_a\mathbf{v}, \quad (\text{S.10})$$

and it also transforms the Hamiltonian into

$$\hat{\mathcal{G}}\hat{H}\hat{\mathcal{G}}^\dagger = \hat{H} + \mathbf{v} \cdot \hat{\mathbf{P}}_{\text{tot}} + \frac{1}{2}M_{\text{tot}}\mathbf{v}^2. \quad (\text{S.11})$$

In terms of the Galilean boost generators $\hat{\mathbf{K}}_G$,

$$\hat{\mathcal{G}} = \exp(-i\mathbf{v} \cdot \hat{\mathbf{K}}_G), \quad (\text{S.12})$$

so under an infinitesimal boost $\mathbf{v} = \vec{\epsilon}$, various operators transform according to

$$\hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}} + \delta\hat{\mathcal{O}}, \quad \delta\hat{\mathcal{O}} = -i\epsilon^i \left[\hat{K}_G^i, \hat{\mathcal{O}} \right]. \quad (\text{S.13})$$

Consequently, the commutation relations with the boost generators follow from the infinitesimal boosts, for example

$$\begin{aligned} \delta\hat{\mathbf{x}}_a &= \vec{\epsilon}t & \implies [\hat{x}_a^i, \hat{K}_G^j] &= i\delta^{ij} \times t, \\ \delta\hat{\mathbf{p}}_a &= \vec{\epsilon}M_a & \implies [\hat{p}_a^i, \hat{K}_G^j] &= i\delta^{ij} \times M_a, \\ \delta\hat{H} &= \vec{\epsilon} \cdot \hat{\mathbf{P}}_{\text{tot}} & \implies [\hat{H}, \hat{K}_G^j] &= i\hat{P}_{\text{tot}}^j. \end{aligned} \quad (\text{S.14})$$

In particular, the Hamiltonian does NOT commute with the Galilean boost generators.

Problem 2(a):

Consider a linear combination $\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu}$ of Lorentz generators with some generic coefficients $N_{\mu\nu} = -N_{\nu\mu}$. The infinitesimal Lorentz transform

$$L^\mu_\nu = \exp(i\epsilon N)^\mu_\nu = \delta^\mu_\nu + i\epsilon N^\mu_\nu + O(\epsilon^2) \quad (\text{S.15})$$

preserves a given momentum p^μ , $L^\mu_\nu P^\nu = P^\mu$ if and only if

$$N^\mu_\nu p^\nu = 0. \quad (\text{S.16})$$

In 3D terms, $N^{ij} = \epsilon^{ijk}a^k$ and $N^{0k} = -N^{k0} = b^k$ for some 3-vectors $\mathbf{a}$ and $\mathbf{b}$, the generator in question is

$$\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} = \mathbf{a} \cdot \hat{\mathbf{J}} + \mathbf{b} \cdot \hat{\mathbf{K}}, \quad (\text{S.17})$$

and

$$\begin{aligned} N^{0\nu}p_\nu &= -N^{0j}p^j = -\mathbf{b} \cdot \mathbf{p}, \\ N^{i\nu}p_\nu &= -N^{ij}p^j + N^{i0}p^0 = -\epsilon^{ijk}p^ja^k - b^iE = (\mathbf{a} \times \mathbf{p} - \mathbf{b}E)^i, \end{aligned} \quad (\text{S.18})$$

so the condition (S.16) becomes

$$\mathbf{a} \times \mathbf{p} - \mathbf{b}E = 0 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{p} = 0. \quad (\text{S.19})$$

Actually, the second condition here is redundant, so the general solution is

$$\text{any } \mathbf{a}, \quad \mathbf{b} = \mathbf{a} \times \frac{\mathbf{p}}{E} \quad (\text{S.20})$$

and hence

$$\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} = \mathbf{a} \cdot \hat{\mathbf{J}} + \frac{(\mathbf{a} \times \mathbf{p}) \cdot \hat{\mathbf{K}}}{E} = \frac{\mathbf{a}}{E} \cdot (E\hat{\mathbf{J}} + \mathbf{p} \times \hat{\mathbf{K}}) \quad \text{for any } \mathbf{a}. \quad (\text{S.21})$$

In other words, the Lorentz symmetries preserving the momentum p^μ have 3 generators, namely the components of the 3-vector

$$\hat{\mathbf{R}} = E\hat{\mathbf{J}} + \mathbf{p} \times \hat{\mathbf{K}}. \quad (4)$$

Quod erat demonstrandum.

Problem 2(b):

Consider a massive particle moving at a slower-than-light speed β in z direction, so the energy is $E = \gamma m$ (where $\gamma = 1/\sqrt{1-\beta^2}$) while the 3-momentum is $\mathbf{p} = (0, 0, \beta\gamma m)$. Consequently, the three components of the $\hat{\mathbf{R}}$ vector for this energy-momentum are

$$\hat{R}^x = \gamma m \hat{J}^x - \beta\gamma m \hat{K}^y, \quad \hat{R}^y = \gamma m \hat{J}^y + \beta\gamma m \hat{K}^x, \quad \hat{R}^z = \gamma m \hat{J}^z. \quad (\text{S.22})$$

These 3 operators generate the little group $G(p)$ of the particle's 4-momentum $p^\mu = (E, \mathbf{p})$. To see that this little group happens to be isomorphic to the 3D rotation group $SO(3)$, we need to find 3 linear combinations $\tilde{J}^{x,y,z}$ of the operators (S.22) which obey the angular-momentum commutation relations

$$[\tilde{J}^i, \tilde{J}^j] = i\epsilon^{ijk} \tilde{J}^k. \quad (\text{S.23})$$

My choice of the (properly normalized) generators $\tilde{J}^{x,y,z}$ is spelled out in eqs. (5). Let's see that they indeed obey the commutation relations (S.23):

$$\begin{aligned} [\hat{J}^z, \tilde{J}^x] &= \gamma [\hat{J}^z, \hat{J}^x] - \beta\gamma [\hat{J}^z, \hat{K}^y] = \gamma \times i\hat{J}^y - \beta\gamma \times (-i\hat{K}^x) \\ &= i(\gamma\hat{J}^y + \beta\gamma\hat{K}^x) = i\tilde{J}^y, \\ [\hat{J}^z, \tilde{J}^y] &= \gamma [\hat{J}^z, \hat{J}^y] + \beta\gamma [\hat{J}^z, \hat{K}^x] = \gamma \times (-i\hat{J}^x) + \beta\gamma \times (+i\hat{K}^y) \\ &= -i(\gamma\hat{J}^x - \beta\gamma\hat{K}^y) = -i\tilde{J}^x, \\ [\tilde{J}^x, \tilde{J}^y] &= \gamma^2 [\hat{J}^x, \hat{J}^y] - \beta\gamma^2 [\hat{K}^y, \hat{J}^y] + \beta\gamma^2 [\hat{J}^x, \hat{K}^x] - \beta^2\gamma^2 [\hat{K}^y, \hat{K}^x] \\ &= \gamma^2 \times i\hat{J}^z - 0 + 0 - \beta^2\gamma^2 \times i\hat{J}^z \\ &= i\hat{J}^z \times (\gamma^2(1 - \beta^2) = 1) = i\hat{J}^z. \end{aligned} \quad (\text{S.24})$$

Quod erat demonstrandum.

Problem 2(c):

For a massless particle, we cannot rescale the little group generators as in eqs. (5) since the $1/m$ factor for the $\tilde{J}^{x,y}$ generators would be infinite. Instead, the best we can do is to use the $1/E$ factor for all three rescaled generators, hence eqs. (6) and (7). Consequently, instead of

eqs. (S.24) for the commutators of the rescaled generators, we get

$$\begin{aligned}
\left[\hat{J}^z, \hat{I}^x \right] &= \left[\hat{J}^z, \hat{J}^x \right] - \left[\hat{J}^z, \hat{K}^y \right] = i\hat{J}^y - (-i\hat{K}^x) = i\hat{I}^y, \\
\left[\hat{J}^z, \hat{I}^y \right] &= \left[\hat{J}^z, \hat{J}^y \right] + \left[\hat{J}^z, \hat{K}^x \right] = (-i\hat{J}^x) - (+i\hat{K}^y) = -i\hat{I}^x, \\
\left[\hat{I}^x, \hat{I}^y \right] &= \left[\hat{J}^x, \hat{J}^y \right] - \left[\hat{K}^y, \hat{J}^y \right] + \left[\hat{J}^x, \hat{K}^x \right] - \left[\hat{K}^y, \hat{K}^x \right] \\
&= i\hat{J}^z - 0 + 0 - i\hat{J}^z = 0,
\end{aligned} \tag{S.25}$$

precisely as in eq. (8).

The commutation relations (8) are different from the angular-momentum commutation relations, and they cannot be brought to the form (S.23) by any finite rescaling of the generators. Consequently, the little group of a light-like momentum $p^\mu = (E, 0, 0, E)$ is NOT isomorphic to the $SO(3)$.

Instead, the commutation relations (8) are similar to the commutation relations between the z component of the angular momentum and the x and y components of the linear momentum,

$$\left[\hat{J}^z, \hat{P}^x \right] = +i\hat{P}^y, \quad \left[\hat{J}^z, \hat{P}^y \right] = -i\hat{P}^x, \quad \left[\hat{P}^x, \hat{P}^y \right] = 0. \tag{S.26}$$

The $\hat{J}^z$ operator generates rotations around the z axis, *i.e.*, within the xy plane, while the $\hat{P}^x$ and $\hat{P}^y$ operators generate translations in that plane. Together, they generate the group ISO(2) of isometries — rotations and translations — in 2 space dimensions.

Thus, we see that the little group $G(p)$ of a light-like momentum of a massless particle is isomorphic to the ISO(2).

Problem 2(d):

Finally, consider a particle moving faster-than-light in the z direction, so its momentum $p^\mu = (p^0, 0, 0, p^3)$ has $p^3 > p^0$. Such a tachyon must have negative mass², so let us denote

$$M_i^2 = -m^2 = -p_\mu p^\mu > 0, \quad \gamma_i = \frac{p^0}{M_i}, \quad \beta = \frac{p^3}{p^0} \tag{S.27}$$

where the subscript i stands for ‘imaginary’ and $\beta > 1$ is the faster-than-light speed. In

these notations, let us rescale the little group generators (4) according to

$$\begin{aligned}\tilde{K}^x &= \frac{+1}{M_i} \hat{R}^y = \beta\gamma_i \hat{K}^x + \gamma_i \hat{J}^y, \\ \tilde{K}^y &= \frac{-1}{M_i} \hat{R}^x = \beta\gamma_i \hat{K}^y - \gamma_i \hat{J}^x, \\ \tilde{J}^z &= \frac{1}{\gamma_i M_i} \hat{R}^z = \hat{J}^z, \quad \text{the helicity.}\end{aligned}\tag{S.28}$$

Consequently, the commutation relations of the rescaled operators become

$$\begin{aligned}[\hat{J}^z, \tilde{K}^x] &= \beta\gamma_i [\hat{J}^z, \hat{K}^x] + \gamma_i [\hat{J}^z, \hat{J}^y] \\ &= \beta\gamma_i (+i\hat{K}^y) + \gamma_i (-i\hat{J}^x) \\ &= +i\tilde{K}^y,\end{aligned}\tag{S.29}$$

$$\begin{aligned}[\hat{J}^z, \tilde{K}^y] &= \beta\gamma_i [\hat{J}^z, \hat{K}^y] - \gamma_i [\hat{J}^z, \hat{J}^x] \\ &= \beta\gamma_i (-i\hat{K}^x) + \gamma_i (+i\hat{J}^y) \\ &= -i\tilde{K}^x,\end{aligned}\tag{S.30}$$

$$\begin{aligned}[\tilde{K}^x, \tilde{K}^y] &= \beta^2\gamma_i^2 [\hat{K}^x, \hat{K}^y] + \beta\gamma_i^2 [\hat{J}^y, \hat{K}^y] - \beta\gamma_i^2 [\hat{K}^x, \hat{J}^x] - \gamma_i^2 [\hat{J}^y, \hat{J}^x] \\ &= \beta^2\gamma_i^2 \times (-i\hat{J}^z) + \beta\gamma_i^2 \times 0 - \beta\gamma_i^2 \times 0 - \gamma_i^2 \times (-i\hat{J}^z) \\ &= -i(\beta^2 - 1)\gamma_i^2 \times \hat{J}^z \\ &= -i\hat{J}^z,\end{aligned}\tag{S.31}$$

where the last equality follows from the kinematic relation

$$(\beta^2 - 1) \times \gamma_i^2 = \frac{p_3^2 - p_0^2}{p_0^2} \times \frac{p_0^2}{M_i^2} = \frac{p_3^2 - p_0^2}{M_i^2} = 1\tag{S.32}$$

Altogether, the generators (S.28) obey the commutation relations

$$[\hat{J}^z, \tilde{K}^x] = +i\tilde{K}^y, \quad [\hat{J}^z, \tilde{K}^y] = -i\tilde{K}^x, \quad [\tilde{K}^x, \tilde{K}^y] = -i\hat{J}^z,\tag{S.33}$$

which are exactly similar to the $SO^+(2, 1)$ commutation relations

$$[\hat{J}^z, \hat{K}^x] = +i\hat{K}^y, \quad [\hat{J}^z, \hat{K}^y] = -i\hat{K}^x, \quad [\hat{K}^x, \hat{K}^y] = -i\hat{J}^z.\tag{S.34}$$

Therefore, the little group $G(p)$ of a tachyonic momentum is isomorphic to the continuous Lorentz group $SO^+(2, 1)$ in $2 + 1$ dimensions.

Problem 3(a):

The quantum state $|p, \lambda\rangle$ has definite momentum p^μ , thus $\hat{P}_\mu |p, \lambda\rangle = p_\mu |p, \lambda\rangle$ and likewise

$$\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma |p, \lambda\rangle = \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}p^\gamma\hat{J}^{\alpha\beta} |p, \lambda\rangle. \quad (\text{S.35})$$

To work out the RHS of this formula, let's spell out the operator

$$\hat{Q}_\mu \stackrel{\text{def}}{=} \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}p^\gamma\hat{J}^{\alpha\beta} \quad (\text{S.36})$$

in components:

$$\hat{Q}_0 = \frac{1}{2}\epsilon_{0ijk}p^k\hat{J}^{ij} = \frac{1}{2}\epsilon^{ijk}p^k \times \epsilon^{ij\ell}\hat{J}^\ell = \mathbf{p} \cdot \hat{\mathbf{J}}, \quad (\text{S.37})$$

$$\begin{aligned} \hat{Q}_i &= \frac{1}{2}\epsilon_{ijk0}p^0\hat{J}^{jk} + \frac{1}{2}\epsilon_{ij0k}p^k\hat{J}^{j0} + \frac{1}{2}\epsilon_{ij0k}p^k\hat{J}^{0j} \\ &= -\frac{1}{2}\epsilon^{ijk}E \times \epsilon^{ij\ell}\hat{J}^\ell + \frac{1}{2}\epsilon^{ijk}p^k \times \hat{K}^j - \frac{1}{2}\epsilon^{ijk}p^k(-\hat{K}^j) \\ &= -E\hat{J}^i + (\hat{\mathbf{K}} \times \mathbf{p})^i. \end{aligned} \quad (\text{S.38})$$

For a massless particle with a lightlike momentum $p^\mu = (E, E\mathbf{v})$, $|\mathbf{v}| = 1$, the components of $\hat{Q}^\mu$ operators become

$$\begin{aligned} \hat{\mathbf{Q}} &= +E\hat{\mathbf{J}} + E\mathbf{v} \times \hat{\mathbf{K}} = E\hat{\mathbf{I}} \\ &\ll \text{for } \hat{\mathbf{I}} \text{ as in eq. (6)} \gg, \end{aligned} \quad (\text{S.39})$$

$$\hat{Q}^0 = E\mathbf{v} \cdot \hat{\mathbf{J}} = E\hat{\lambda}.$$

When this operator acts on the particle state $|p, \lambda\rangle$ which has definite helicity and is also annihilated by the two transverse generators $\hat{\mathbf{I}}_\perp$ as in eq. (9), we get

$$\hat{Q}^0 |p, \lambda\rangle = E\lambda |p, \lambda\rangle = \lambda p^0 |p, \lambda\rangle \quad (\text{S.40})$$

and also

$$\hat{\mathbf{I}} |p, \lambda\rangle = \lambda \mathbf{v} |p, \lambda\rangle, \quad (\text{S.41})$$

hence

$$\hat{\mathbf{Q}} |p, \lambda\rangle = E\lambda \mathbf{v} |p, \lambda\rangle = \lambda \mathbf{p} |p, \lambda\rangle. \quad (\text{S.42})$$

Altogether, in 4d terms we get

$$\hat{Q}^\mu |p, \lambda\rangle = \lambda p^\mu |p, \lambda\rangle \quad (\text{S.43})$$

and hence (after lowering the index μ on both sides)

$$\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma |p, \lambda\rangle = \lambda\hat{P}_\mu |p, \lambda\rangle. \quad (10)$$

Quod erat demonstrandum.

Problem 3(b):

Consider a *continuous Lorentz transform* $x^\mu \rightarrow x'^\mu = L^\mu_\nu x^\nu$ acting on the $|p, \lambda\rangle$ state of a massless particle. The operators on both sides of both sides of eq. (10) transform as Lorentz vectors,

$$\hat{\mathcal{D}}(L)\hat{P}_\nu\hat{\mathcal{D}}^\dagger(L) = L_\nu^\mu\hat{P}_\mu, \quad \hat{\mathcal{D}}(L)\left(\frac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)\hat{\mathcal{D}}^\dagger(L) = L_\nu^\mu\left(\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right). \quad (\text{S.44})$$

Consequently, the transformed state

$$\hat{\mathcal{D}}(L)|p, \lambda\rangle = |Lp, ??\rangle \quad (\text{S.45})$$

satisfies the same eq. (10) as the original state $|p, \lambda\rangle$. Indeed,

$$\begin{aligned} L_\nu^\mu\left(\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|Lp, ??\rangle &= \hat{\mathcal{D}}(L)\left(\frac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)\hat{\mathcal{D}}^\dagger(L)\times\hat{\mathcal{D}}|p, \lambda\rangle \\ &= \hat{\mathcal{D}}(L)\times\left(\frac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|p, \lambda\rangle \\ \langle\langle \text{by eq. (10)} \rangle\rangle &= \hat{\mathcal{D}}(L)\times\lambda\hat{P}_\nu|p, \lambda\rangle \\ &= \lambda\times\hat{\mathcal{D}}(L)\hat{P}_\nu\hat{\mathcal{D}}^\dagger(L)\times\hat{\mathcal{D}}|p, \lambda\rangle \\ &= \lambda\times L_\nu^\mu\hat{P}_\mu|Lp, ??\rangle, \end{aligned} \quad (\text{S.46})$$

and hence

$$\left(\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|Lp, ??\rangle = \lambda\hat{P}_\mu|Lp, ??\rangle. \quad (\text{S.47})$$

Note that this equation for the transformed state $|Lp, ??\rangle$ has exactly the same helicity eigenvalue λ as the original eq. (10), which means that the transformed state has the same

helicity as the original state. And since the momentum and the helicity completely determine the quantum state of a particle up to an overall phase, it follows that

$$\hat{\mathcal{D}}(L) |p, \lambda\rangle = |Lp, \text{same } \lambda\rangle \times e^{i \text{phase}}. \quad (11)$$

Thus, *for massless particles, the continuous Lorentz transforms preserve helicity!*

Problem 4(a):

The Lorentz generators $\hat{J}^i$ and $\hat{K}^i$ obey commutation relations

$$[\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk} \hat{J}^k, \quad [\hat{J}^i, \hat{K}^j] = i\epsilon^{ijk} \hat{K}^k, \quad [\hat{K}^i, \hat{K}^j] = -i\epsilon^{ijk} \hat{J}^k. \quad (2)$$

Consequently, for two components of $\hat{\mathbf{J}}_+$ or two components of $\hat{\mathbf{J}}_-$ we have

$$\begin{aligned} [\hat{J}_\pm^i, \hat{J}_\pm^j] &= \frac{1}{4} [\hat{J}^i, \hat{J}^j] \pm \frac{i}{4} [\hat{K}^i, \hat{J}^j] \pm \frac{i}{4} [\hat{J}^i, \hat{K}^j] - \frac{1}{4} [\hat{K}^i, \hat{K}^j] \\ &= \frac{1}{4} i\epsilon^{ijk} \hat{J}^k \pm \frac{i}{4} i\epsilon^{ijk} \hat{K}^k \pm \frac{i}{4} i\epsilon^{ijk} \hat{K}^k - \frac{1}{4} (-i)\epsilon^{ijk} \hat{J}^k \\ &= \frac{i}{2} \epsilon^{ijk} \hat{J}^k \mp \frac{1}{2} \epsilon^{ijk} \hat{K}^k = i\epsilon^{ijk} \times \left(\frac{1}{2} \hat{J}^k \pm \frac{i}{2} \hat{K}^k \right) \\ &= i\epsilon^{ijk} \hat{J}_\pm^k, \end{aligned} \quad (\text{S.48})$$

while for one component of $\hat{\mathbf{J}}_+$ and one component of $\hat{\mathbf{J}}_-$ (or the other way around) we get

$$\begin{aligned} [\hat{J}_\pm^i, \hat{J}_\mp^j] &= \frac{1}{4} [\hat{J}^i, \hat{J}^j] \pm \frac{i}{4} [\hat{K}^i, \hat{J}^j] \mp \frac{i}{4} [\hat{J}^i, \hat{K}^j] + \frac{1}{4} [\hat{K}^i, \hat{K}^j] \\ &= \frac{1}{4} i\epsilon^{ijk} \hat{J}^k \pm \frac{i}{4} i\epsilon^{ijk} \hat{K}^k \mp \frac{i}{4} i\epsilon^{ijk} \hat{K}^k + \frac{1}{4} (-i)\epsilon^{ijk} \hat{J}^k \\ &= \frac{i}{4} \epsilon^{ijk} (\hat{J}^k \pm i\hat{K}^k \mp i\hat{K}^k - \hat{J}^k) \\ &= 0. \end{aligned} \quad (\text{S.49})$$

Problem 4(b):

Let's start by calculating the squares of the matrices (17) representing the boost $B(r, \mathbf{n})$,

$$[M_{\mathbf{2}}(B)]^2 = \exp(-r\mathbf{n} \cdot \boldsymbol{\sigma}) \quad \text{and} \quad [M_{\overline{\mathbf{2}}}(B)]^2 = \exp(+r\mathbf{n} \cdot \boldsymbol{\sigma}). \quad (\text{S.50})$$

For any unit vector $\mathbf{n}$,

$$(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \mathbf{n}^2 = 1, \quad (\text{S.51})$$

hence for any integer $k \geq 0$

$$(\mathbf{n} \cdot \boldsymbol{\sigma})^k = \begin{cases} 1 & \text{for even } k, \\ (\mathbf{n} \cdot \boldsymbol{\sigma}) & \text{for odd } k. \end{cases} \quad (\text{S.52})$$

Consequently, expanding the exponentials in eq. (S.50) into powers of the rapidity r yields

$$\begin{aligned} \exp(\mp r\mathbf{n} \cdot \boldsymbol{\sigma}) &= \sum_{k=0}^{\infty} \frac{(\mp r)^k}{k!} \times (\mathbf{n} \cdot \boldsymbol{\sigma})^k \\ &= \sum_{\text{even } k} \frac{(\mp r)^k}{k!} \times 1 + \sum_{\text{odd } k} \frac{(\mp r)^k}{k!} \times (\mathbf{n} \cdot \boldsymbol{\sigma}) \\ &= \cosh(r) \times 1 \mp \sinh(r) \times (\mathbf{n} \cdot \boldsymbol{\sigma}) \\ &\quad \langle\langle \text{translating from the rapidity to the } \beta \text{ and } \gamma \text{ parameters (16)} \rangle\rangle \\ &= \gamma \times 1 \mp \beta\gamma \times (\mathbf{n} \cdot \boldsymbol{\sigma}) \\ &= \gamma \times (1 \mp \beta\mathbf{n} \cdot \boldsymbol{\sigma}). \end{aligned} \quad (\text{S.53})$$

Thus,

$$[M_{\mathbf{2}}(B)]^2 = \gamma(1 - \beta\mathbf{n} \cdot \boldsymbol{\sigma}), \quad [M_{\overline{\mathbf{2}}}(B)]^2 = \gamma(1 + \beta\mathbf{n} \cdot \boldsymbol{\sigma}), \quad (\text{S.54})$$

and therefore

$$M_{\mathbf{2}}(B) = \sqrt{\gamma} \times \sqrt{1 - \beta\mathbf{n} \cdot \boldsymbol{\sigma}}, \quad M_{\overline{\mathbf{2}}}(B) = \sqrt{\gamma} \times \sqrt{1 + \beta\mathbf{n} \cdot \boldsymbol{\sigma}}. \quad (11)$$

Quod erat demonstrandum.

Problem 4(c):

Among the 3 Pauli matrices $\boldsymbol{\sigma}$, the σ_1 and the σ_3 matrices are real while the σ_2 is imaginary. At the same time, the σ_1 and the σ_3 anticommute with the σ_2 while the σ_2 commutes with itself. Consequently,

$$\sigma_2(\sigma_{1,3})^* \sigma_2 = +\sigma_2 \sigma_{1,3} \sigma_2 = -\sigma_{1,3} \sigma_2 \sigma_2 = -\sigma_{1,3}, \quad (\text{S.55})$$

while

$$\sigma_2(\sigma_2)^* \sigma_2 = -\sigma_2 \sigma_2 \sigma_2 = -\sigma_2, \quad (\text{S.56})$$

thus for all 3 Pauli matrices

$$\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}. \quad (\text{S.57})$$

Now let's apply this identity to the $\mathbf{2}$ and the $\overline{\mathbf{2}}$ representations of the same Lorentz symmetry L . Any *continuous* Lorentz symmetry must be generated by some linear combinations of the angular momenta and boost generators, thus

$$\hat{L} = \exp(-i\mathbf{a} \cdot \hat{\mathbf{J}} - i\mathbf{b} \cdot \hat{\mathbf{K}}) \quad (\text{S.58})$$

for some real 3-vectors $\mathbf{a}$ and $\mathbf{b}$. In the $\mathbf{2}$ representation, the $\hat{J}^i$ generators act as $\frac{1}{2}\sigma^i$ matrices while the $\hat{K}^i$ generators act as $-\frac{i}{2}\sigma^i$, thus

$$\mathbf{a} \cdot \hat{\mathbf{J}} + \mathbf{b} \cdot \hat{\mathbf{K}} \quad \text{acts as} \quad \frac{1}{2}(\mathbf{a} - i\mathbf{b}) \cdot \boldsymbol{\sigma} \quad (\text{S.59})$$

and hence

$$M \stackrel{\text{def}}{=} M_{\mathbf{2}}(L) = \exp\left(-\frac{i}{2}(\mathbf{a} - i\mathbf{b}) \cdot \boldsymbol{\sigma}\right). \quad (\text{S.60})$$

Likewise, in the $\overline{\mathbf{2}}$ representation, the $\hat{J}^i$ generators also act as $\frac{1}{2}\sigma^i$ matrices but the $\hat{K}^i$

generators act as $+\frac{i}{2}\sigma^i$, thus

$$\mathbf{a} \cdot \hat{\mathbf{J}} + \mathbf{b} \cdot \hat{\mathbf{K}} \quad \text{acts as} \quad \frac{1}{2}(\mathbf{a} + i\mathbf{b}) \cdot \boldsymbol{\sigma} \quad (\text{S.61})$$

and hence

$$\overline{M} \stackrel{\text{def}}{=} M_{\overline{\mathbf{2}}}(L) = \exp\left(-\frac{i}{2}(\mathbf{a} + i\mathbf{b}) \cdot \boldsymbol{\sigma}\right). \quad (\text{S.62})$$

In terms of the complex 3-vector $\mathbf{c} = \frac{1}{2}\mathbf{a} - \frac{i}{2}\mathbf{b}$,

$$M = \exp(-i\mathbf{c} \cdot \boldsymbol{\sigma}) \quad \text{and} \quad \overline{M} = \exp(-i\mathbf{c}^* \cdot \boldsymbol{\sigma}), \quad (\text{S.63})$$

and thanks to eq. (S.57), and two such matrices are related to each other as

$$\overline{M} = \sigma_2 M^* \sigma_2 \quad \text{and} \quad M = \sigma_2 \overline{M}^* \sigma_2. \quad (\text{S.64})$$

To prove this relation, note that eq. (S.57) implies

$$\sigma_2(-i\mathbf{c} \cdot \boldsymbol{\sigma})^* \sigma_2 = +i\mathbf{c}^* \cdot (\sigma_2 \boldsymbol{\sigma}^* \sigma_2) = -i\mathbf{c}^* \cdot \boldsymbol{\sigma}, \quad (\text{S.65})$$

hence

$$\sigma_2[(-i\mathbf{c} \cdot \boldsymbol{\sigma})^2]^* \sigma_2 = \sigma_2(-i\mathbf{c} \cdot \boldsymbol{\sigma})^* \sigma_2 \times \sigma_2(-i\mathbf{c} \cdot \boldsymbol{\sigma})^2 \sigma_2 = (-i\mathbf{c}^* \cdot \boldsymbol{\sigma}) \times (-i\mathbf{c}^* \cdot \boldsymbol{\sigma}) = (-i\mathbf{c}^* \cdot \boldsymbol{\sigma})^2, \quad (\text{S.66})$$

and likewise for any integer power n ,

$$\sigma_2[(-i\mathbf{c} \cdot \boldsymbol{\sigma})^n]^* \sigma_2 = (-i\mathbf{c}^* \cdot \boldsymbol{\sigma})^n. \quad (\text{S.67})$$

Consequently, for a power series like the exponential we have

$$\begin{aligned} \sigma_2 M^* \sigma_2 &= \sigma_2 \left(\sum_{n=0}^{\infty} \frac{1}{n!} (-i\mathbf{c} \cdot \boldsymbol{\sigma})^n \right)^* \sigma_2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_2 ((-i\mathbf{c} \cdot \boldsymbol{\sigma})^n)^* \sigma_2 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (-i\mathbf{c}^* \cdot \boldsymbol{\sigma})^n = \overline{M}, \end{aligned} \quad (\text{S.68})$$

and likewise $\sigma_2 \overline{M}^* \sigma_2 = M$. *Quod erat demonstrandum.*

Problem 4(d):

Let's start with reality. The matrix $V = V^\mu \overline{\sigma}_\mu$ is hermitian if and only if the 4-vector V^μ is real. For any matrix $M \in SL(2, \mathbf{C})$, the transform

$$V \rightarrow V' = MVM^\dagger \quad (\text{S.69})$$

preserves hermiticity: if V is hermitian, then so is V' ; indeed

$$(V')^\dagger = (MVM^\dagger)^\dagger = (M^\dagger)^\dagger V^\dagger M^\dagger = MVM^\dagger = V'. \quad (\text{S.70})$$

In terms of the 4-vectors, this means that if the V^μ is real then the $V'^\mu = L^\mu_\nu V^\nu$ is also real. In other words, the 4×4 matrix $L^\mu_\nu(M)$ is real.

Next, let's prove that $L^\mu_\nu(M) \in O(3,1)$ — it preserves the Lorentz metric $g^{\alpha\beta}$, or equivalently, for any V^μ , $g^{\alpha\beta} V'_\alpha V'_\beta = g^{\alpha\beta} V_\alpha V_\beta$. In terms of the 2×2 matrix $V = V^\mu \overline{\sigma}_\mu$, the Lorentz square of the 4-vector becomes the determinant:

$$g^{\alpha\beta} V_\alpha V_\beta = \det(V = V_\mu \sigma^\mu). \quad (\text{S.71})$$

Indeed, from the explicit form of the 4 matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{S.72})$$

we have

$$V = V^\mu \overline{\sigma}_\mu = \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix}$$

and hence

$$\det(V) = (V_0 + V_3)(V_0 - V_3) - (V_1 - iV_2)(V_1 + iV_2) = V_0^2 - V_3^2 - V_1^2 - V_2^2 = g^{\alpha\beta} V_\alpha V_\beta. \quad (\text{S.73})$$

The determinant of a matrix product is the product of the individual matrices' determinants. Hence, for the transform (S.69),

$$\det(V') = \det(M) \times \det(V) \times \det(M^\dagger) = \det(V) \times |\det(M)|^2. \quad (\text{S.74})$$

The M matrices of interest to us belong to the $SL(2, \mathbf{C})$ group — they are complex matrices with unit determinants. There are no other restrictions, but $\det(M) = 1$ is enough to assure

$\det(V') = \det(V)$, cf. eq. (S.74). Thanks to the relation (S.71), this means

$$g^{\alpha\beta} V'_\alpha V'_\beta = \det(V') = \det(V) = g^{\alpha\beta} V_\alpha V_\beta \quad (\text{S.75})$$

— which proves that the matrix $L^\mu_\nu(M)$ is indeed Lorentzian.

Problem 4(e):

To prove that the Lorentz transform $L^\mu_\nu(M)$ is orthochronous, we need to show that for any V_μ in the forward light cone — that is, $V^2 > 0$ and $V_0 > 0$ — the V'_μ is also in the forward light cone. In matrix terms, $V^2 > 0$ means $\det(V) > 0$ while $V_0 > 0$ means $\text{tr}(V) > 0$; together, these two conditions mean that the 2×2 hermitian matrix V is positive-definite. The transform (S.69) preserves positive definiteness: if for any complex 2-vector $\xi \neq 0$ we have $\xi^\dagger V \xi > 0$, then

$$\xi^\dagger V' \xi = \xi^\dagger M V M^\dagger \xi = (M^\dagger \xi)^\dagger V (M^\dagger \xi) > 0. \quad (\text{S.76})$$

(Note that $M^\dagger \xi \neq 0$ for any $\xi \neq 0$ because $\det(M) \neq 0$.) Thus, for any $M \in SL(2, \mathbf{C})$ the Lorentz transform $V^\mu \rightarrow V'^\mu$ preserves the forward light cone — which means that the $L^\mu_\nu(M)$ matrix is orthochronous, $L^\mu_\nu(M) \in O^+(3, 1)$.

The simplest proof that the Lorentz transform (22) is proper — $\det(L) = +1$ — for any $SL(2, \mathbf{C})$ matrix M is topological: The $SL(2, \mathbf{C})$ group manifold — which spans all matrices of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{complex } a, b, c, d, \quad ad - bc = 1 \quad (\text{S.77})$$

is connected, so all such matrices are continuously connected to the $\mathbf{1}_{2 \times 2}$ matrix. It is easy to see that for $M = 1$, the Lorentz transform $L(1)$ is trivial, $L^\mu_\nu = \delta^\mu_\nu$, hence all Lorentz transforms of the form (16) are continuously connected to the trivial transform. Consequently, they all must be continuous Lorentz transforms and therefore proper and orthochronous.

An alternative proof is more involved. It involves decomposition of M into a product of a unitary U and a Hermitian H , 2 Lemmas showing that $L(U)$ is a pure 3D rotation

(without reflections) while $L(H)$ is a pure Lorentz boost, and the group law from part (f) which tells us that

$$L(M = UH) = L(U) \times L(H) = \text{rotation} \times \text{boost}, \quad (\text{S.78})$$

which makes it a continuous Lorentz transform. But this is a rather long proof, so I leave it as optional exercise for the interested students.

Problem 4(f):

Let $L_1 = L(M_1)$, $L_2 = L(M_2)$ and $L_{12} = L(M_2 M_1)$ be Lorentz transforms constructed according to eq. (19–22) for some $SL(2, \mathbf{C})$ matrices M_1 and M_2 and their product $M_2 M_1$. We want to prove that L_{12} obtains from a product of consecutive Lorentz transforms, first L_1 and then L_2 , so let's consider how all these transforms act on some 4-vector V^μ . On one hand,

$$(L_{12}V)^\nu \bar{\sigma}_\nu = (M_2 M_1) \times (V^\nu \bar{\sigma}_\nu) \times (M_2 M_1)^\dagger = M_2 M_1 \times (V^\nu \bar{\sigma}_\nu) \times M_1^\dagger M_2^\dagger. \quad (\text{S.79})$$

On the other hand,

$$\begin{aligned} (L_2 L_1 V)^\nu \bar{\sigma}_\nu &= M_2 \times ((L_1 V)^\nu \bar{\sigma}_\nu) \times M_2^\dagger = M_2 \times (M_1 \times (V^\nu \bar{\sigma}_\nu) \times M_1^\dagger) \times M_2^\dagger \\ \text{also} &= M_2 M_1 \times (V^\nu \bar{\sigma}_\nu) \times M_1^\dagger M_2^\dagger. \end{aligned}$$

Thus we see that

$$(L_{12}V)^\nu \bar{\sigma}_\nu = (L_2 L_1 V)^\nu \bar{\sigma}_\nu \quad (\text{S.80})$$

and therefore

$$(L_{12}V)^\nu = (L_2 L_1 V)^\nu. \quad (\text{S.81})$$

Moreover, this holds true for any 4-vector V^μ , hence the Lorentz transforms $L_{12} = L(M_2 M_1)$ and $L_2 L_1 = L(M_2) L(M_1)$ must be equal to each other, *quod erat demonstrandum*.

Problem 4(g):

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$, or in (j) notations, $(\frac{1}{2}) \otimes (\frac{1}{2}) = (1) \oplus (0)$. Furthermore, the triplet $\mathbf{3} = (1)$ is symmetric with respect to permutations of the two doublets while the singlet $\mathbf{1} = (0)$ is antisymmetric.

For two separate and independent types of angular momenta $\mathbf{J}_+$ and $\mathbf{J}_-$ we combine the j_+ quantum numbers independently from the j_- and the j_- quantum numbers independently from the j_+ . For two bi-spinors, this gives us

$$(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0). \quad (\text{S.82})$$

Furthermore, the symmetric part of this product should be either symmetric with respect to both the j_+ and the j_- indices or antisymmetric with respect to both indices, thus

$$[(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})]_{\text{sym}} = (1, 1) \oplus (0, 0). \quad (\text{S.83})$$

Likewise, the antisymmetric part is either symmetric with respect to the j_+ but antisymmetric with respect to the j_- or the other way around, thus

$$[(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})]_{\text{antisym}} = (1, 0) \oplus (0, 1). \quad (\text{S.84})$$

From the $SO^+(3, 1)$ point of view, the bi-spinor $(\frac{1}{2}, \frac{1}{2})$ is the Lorentz vector. A general 2-index Lorentz tensor transforms like a product of two such vectors, so from the $SL(2, \mathbf{C})$ point of view it's a product of two bi-spinors, which decomposes to irreducible multiplets according to eq. (S.82).

The Lorentz symmetry respects splitting of a general 2-index tensor into a symmetric tensor $T^{\mu\nu} = +T^{\nu\mu}$ and an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$. The symmetric tensor corresponds to a symmetrized square of a bi-spinor, which decomposes into irreducible multiplets according to eq. (S.83). The singlet $(0, 0)$ component is the Lorentz-invariant trace T^μ_μ while the $(1, 1)$ irreducible multiplet is the traceless part of the symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ decomposes according to eq. (S.84). Here, the irreducible components $(1, 0)$ and $(0, 1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu} = \pm iF^{\kappa\lambda}$, or in 3D terms, $\mathbf{E} = \pm i\mathbf{B}$.