Problem **1**(a):

Suppose the γ^μ matrices obey the anticommutation relations (1). Then:

$$\begin{aligned}
\beta^2 &= \gamma^0\gamma^0 = g^{00} = +1, && \text{(S.1)}\\
\{\beta,\alpha^i\} &= \{\gamma^0,\gamma^0\gamma^i\} = \gamma^0\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0 \\
&= \gamma^0\times\{\gamma^0,\gamma^i\} = \gamma^0\times 0 = 0, && \text{(S.2)}\\
\alpha^i\alpha^j &= \gamma^0\gamma^i\gamma^0\gamma^j = -\gamma^0\gamma^0\gamma^i\gamma^j = -\gamma^i\gamma^j, && \text{(S.3)}\\
&\Downarrow \\
\{\alpha^i,\alpha^j\} &= -\{\gamma^i,\gamma^j\} = -2g^{ij} = +2\delta^{ij}, && \text{(S.4)}
\end{aligned}$$

quod erat demonstrandum.

Conversely, suppose $\beta^2=1$, $\{\beta,\alpha^i\}=0$, and $\{\alpha^i,\alpha^j\}=+2\delta^{ij}$, and let us *define* $\gamma^0 \stackrel{\text{def}}{=} \beta$ and $\gamma^i \stackrel{\text{def}}{=} \beta\alpha^i$. In this case:

$$\begin{aligned}
\gamma^0\gamma^0 &= \beta\beta = +1 = g^{00}, && \text{(S.5)}\\
\{\gamma^0,\gamma^i\} &= \{\beta,\beta\alpha^i\} = \beta\times\{\beta,\alpha^i\} = \beta\times 0 = 0 = 2g^{0i}, && \text{(S.6)}\\
\gamma^i\gamma^j &= \beta\alpha^i\beta\alpha^j = -\beta\beta\alpha^i\alpha^j = -\alpha^i\alpha^j, && \text{(S.7)}\\
&\Downarrow \\
\{\gamma^i,\gamma^j\} &= -\{\alpha^i,\alpha^j\} = -2\delta^{ij} = +2g^{ij}, && \text{(S.8)}
\end{aligned}$$

and therefore

$$\forall\mu,\nu: \quad \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\,. \qquad (1)$$

Quod erat demonstrandum.

Problem **1**(b):

$$\gamma^\alpha \gamma_\alpha \;=\; \tfrac{1}{2}\{\gamma^\alpha, \gamma^\beta\} g_{\alpha\beta} \;=\; g^{\alpha\beta} g_{\alpha\beta} \;=\; 4; \tag{S.9}$$

$$\begin{aligned} \gamma^\alpha \gamma^\nu \gamma_\alpha \;&=\; (\gamma^\alpha \gamma^\nu \;=\; 2g^{\nu\alpha} \;-\; \gamma^\nu \gamma^\alpha)\gamma_\alpha \\ &=\; 2\gamma^\nu \;-\; \gamma^\nu(\gamma^\alpha \gamma_\alpha \;=\; 4) \;=\; -2\gamma^\nu\,; \end{aligned} \tag{S.10}$$

$$\begin{aligned} \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha \;&=\; (\gamma^\alpha \gamma^\mu \;=\; 2g^{\mu\alpha} \;-\; \gamma^\mu \gamma^\alpha)\gamma^\nu \gamma_\alpha \\ &=\; 2\gamma^\nu \gamma^\mu \;-\; \gamma^\mu(\gamma^\alpha \gamma^\nu \gamma_\alpha \;=\; -2\gamma^\nu) \\ &=\; 2\gamma^\nu \gamma^\mu \;+\; 2\gamma^\mu \gamma^\nu \;=\; 4g^{\mu\nu}\,; \end{aligned} \tag{S.11}$$

$$\begin{aligned} \gamma^\alpha \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\alpha \;&=\; (\gamma^\alpha \gamma^\lambda \;=\; 2g^{\mu\alpha} \;-\; \gamma^\lambda \gamma^\alpha)\gamma^\mu \gamma^\nu \gamma_\alpha \\ &=\; 2\gamma^\mu \gamma^\nu \gamma^\lambda \;-\; \gamma^\lambda(\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha \;=\; 4g^{\mu\nu}) \\ &=\; (2\gamma^\mu \gamma^\nu \;-\; 4g^{\mu\nu} \;=\; -2\gamma^\nu \gamma^\mu)\gamma^\lambda \\ &=\; -2\gamma^\nu \gamma^\mu \gamma^\lambda. \end{aligned} \tag{S.12}$$

Problem **1**(c):
First, a Lemma: for any Lorentz vector a^μ, the $\not{a} \stackrel{\text{def}}{=} \gamma^\mu a_\mu$ matrix squares to

$$\not{a}\not{a} \;=\; \gamma^\mu a_\mu \gamma^\nu a_\nu \;=\; a_\mu a_\nu \times \Big(\gamma^\mu \gamma^\nu \;=\; g^{\mu\nu} - 2iS^{\mu\nu}\Big) \;=\; a^2 \;-\; i[a_\mu, a_\nu] \times S^{\mu\nu} \tag{S.13}$$

(where the last equality comes from $S^{\mu\nu} = -S^{\nu\mu}$.) For vectors a^μ whose components commute with each other, this formula simplifies to $\not{a}^2 = a^2$, in particular for the ordinary derivatives ∂^μ, $\not{\partial}^2 = \partial^2$. However, the covariant derivatives D_μ do not commute with each other. Instead, $[D_\mu, D_\nu] = iqF_{\mu\nu}(x)$ where q is the electric charge of the field on which D_μ act; for the electron field $\Psi(x)$, $q = -e$ and hence $[D_\mu, D_\nu]\Psi(x) = -ieF_{\mu\nu}(x)\Psi(x)$. Hence, according to the lemma (S.13),

$$\not{D}^2 \Psi \;=\; D^2 \Psi \;-\; eF_{\mu\nu} S^{\mu\nu} \Psi. \tag{S.14}$$

Now, suppose the electron field $\Psi(x)$ satisfies the covariant Dirac equation $(i\not{D} - m)\Psi = 0$.

Then for any differential operator $\mathcal{D}$, $\mathcal{D} \times (i\not{D} - m)\Psi = 0$, and in particular

$$(-i\not{D} - m) \times (i\not{D} - m)\Psi \;=\; 0. \tag{S.15}$$

The LHS of this formula amounts to

$$(-i\not{D} - m) \times (i\not{D} - m)\Psi \;=\; \big(\not{D}^2 \,+\, m^2\big)\Psi \;=\; \big(D^2 \,-\, eF_{\mu\nu}S^{\mu\nu} \,+\, m^2\big)\Psi, \tag{S.16}$$

which immediately leads to eq. (2).

Problem **1**(d):
The anti-commutation relations (1) imply $\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is '+' for $\mu = \nu$ and '−' otherwise. Hence for any product Γ of the γ matrices, $\gamma^\mu\Gamma = (-1)^n\Gamma\gamma^\mu$, where n is the number of $\gamma^{\nu\neq\mu}$ factors of Γ. For the $\Gamma = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n = 3$ for any $\mu = 0, 1, 2, 3$, hence $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$.

As to the spin matrices, $\gamma^5\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^5\gamma^\nu = +\gamma^\mu\gamma^\nu\gamma^5$ and therefore $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$.

Problem **1**(e):
First, the hermiticity:

$$\begin{aligned}
\big(\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\big)^\dagger \;&=\; -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \;=\; -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(+\gamma^0) \\
&=\; +i\gamma^3\gamma^2\gamma^1\gamma^0 \;=\; +i(\gamma^3\gamma^2\gamma^1) \times \gamma^0 \;=\; +i(-1)^3\gamma^0 \times \gamma^3\gamma^2\gamma^1 \\
&=\; +i(-1)^3\gamma^0 \times (-1)^2\gamma^1 \times \gamma^3\gamma^2 \;=\; +i(-1)^3\gamma^0 \times (-1)^2\gamma^1 \times (-1)\gamma^2\gamma^3 \\
&=\; (-1)^6 \times (+i\gamma^0\gamma^1\gamma^2\gamma^3) \;\equiv\; +1 \times \gamma^5.
\end{aligned} \tag{S.17}$$

Second, the square:

$$\begin{aligned}
(\gamma^5)^2 \;&=\; \gamma^5(\gamma^5)^\dagger \;=\; (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) \;=\; -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0 \\
&=\; +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = \; -\gamma^0(\gamma^1\gamma^1)\gamma^0 \;=\; +\gamma^0\gamma^0 \;=\; +1.
\end{aligned} \tag{S.18}$$

Problem **1**(f):

Since the four Dirac matrices γ^0, γ^1, γ^2, γ^3 all anticommute with each other, we have

$$\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \;=\; \gamma^{[0}\gamma^1\gamma^2\gamma^{3]} \;=\; 24\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -24i\gamma^5. \tag{S.19}$$

To prove the other identity, we note that a totally antisymmetric product $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$ vanishes unless the Lorentz indices $\kappa, \lambda, \mu, \nu$ are all distinct — which makes them $0, 1, 2, 3$ in some order. For such indices, the anticommutativity of the Dirac matrices implies $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = -\epsilon^{\kappa\lambda\mu\nu} \times \gamma^0\gamma^1\gamma^2\gamma^3$ (note that $\epsilon^{0123} = -1$), and hence

$$\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} \;=\; -24\epsilon^{\kappa\lambda\mu\nu} \times \gamma^0\gamma^1\gamma^2\gamma^3 \;=\; +24i\epsilon^{\kappa\lambda\mu\nu} \times \gamma^5. \tag{S.20}$$

Problem **1**(g):

$$\begin{aligned} 6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5 &= \tfrac{6}{24}\,\gamma_\kappa\,\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} \\ &= \tfrac{1}{4}\,\gamma_\kappa\,\Big(\gamma^\kappa\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;-\; \gamma^{[\lambda)}\gamma^\kappa\gamma^{(\mu}\gamma^{\nu]} \;+\; \gamma^{[\lambda}\gamma^{\mu)}\gamma^\kappa\gamma^{(\nu]} \;-\; \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\gamma^\kappa\Big) \\ &= \tfrac{1}{4}\,\Big(4\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;+\; 2\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;+\; 4g^{[\lambda\mu}\gamma^{\nu]} \;+\; 2\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\Big) \\ &= \tfrac{1}{4}(4+2+0-2) \times \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;=\; \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}. \end{aligned} \tag{S.21}$$

Problem **1**(h):

Proof by inspection: In the Weyl basis and in the 2×2 block form, the 16 matrices are

$$\begin{gathered} \mathbf{1}_{4\times4} \;=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \qquad \gamma^0 \;=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i \;=\; \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ \tfrac{1}{2}\gamma^{[i}\gamma^{j]} \;=\; -i\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \tfrac{1}{2}\gamma^{[0}\gamma^{i]} \;=\; \begin{pmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{pmatrix}, \\ \gamma^5\gamma^0 \;=\; \begin{pmatrix} 0 & -\mathbf{1} \\ +\mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^5\gamma^i \;=\; \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 \;=\; \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & +\mathbf{1} \end{pmatrix}, \end{gathered} \tag{S.22}$$

and their linear independence is self-evident. Since there are only 16 independent 4×4 matrices altogether, any such matrix Γ is a linear combination of the matrices (S.22). *Quod erat demonstrandum.*

Algebraic Proof: Without making any assumption about the matrix form of the γ^μ operators, let us consider the Clifford algebra (1). Using $\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is $+$ for $\mu = \nu$ and $-$ for $\mu \neq \nu$, we may re-order any product of the γ matrices as $\pm\gamma^0\cdots\gamma^0\,\gamma^1\cdots\gamma^1\,\gamma^2\cdots\gamma^2\,\gamma^3\cdots\gamma^3$. Moreover, since each γ^μ squares to $+1$ or -1, we may further simplify the product in question to to $\pm(\gamma^0 \text{ or } 1)\times(\gamma^1 \text{ or } 1)\times(\gamma^2 \text{ or } 1)\times(\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 matrices: 1, or a γ^μ, or a $\gamma^\mu\gamma^\nu$ for $\mu \neq \nu$ which equals to $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}$, or a $\gamma^\lambda\gamma^\mu\gamma^\nu$ for 3 different λ, μ, ν which equals to $\frac{1}{6}\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = i\epsilon^{\lambda\mu\nu\rho}\gamma^5\gamma_\rho$ (*cf.* part (g)), or $\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5$. Consequently, any operator Γ algebraically constructed of the γ's is a linear combination of these 16 matrices.

Incidentally, this proof explains why the Dirac matrices are 4×4 in $d = 4$ spacetime dimensions: the 16 linearly-independent products of Dirac matrices require matrix size to be $\sqrt{16} = 4$.

Technically, we may also use matrices of size $4n \times 4n$, but then we would have $\gamma^\mu = \gamma^\mu_{4\times4} \otimes \mathbf{1}_{n\times n}$, and ditto for all their products. Physically, this means combining the Dirac spinor index with some other index $i = 1, \ldots, n$ which has nothing to do with Lorentz symmetry. Nobody wants such an index confusion, so physicists always stick to 4×4 Dirac matrices in 4 spacetime dimensions.

Problem **2**(a):

I have already written down the γ^5 and the $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$ in the Weyl basis. (S.22), but here is the detailed calculation, in case you need it:

$$\begin{aligned}\gamma^5 &= i\begin{pmatrix}0 & \sigma^0\\ \overline{\sigma}^0 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^1\\ \overline{\sigma}^1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^2\\ \overline{\sigma}^2 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^3\\ \overline{\sigma}^3 & 0\end{pmatrix}\\ &= i\begin{pmatrix}\sigma^0\overline{\sigma}^1\sigma^2\overline{\sigma}^3 & 0\\ 0 & \overline{\sigma}^0\sigma^1\overline{\sigma}^2\sigma^3\end{pmatrix} = \begin{pmatrix}+i\sigma^1\sigma^2\sigma^3 & 0\\ 0 & -i\sigma^1\sigma^2\sigma^3\end{pmatrix}\\ &= \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}.\end{aligned} \tag{5}$$

Problem **2**(b):

In the Weyl basis for the γ matrices,

$$\gamma^\mu\gamma^\nu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \overline{\sigma}^\nu & 0 \end{pmatrix} \;=\; \begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu \end{pmatrix}, \tag{S.23}$$

and hence

$$S^{ij} \;=\; \frac{i}{4}\begin{pmatrix} -\sigma^{[i}\sigma^{j]} & 0 \\ 0 & -\sigma^{[i}\sigma^{j]} \end{pmatrix} \;=\; \frac{\epsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{S.24}$$

while

$$S^{0k} \;=\; -S^{k0} \;=\; \tfrac{i}{2}\gamma^0\gamma^k \;=\; \frac{i}{2}\begin{pmatrix} -\sigma^k & 0 \\ 0 & +\sigma^k \end{pmatrix}. \tag{S.25}$$

By inspection, these spin matrices agree with the $\mathbf{2}+\overline{\mathbf{2}}$ reducible representation of the Lorentz generators,

$$S^{ij} \;=\; \epsilon^{ijk}\begin{pmatrix} \hat{J}^k_{\mathbf{2}} & 0 \\ 0 & \hat{J}^k_{\overline{\mathbf{2}}} \end{pmatrix} \quad \text{while} \quad S^{0i} \;=\; -S^{i0} \;=\; \begin{pmatrix} \hat{K}^i_{\mathbf{2}} & 0 \\ 0 & \hat{K}^i_{\overline{\mathbf{2}}} \end{pmatrix}, \tag{S.26}$$

or in one formula

$$S^{\mu\nu} \;=\; \begin{pmatrix} \hat{J}^{\mu\nu}_{\mathbf{2}} & 0 \\ 0 & \hat{J}^{\mu\nu}_{\overline{\mathbf{2}}} \end{pmatrix}. \tag{6}$$

Problem **2**(c):

In the previous homework we saw that for any continuous Lorentz transform L,

$$M_R(L) \;=\; \sigma_2 M_L^*(L)\sigma_2 \quad \text{and} \quad M_L(L) \;=\; \sigma_2 M_R^*\sigma_2\,. \tag{S.27}$$

Consequently, considering the transformation laws (8) of the Weyl spinors $\psi_L(x)$ and $\psi_R(x)$ and their complex conjugates, we see that

$$\begin{aligned} \psi'_L(x') \;&=\; M_L \times \psi_L(x), \\ \psi'^*_L(x') \;&=\; M_L^* \times \psi_L(x), \\ \sigma_2 \times \psi'^*_L(x') \;&=\; \sigma_2 \times M_L^* \times \psi_L^*(x) \;=\; \sigma_2 M_L^*\sigma_2 \times \sigma_2\psi_L^*(x) \;=\; M_R \times \sigma_2\psi_L^*(x), \end{aligned} \tag{S.28}$$

thus $\sigma_2 \times \psi_L^*(x)$ transforms under the continuous Lorentz transforms exactly like the $\psi_R(x)$.

Likewise,

$$\begin{aligned}
\psi'_R(x') &= M_R \times \psi_R(x), \\
\psi'^*_R(x') &= M^*_R \times \psi_R(x), \\
\sigma_2 \times \psi'^*_R(x') &= \sigma_2 \times M^*_R \times \psi^*_R(x) \;=\; \sigma_2 M^*_R \sigma_2 \times \sigma_2 \psi^*_R(x) \;=\; M_L \times \sigma_2 \psi^*_R(x),
\end{aligned} \tag{S.29}$$

thus $\sigma_2 \times \psi^*_R(x)$ transforms under the continuous Lorentz transforms exactly like the $\psi_L(x)$.

Problem **2**(d):
Given the γ^μ matrices (3) and the decomposition (8) of the Dirac spinor field $\Psi(x)$ into 2 Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$, we have

$$(i\gamma^\mu\partial_\mu \,-\, m)\Psi = \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\overline{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -m\psi_L \,+\, i\sigma^\mu\partial_\mu\psi_R \\ i\overline{\sigma}^\mu\partial_\mu\psi_L \,-\, m\psi_R \end{pmatrix} \tag{S.30}$$

while

$$\overline{\Psi} \;=\; \Psi^\dagger\gamma^0 \;=\; (\,\psi^\dagger_L \quad \psi^\dagger_R\,) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;=\; (\,\psi^\dagger_R \quad \psi^\dagger_L\,)\,. \tag{S.31}$$

Consequently, the Dirac Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= \overline{\Psi}\,(i\gamma^\mu\partial_\mu \,-\, m)\Psi \\
&= \psi^\dagger_R\Big(-m\psi_L \,+\, i\sigma^\mu\partial_\mu\psi_R\Big) \,+\, \psi^\dagger_L\Big(i\overline{\sigma}^\mu\partial_\mu\psi_L \,-\, m\psi_R\Big) \\
&= i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \,+\, i\psi^\dagger_L\overline{\sigma}^\mu\partial_\mu\psi_L \,-\, m\,\psi^\dagger_L\psi_R \,-\, m\,\psi^\dagger_R\psi_L\,.
\end{aligned} \tag{S.32}$$

Problem **2**(e):
For the massless fermions — and only for the massless fermions, — the last two terms in the Lagrangian (S.32) go away, and we are left with

$$\mathcal{L}_{\text{massless}} \;=\; i\psi^\dagger_L\overline{\sigma}^\mu\partial_\mu\psi_L \,+\, i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \tag{S.33}$$

where the two Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ are completely independent from each

other. In particular, they obey independent Weyl equations

$$i\overline{\sigma}^\mu \partial_\mu \psi_L \;=\; 0 \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R \;=\; 0. \tag{S.34}$$

On the other hand, for $m \neq 0$ the two last terms in the Lagrangian (S.32) connect the ψ_L and ψ_R to each other and we cannot have one without the other. In particular, their equations of motion become mixed,

$$i\overline{\sigma}^\mu \partial_\mu \psi_L \;=\; m\psi_R \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R \;=\; m\psi_L\,. \tag{S.35}$$

Problem **3**(a):
As we saw in class, the Dirac equation $(i\,\partial\!\!\!/ - m)\Psi = 0$ implies the Klein–Gordon equation $(\partial^2 + m^2)\Psi = 0$, hence any plane-wave solution of the Dirac equation must have on-shell momentum p^μ with $p^2 = m^2$. But there is more to the Dirac equation than the Klein–Gordon equation, hence eqs. (11) for the spinor coefficients of the $e^{\mp ipx}$ plane waves. Indeed, for $\Psi_\alpha(x) = e^{-ipx}u_\alpha$ the Dirac equation becomes

$$(i\,\partial\!\!\!/ - m)e^{-ipx}u \;=\; e^{-ipx} \times (p\!\!\!/ - m)u, \tag{S.36}$$

so the constant spinor u_α must obey the matrix equation $(p\!\!\!/ - m)u = 0$. Likewise, for $\Psi_\alpha(x) = e^{+ipx}v_\alpha$ the Dirac equation becomes

$$(i\,\partial\!\!\!/ - m)e^{+ipx}v \;=\; e^{+ipx} \times (-p\!\!\!/ - m)v, \tag{S.37}$$

so the spinor v_α must obey the matrix equation $(p\!\!\!/ + m)v = 0$. Conversely, for any on-shell momentum p and any spinors u_α and v_α obeying the matrix equations (11) the plane waves $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ obey the Dirac equation.

Problem **3**(b):

For $\mathbf{p} = \mathbf{0}$, $p^0 = +m$ and $\not{p} - m = m(\gamma^0 - 1)$. Hence, the $u(\mathbf{p} = \mathbf{0}, s)$ spinors satisfy $(\gamma^0 - 1)u = 0$, or in the Weyl basis

$$\begin{pmatrix} -\mathbf{1}_{2\times2} & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & -\mathbf{1}_{2\times2} \end{pmatrix} u \;=\; 0 \quad\Longrightarrow\quad u \;=\; \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \tag{S.38}$$

where ζ is an arbitrary two-component spinor. Its normalization follows from $u^\dagger u = 2\,\zeta^\dagger\zeta$, so if we want $u^\dagger u = 2E_{\mathbf{p}} = 2m$ (for $\mathbf{p} = 0$), then we need $\zeta^\dagger\zeta = m$. Equivalently, we want $\zeta = \sqrt{m}\,\xi$ — and hence u as in eq. (13) — for a conventionally normalized spinor ξ with $\xi^\dagger\xi = 0$.

Note that there are two independent choices of ξ, normalized to $\xi_s^\dagger\xi_{s'} = \delta_{s,s'}$, so they give rise to two independent $u_\alpha(\mathbf{0}, s)$ spinors normalized to $u^\dagger(\mathbf{0}, s)u(\mathbf{0}, s') = 2m\delta_{s,s'} = 2E_{\mathbf{p}}\delta_{s,s'}$. They correspond to the two spin states of the $\mathbf{p} = 0$ electron. In terms of the spin vector, $\mathbf{S} = \frac{1}{2}\xi_s^\dagger\boldsymbol{\sigma}\xi_s$.

Problem **3**(c):

The Dirac equation is Lorentz-covariant, so we may obtain solutions for all $p^\mu = (+E_{\mathbf{p}}, \mathbf{p})$ by simply Lorentz-boosting the solutions (13) from the rest frame where $p_0^\mu = (+m, \mathbf{0})$. Thus,

$$u(p, s) \;=\; M_D(B)\,u(p_0, s) \;=\; \begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix} \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{m}\,M_L\,\xi_s \\ \sqrt{m}\,M_R\,\xi_s \end{pmatrix} \tag{S.39}$$

where M_D, M_L, and M_R are respectively Dirac-spinor, LH–Weyl-spinor, and RH–Weyl-spinor representations or the Lorentz boost B from $p^\mu = (m, \mathbf{0})$ to $p^\mu = (E, \mathbf{p})$. We saw in in the previous homework that for a boost of velocity β in the direction $\mathbf{n}$,

$$M_L \;=\; \sqrt{\gamma} \times \sqrt{1 - \beta\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \quad M_R \;=\; \sqrt{\gamma} \times \sqrt{1 + \beta\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{S.40}$$

For the boost in question

$$\gamma \;=\; \frac{E}{m}\,, \quad \gamma\beta\mathbf{n} \;=\; \frac{\mathbf{p}}{m}\,, \tag{S.41}$$

hence

$$\sqrt{m}\,M_L \;=\; \sqrt{E\,-\,\mathbf{p}\cdot\boldsymbol{\sigma}}\,, \quad \sqrt{m}\,M_R \;=\; \sqrt{E\,+\,\mathbf{p}\cdot\boldsymbol{\sigma}}\,. \tag{S.42}$$

Plugging these formulae into eq. (S.39) immediately gives us

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}. \tag{14}$$

Problem **3**(d):

The negative-frequency solutions $e^{+ipx}v_\alpha(p,s)$ have Dirac spinors v_α satisfying $(\not{p}+m)v=0$. For a particle at rest, $p^\mu=(+m,\mathbf{0})$, this equation becomes $m(\gamma^0+1)v=0$, or in the Weyl basis

$$\begin{pmatrix} \mathbf{1}_{2\times2} & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & \mathbf{1}_{2\times2} \end{pmatrix} v(\mathbf{p}=\mathbf{0},s) \;=\; 0 \quad\Longrightarrow\quad v(\mathbf{p}=\mathbf{0},s) \;=\; \sqrt{m}\begin{pmatrix} +\eta_s \\ -\eta_s \end{pmatrix} \tag{S.43}$$

for some two-component spinor η_s. As in part (b), the $\sqrt{m}$ factor translates the normalization $v^\dagger(p,s)v(p,s')=2E_p\delta_{s,s'}=2m\delta_{s,s'}$ (for $\mathbf{p}=\mathbf{0}$) to $\eta_s^\dagger\eta_{s'}=\delta_{s,s'}$.

For $\mathbf{p}\neq\mathbf{0}$ we proceed similarly to part (c), namely Lorentz-boost the rest-frame solution (S.43) to the frame where $p^\mu=(+E_\mathbf{p},\mathbf{p})$:

$$\begin{aligned} v(p,s) \;&=\; M_D(B)\,v(p_0,s) \;=\begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix}\begin{pmatrix} +\sqrt{m}\,\eta_s \\ -\sqrt{m}\,\eta_s \end{pmatrix} \\ &= \begin{pmatrix} +\sqrt{m}\,M_L\,\eta_s \\ -\sqrt{m}\,M_R\,\eta_s \end{pmatrix} \;=\begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix}. \end{aligned} \tag{S.44}$$

precisely as in eq. (16).

Problem **3**(e):

Physically, a hole in the Fermi sea has opposite free energy, opposite momentum, opposite spin, *etc.*, from the missing fermion (see my notes for the explanation). A positron is a hole in the Dirac sea of electrons, so it should have opposite $p^\mu = (E, \mathbf{p})$ from the missing electron state — that's why the $v(p,s)$ spinors accompany the $e^{+ipx} = e^{+iEt - i\mathbf{px}}$ plane waves instead of the $e^{-ipx} = E^{-iEt+i\mathbf{px}}$ factors of the $u(p,s)$ spinors. Likewise, the positron should have the opposite spin state from the missing electron, and that's why the η_s should have the opposite spin from the ξ_s. More accurately, the η_s should carry the opposite spin vector $\eta_s^\dagger \mathbf{S} \eta_s$ from the $\xi_s^\dagger \mathbf{S} \xi_s$.

The solution to this spin relation is $\eta_s = \sigma_2 \xi_s^*$ (where $*$ denotes complex conjugation). Indeed, let

$$\eta \;=\; \sigma_2 \xi^* \quad\Longrightarrow\quad \eta^\dagger \;=\; \xi^\top \sigma_2 \,, \tag{S.45}$$

and let's use the formula

$$\sigma_2 \boldsymbol{\sigma}^* \sigma_2 \;=\; -\boldsymbol{\sigma} \tag{S.46}$$

from the previous homework. Consequently,

$$\eta^\dagger \boldsymbol{\sigma} \eta \;=\; \xi^\top \sigma_2 \boldsymbol{\sigma} \sigma_2 \xi^* \;=\; \Big(\xi^\dagger \sigma_2 \boldsymbol{\sigma}^* \sigma_2 \xi\Big)^* \;=\; \Big(-\xi^\dagger \boldsymbol{\sigma} \xi\Big)^* \;=\; -\xi^\dagger \boldsymbol{\sigma} \xi, \tag{S.47}$$

or in other words

$$\eta^\dagger \mathbf{S} \eta \;=\; \tfrac{1}{2} \eta^\dagger \boldsymbol{\sigma} \eta \;=\; -\xi^\dagger \mathbf{S} \xi \,. \tag{S.48}$$

And this is why we set $\eta_s = \sigma_2 \xi_s^*$.

Now consider implication of this relation between the ξ_s and η_s spinors for the plane-wave factors $u_\alpha(p,s)$ and $v_\alpha(p,s)$. Thanks to eq. (S.46) we have

$$\sigma_2 \times \sqrt{E \mp \mathbf{p}\cdot\boldsymbol{\sigma}} \times \sigma_2 \;=\; \Big(\sqrt{E \pm \mathbf{p}\cdot\boldsymbol{\sigma}}\Big)^* , \tag{S.49}$$

hence in light of the explicit formulae (15) and (16),

$$\begin{aligned} v(p,s) \;&=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \end{pmatrix} \\ &=\; \begin{pmatrix} +\sigma_2\sigma_2\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \\ -\sigma_2\sigma_2\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \end{pmatrix} \;=\; \begin{pmatrix} +\sigma_2\left(\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\right)^*\,\xi_s^* \\ -\sigma_2\left(\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\right)^*\,\xi_s^* \end{pmatrix} \\ &=\; \begin{pmatrix} 0 & +\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \times \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ +\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}^* \\ &=\; \gamma^2 u^*(p,s). \end{aligned} \tag{S.50}$$

This verifies the first eq. (17). We may verify the second eq. (17) in a similar manner, but it's easier to use γ^2 being imaginary and squaring to -1, hence $\gamma^2(\gamma^2)^* = -\gamma^2\gamma^2 = +1$, and therefore

$$\gamma^2 v^*(p,s) \;=\; \gamma^2\left(\gamma^2 u^*(p,s)\right)^* \;=\; \gamma^2\left(\gamma^2\right)^* u(p,s) \;=\; +u(p,s). \tag{S.51}$$

Problem **3**(f):

The 3D spinors ξ_λ of definite helicity $\lambda = \mp\frac{1}{2}$ satisfy

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\mp \;=\; \mp|\mathbf{p}|\times\xi_\mp\,. \tag{S.52}$$

Plugging these ξ_λ into the positive-energy Dirac spinors (15), we obtain

$$u(p,\lambda=\mp\tfrac{1}{2}) \;=\; \begin{pmatrix} \sqrt{E\pm|\mathbf{p}|}\times\xi_\mp \\ \sqrt{E\mp|\mathbf{p}|}\times\xi_\mp \end{pmatrix}. \tag{S.53}$$

In the ultra-relativistic limit $E\approx|\mathbf{p}|\gg m$, the square roots here simplify to $\sqrt{E+|\mathbf{p}|}\approx\sqrt{2E}$ and $\sqrt{E-|\mathbf{p}|}\approx 0$ (in comparison with the other root). Consequently, eq. (S.53) simplifies to

$$u(p,L) \;\approx\; \sqrt{2E}\begin{pmatrix}\xi_L\\0\end{pmatrix}, \qquad u(p,R) \;\approx\; \sqrt{2E}\begin{pmatrix}0\\\xi_R\end{pmatrix}. \tag{S.54}$$

In other words, the ultra-relativistic positive-energy Dirac spinors of definite helicity are chiral — dominated by the LH Weyl components for the left helicity or by the RH Weyl components for the right helicity.

Now consider the negative-energy Dirac spinors (16). The η_s spinors have exactly opposite spins from the ξ_s, so their helicities are also opposite from the ξ_s. Thus,

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\eta_\mp \;=\; \pm|\mathbf{p}|\times\eta_\mp \tag{S.55}$$

— note the opposite sign from eq. (S.52). Therefore, the negative-energy Dirac spinors v of definite helicity are

$$v(p,\lambda=\mp\tfrac{1}{2}) \;=\; \begin{pmatrix} +\sqrt{E\mp|\mathbf{p}|}\times\eta_\mp \\ -\sqrt{E\pm|\mathbf{p}|}\times\eta_\mp \end{pmatrix}, \tag{S.56}$$

and in the ultra-relativistic limit they become

$$v(p,L) \;\approx\; -\sqrt{2E}\begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \qquad v(p,R) \;\approx\; +\sqrt{2E}\begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \tag{S.57}$$

Again, the ultra-relativistic negative-energy spinors are chiral, but this time the chirality is opposite from the helicity — the left-helicity spinor has dominant RH Weyl components while the right-helicity spinor has dominant LH Weyl components.

Problem **4**(a):
In problem (3.b–c), we saw that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}, \quad v(p,s) \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \tag{S.58}$$

where ξ_s and η_s are 2-component $SU(2)$ spinors normalized to

$$\xi_s^\dagger\xi_{s'} \;=\; \eta_s^\dagger\eta_{s'} \;=\; \delta_{s,s'}\,, \quad \eta_s \;=\; \sigma_2\xi_s^*\,. \tag{S.59}$$

Before we check eqs. (20), let's check the normalization (19) conditions for the spinors (S.58):

$$\begin{aligned} u^\dagger(p,s)u(p,s') \;&=\; \xi_s^\dagger\Big((\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 \,+\, (\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2\Big)\xi_{s'} \;=\; \xi_s^\dagger(2E)\xi_{s'} \;=\; 2E\delta_{s,s'}\,, \\ v^\dagger(p,s)v(p,s') \;&=\; \eta_s^\dagger\Big((+\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 \,+\, (-\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2\Big)\eta_{s'} \;=\; \eta_s^\dagger(+2E)\eta_{s'} \;=\; 2E\delta_{s,s'}\,, \end{aligned} \tag{S.60}$$

because

$$(\pm\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 \;+\; (\pm'\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2 \;=\; (E-\mathbf{p}\cdot\boldsymbol{\sigma}) \;+\; (E+\mathbf{p}\cdot\boldsymbol{\sigma}) \;=\; 2E. \tag{S.61}$$

And now that we have verified that the spinors (S.58) are properly normalized, let's consider the Lorentz invariant products $\bar{u}u$ and $\bar{v}v$. For the $u(p,s)$ and $v(p,s)$ as in eqs. (S.58), the $\bar{u}$ and $\bar{v}$ are given by

$$\begin{aligned}
\bar{u}(p,s) \;&=\; u^\dagger(p,s)\gamma^0 \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & 0 \end{pmatrix} \\
&= \big(\,\xi_s^\dagger\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,,\quad \xi_s^\dagger\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\big)\,, \\
\bar{v}(p,s) \;&=\; v^\dagger(p,s)\gamma^0 \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & 0 \end{pmatrix} \\
&= \big(\,-\eta_s^\dagger\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,,\quad +\eta_s^\dagger\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\big)\,.
\end{aligned} \tag{S.62}$$

Consequently,

$$\begin{aligned}
\bar{u}(p,s)\,u(p,s') \;&=\; \xi_s^\dagger\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\times\xi_{s'} \\
&\quad+\; \xi_s^\dagger\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\times\xi_{s'} \\
&=\; 2m\times\xi_s^\dagger\xi_{s'} \;=\; 2m\delta_{s,s'}
\end{aligned} \tag{S.63}$$

because

$$\begin{aligned}
\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \;&=\; \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \\
&=\; \sqrt{E^2-(\mathbf{p}\cdot\boldsymbol{\sigma})^2} \;=\; \sqrt{E^2-\mathbf{p}^2} \;=\; m.
\end{aligned} \tag{S.64}$$

Likewise,

$$\begin{aligned}
\bar{v}(p,s)\,v(p,s') \;&=\; -\,\eta_s^\dagger\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\times\eta_{s'} \\
&\quad-\;\eta_s^\dagger\times\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\times\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\times\eta_{s'} \\
&=\; -\,2m\times\eta_s^\dagger\eta_{s'} \;=\; -2m\delta_{s,s'}\,.
\end{aligned} \tag{S.65}$$

Problem **4**(b):

In matrix notations (column × row = matrix), we have

$$
\begin{aligned}
u(p,s)\times\overline{u}(p,s) \;&=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\end{pmatrix}\times\left(\xi_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,\,\xi_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right)\\
&=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\;\xi_s\times\xi_s^\dagger\,\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\;\xi_s\times\xi_s^\dagger\,\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\;\xi_s\times\xi_s^\dagger\,\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\;\xi_s\times\xi_s^\dagger\,\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix},
\end{aligned}
\tag{S.66}
$$

$$
\begin{aligned}
v(p,s)\times\overline{v}(p,s) \;&=\; \begin{pmatrix}+\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\\ -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\end{pmatrix}\times\left(-\eta_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,+\eta_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right)\\
&=\; \begin{pmatrix}-\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}.
\end{aligned}
\tag{S.67}
$$

Summing over two spin polarizations replaces $\xi_s\times\xi_s^\dagger$ with $\sum_s \xi_s\times\xi_s^\dagger = \mathbf{1}_{2\times2}$ and likewise $\eta_s\times\eta_s^\dagger$ with $\sum_s \eta_s\times\eta_s^\dagger = \mathbf{1}_{2\times2}$. Consequently,

$$
\begin{aligned}
\sum_s u(p,s)\times\overline{u}(p,s) \;&=\\
&=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s\times\xi_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s\times\xi_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s\times\xi_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s\times\xi_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}\\
&=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\times\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\times\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\times\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\times\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}\\
&=\; \begin{pmatrix}m & E-\mathbf{p}\boldsymbol{\sigma}\\ E+\mathbf{p}\boldsymbol{\sigma} & m\end{pmatrix} \;=\; m\times\mathbf{1}_{4\times4} \;+\; E\times\gamma^0 \;-\; \mathbf{p}\cdot\vec{\gamma}\\
&=\; \not{p} + m.
\end{aligned}
\tag{S.68}
$$

$$\sum_s v(p,s) \times \overline{v}(p,s) \;=$$

$$= \begin{pmatrix} -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \left[\sum_s \eta_s \times \eta_s^\dagger\right] \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \left[\sum_s \eta_s \times \eta_s^\dagger\right] \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \left[\sum_s \eta_s \times \eta_s^\dagger\right] \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \left[\sum_s \eta_s \times \eta_s^\dagger\right] \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} -m & E-\mathbf{p}\boldsymbol{\sigma} \\ E+\mathbf{p}\boldsymbol{\sigma} & -m \end{pmatrix} \;=\; -m \times \mathbf{1}_{4\times 4} \;+\; E \times \gamma^0 \;-\; \mathbf{p}\cdot\vec{\gamma}$$

$$= \;\not{p} \,-\, m. \tag{S.69}$$

Quod erat demonstrandum.