

Problem 1(a):

The γ^0 matrix commutes with itself but anticommutes with the space-indexed $\gamma^{1,2,3}$. At the same time, the parity reflects the space coordinates but not the time coordinate, $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$ but $t \rightarrow t' = +t$, hence the new space and time derivatives are related to the old derivatives as $\nabla' = -\nabla$ but $\partial'_0 = +\partial$. Together, these two facts give us

$$\not{\partial}' \times \gamma^0 = (\gamma^0 \partial'_0 + \vec{\gamma} \cdot \nabla') \gamma^0 = \gamma^0 (\gamma^0 \partial'_0 - \vec{\gamma} \cdot \nabla) = \gamma^0 (+\gamma^0 \partial_0 + \vec{\gamma} \cdot \nabla) = \gamma^0 \times \not{\partial} \quad (\text{S.1})$$

and hence

$$(i \not{\partial}' - m) \times \gamma^0 = \gamma^0 \times (i \not{\partial} - m). \quad (\text{S.2})$$

Combining this formula with eq. (1) for the Dirac field, we find

$$(i \not{\partial} - m)' \Psi'(x') = (i \not{\partial}' - m)(\pm \gamma^0 \Psi(x)) = \pm (i \not{\partial}' - m) \gamma^0 \Psi(x) = \pm \gamma^0 (i \not{\partial} - m) \Psi(x) \quad (\text{S.3})$$

— the $(i \not{\partial} - m) \Psi(x)$ transforms under parity precisely like the $\Psi(x)$ field itself. In other words, the Dirac equation transforms *covariantly*.

Now consider the Dirac Lagrangian. Taking the Hermitian conjugate of eq. (1) we find

$$\Psi'^{\dagger}(-\mathbf{x}, t) = \pm \Psi^{\dagger}(\mathbf{x}, t) \gamma^{0\dagger} = \pm \Psi^{\dagger}(\mathbf{x}, t) \gamma^0 \quad (\text{S.4})$$

and hence

$$\overline{\Psi}'(-\mathbf{x}, t) = \pm \overline{\Psi}(\mathbf{x}, t) \gamma^0. \quad (\text{S.5})$$

Consequently, the Dirac Lagrangian $\mathcal{L} = \overline{\Psi}(i \not{\partial} - m)\Psi$ transforms into

$$\begin{aligned} \mathcal{L}(x') &= \overline{\Psi}'(x') \times (i \not{\partial} - m)' \Psi'(x') \\ &= \pm \overline{\Psi}(x) \gamma^0 \times \pm \gamma^0 (i \not{\partial} - m) \Psi(x) \\ &= + \overline{\Psi}(x) \times (i \not{\partial} - m) \Psi(x) \\ &= \mathcal{L}(x). \end{aligned} \quad (\text{S.6})$$

In other words, the Dirac Lagrangian is invariant modulo $x \rightarrow x' = (-\mathbf{x}, +t)$, and the Dirac action $S = \int d^4x \mathcal{L}$ is invariant.

Problem 1(b):

The linear momentum $\mathbf{p}$ is a polar vector while the angular momentum — orbital, or spin, or whatever — is an axial vector. Therefore, when the parity symmetry acts on a particle state with momentum $\mathbf{p}$ and spin s , it reverses $\mathbf{p} \rightarrow -\mathbf{p}$ but leaves the spin state as it is $s \rightarrow +s$. The same rules apply to the plane waves of definite momentum and spin, hence for the Dirac spinors

$$u(\mathbf{p}, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ +\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix}, \quad v(\mathbf{p}, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} \quad (\text{S.7})$$

from the previous homework, we have

$$\begin{aligned} \mathbf{P} : u(\mathbf{p}, s) &\rightarrow u(-\mathbf{p}, +s) = \begin{pmatrix} +\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{2 \times 2} & \mathbf{1}_{2 \times 2} \\ \mathbf{1}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ +\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix} \\ &= \gamma^0 \times u(\mathbf{p}, s), \end{aligned} \quad (\text{S.8})$$

$$\begin{aligned} \mathbf{P} : v(\mathbf{p}, s) &\rightarrow v(-\mathbf{p}, +s) = \begin{pmatrix} +\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{2 \times 2} & -\mathbf{1}_{2 \times 2} \\ -\mathbf{1}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} \\ &= -\gamma^0 \times v(\mathbf{p}, s), \end{aligned} \quad (\text{S.9})$$

quod erat demonstrandum.

Problem 1(c):

Let's apply parity to the quantum Dirac field

$$\hat{\Psi}(\mathbf{x}, t) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(e^{-itE_{\mathbf{p}} + i\mathbf{x} \cdot \mathbf{p}} \times u(\mathbf{p}, s) \times \hat{a}_{\mathbf{p}, s} + e^{+itE_{\mathbf{p}} - i\mathbf{x} \cdot \mathbf{p}} \times v(\mathbf{p}, s) \times \hat{b}_{\mathbf{p}, s}^\dagger \right). \quad (\text{S.10})$$

Since everything besides the $\hat{a}_{\mathbf{p}, s}$ and $\hat{b}_{\mathbf{p}, s}^\dagger$ operators in this expansion is a c-number, sand-

wiching the field between two parity operators gives us

$$\hat{\mathbf{P}} \hat{\Psi}(\mathbf{x}, t) \hat{\mathbf{P}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(\begin{array}{l} e^{-itE_{\mathbf{p}} + i\mathbf{x} \cdot \mathbf{p}} \times u(\mathbf{p}, s) \times \hat{\mathbf{P}} \hat{a}_{\mathbf{p},s} \hat{\mathbf{P}} \\ + e^{+itE_{\mathbf{p}} - i\mathbf{x} \cdot \mathbf{p}} \times v(\mathbf{p}, s) \times \hat{\mathbf{P}} \hat{b}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} \end{array} \right). \quad (\text{S.11})$$

At the same time, this expansion should match the the right hand side of eq. (2), for which we have

$$\begin{aligned} \pm \gamma^0 \hat{\Psi}(-\mathbf{x}, t) &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(\begin{array}{l} \pm e^{-itE_{\mathbf{p}} - i\mathbf{x} \cdot \mathbf{p}} \times \gamma^0 u(\mathbf{p}, s) \times \hat{a}_{\mathbf{p},s} \\ \pm e^{+itE_{\mathbf{p}} + i\mathbf{x} \cdot \mathbf{p}} \times \gamma^0 v(\mathbf{p}, s) \times \hat{b}_{\mathbf{p},s}^\dagger \end{array} \right) \\ \langle\langle \text{using part (b)} \rangle\rangle & \\ &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(\begin{array}{l} \pm e^{-itE_{\mathbf{p}} - i\mathbf{x} \cdot \mathbf{p}} \times u(-\mathbf{p}, s) \times \hat{a}_{\mathbf{p},s} \\ \mp e^{+itE_{\mathbf{p}} + i\mathbf{x} \cdot \mathbf{p}} \times v(-\mathbf{p}, s) \times \hat{b}_{\mathbf{p},s}^\dagger \end{array} \right) \quad (\text{S.12}) \\ \langle\langle \text{changing } \int \text{ variable } \mathbf{p} \rightarrow -\mathbf{p} \rangle\rangle & \\ &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(\begin{array}{l} \pm e^{-itE_{\mathbf{p}} + i\mathbf{x} \cdot \mathbf{p}} \times u(\mathbf{p}, s) \times \hat{a}_{-\mathbf{p},s} \\ \mp e^{+itE_{\mathbf{p}} - i\mathbf{x} \cdot \mathbf{p}} \times v(\mathbf{p}, s) \times \hat{b}_{-\mathbf{p},s}^\dagger \end{array} \right). \end{aligned}$$

By eq. (2), the right hand sides of eqs. (S.11) and (S.12) must be equal to each other. Since the Dirac plane waves $e^{-ipx}u(p, s)$ and $e^{+ipx}v(p, s)$ are linearly independent from each other, this means

$$\hat{\mathbf{P}} \hat{a}_{\mathbf{p},s} \hat{\mathbf{P}} = \pm \hat{a}_{-\mathbf{p},s} \quad \text{and} \quad \hat{\mathbf{P}} \hat{b}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} = \mp \hat{b}_{-\mathbf{p},s}^\dagger. \quad (4a)$$

The rest of eq. (4) follows by hermitian conjugation: Since $\hat{\mathbf{P}}^\dagger = \hat{\mathbf{P}}^{-1} = \hat{\mathbf{P}}$,

$$\begin{aligned} \hat{\mathbf{P}} \hat{a}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} &= \left(\hat{\mathbf{P}} \hat{a}_{\mathbf{p},s} \hat{\mathbf{P}} \right)^\dagger = \pm \hat{a}_{-\mathbf{p},s}^\dagger, \\ \hat{\mathbf{P}} \hat{b}_{\mathbf{p},s} \hat{\mathbf{P}} &= \left(\hat{\mathbf{P}} \hat{b}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} \right)^\dagger = \mp \hat{b}_{-\mathbf{p},s}. \end{aligned} \quad (7b)$$

Finally, eqs. (5) follow from eqs. (4) and from parity-invariance of the vacuum state, $\hat{\mathbf{P}} |0\rangle = |0\rangle$. Indeed,

$$\hat{\mathbf{P}} |F(\mathbf{p}, s)\rangle = \hat{\mathbf{P}} \times \hat{a}_{\mathbf{p},s}^\dagger |0\rangle = \hat{\mathbf{P}} \hat{a}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} \times \hat{\mathbf{P}} |0\rangle$$

$$= \pm \hat{a}_{-\mathbf{p},+s}^\dagger \times |0\rangle = \pm |F(-\mathbf{p},+s)\rangle, \quad (\text{S.13})$$

$$\begin{aligned} \hat{\mathbf{P}} |\overline{F}(\mathbf{p},s)\rangle &= \hat{\mathbf{P}} \times \hat{b}_{\mathbf{p},s}^\dagger |0\rangle = \hat{\mathbf{P}} \hat{b}_{\mathbf{p},s}^\dagger \hat{\mathbf{P}} \times \hat{\mathbf{P}} |0\rangle \\ &= \mp \hat{b}_{-\mathbf{p},+s}^\dagger \times |0\rangle = \mp |\overline{F}(-\mathbf{p},+s)\rangle. \end{aligned} \quad (\text{S.14})$$

Problem 2(a):

Consider a state $\hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle$ of one fermion and one antifermion with definite reduced momentum and spins. The charge conjugation operator $\hat{\mathbf{C}}$ turns this state into

$$\begin{aligned} \hat{\mathbf{C}} \times \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle &= \hat{\mathbf{C}} \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{\mathbf{C}} \times \hat{\mathbf{C}} \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) \hat{\mathbf{C}} \times |0\rangle \\ &= \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \times \hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle \\ &= -\hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_2) \times \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_1) |0\rangle. \end{aligned} \quad (\text{S.15})$$

Let's plug this formula into eq. (6):

$$\begin{aligned} \hat{\mathbf{C}} \times |B(\mathbf{p}_{\text{tot}}=0)\rangle &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times \hat{\mathbf{C}} \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle \\ &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times -\hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_2) \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_1) |0\rangle \\ &\quad \langle\langle \text{change variables } \mathbf{p}_{\text{red}} \rightarrow -\mathbf{p}_{\text{red}} \text{ and } s_1 \leftrightarrow s_2 \rangle\rangle \\ &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} -\psi(-\mathbf{p}_{\text{red}}, s_2, s_1) \times \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle. \end{aligned} \quad (\text{S.16})$$

In terms of the bound state's wave function ψ , this action of the C-parity operator $\hat{\mathbf{C}}$ is equivalent to

$$\hat{\mathbf{C}} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) = -\psi(-\mathbf{p}_{\text{red}}, s_2, s_1). \quad (\text{S.17})$$

For a bound state with a definite orbital angular momentum L ,

$$\psi(-\mathbf{p}_{\text{red}}, s_1, s_2) = (-1)^L \times \psi(+\mathbf{p}_{\text{red}}, s_1, s_2). \quad (\text{S.18})$$

Likewise, for a bound state with a definite net spin S ,

$$\psi(\mathbf{p}_{\text{red}}, s_2, s_1) = (-1)^{1-S} \psi(\mathbf{p}_{\text{red}}, s_1, s_2). \quad (\text{S.19})$$

Plugging these two formulae into eq. (S.17) for the C-parity, we obtain

$$\begin{aligned} \hat{\mathbf{C}}\psi(\mathbf{p}_{\text{red}}, s_1, s_2) &= -\psi(-\mathbf{p}_{\text{red}}, s_2, s_1) \\ &= -(-1)^L \psi(+\mathbf{p}_{\text{red}}, s_2, s_1) \\ &= -(-1)^L (-1)^{1-S} \psi(+\mathbf{p}_{\text{red}}, s_1, s_2). \end{aligned} \quad (\text{S.20})$$

In other words, the bound state has definite C-parity

$$C = -(-1)^L (-1)^{1-S} = (-1)^L \times (-1)^S, \quad (\text{S.21})$$

Quod erat demonstrandum.

Problem 2(b):

Now consider how the P-parity (reflection of space) acts on the one-fermion+one-antifermions state $\hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle$:

$$\begin{aligned} \hat{\mathbf{P}} \times \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle &= \hat{\mathbf{P}} \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{\mathbf{P}} \times \hat{\mathbf{P}} \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) \hat{\mathbf{P}} \times |0\rangle \\ &= (\pm 1) \hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_1) \times (\mp 1) \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_2) |0\rangle \\ &= -\hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_2) |0\rangle. \end{aligned} \quad (\text{S.22})$$

where the overall $-$ sign comes from the opposite intrinsic parities of the fermion and the antifermion. Again, we plug this formula into eq. (6) and then change the integration variable $\mathbf{p}_{\text{red}} \rightarrow -\mathbf{p}_{\text{red}}$ — but this time we do not swap the spins s_1 and s_2 :

$$\begin{aligned} \hat{\mathbf{P}} \times |B(\mathbf{p}_{\text{tot}}) = 0\rangle &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times \hat{\mathbf{P}} \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle \\ &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times -\hat{a}^\dagger(-\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(+\mathbf{p}_{\text{red}}, s_2) |0\rangle \\ &= \int \frac{d^3 \mathbf{p}_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2) \times \hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2) |0\rangle, \end{aligned} \quad (\text{S.23})$$

In terms of the wave-function ψ , this action of the P-parity operator means

$$\hat{\mathbf{P}}\psi(\mathbf{p}_{\text{red}}, s_1, s_2) = -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2). \quad (\text{S.24})$$

For a bound state with a definite angular momentum, this gives us

$$\hat{\mathbf{P}}\psi(\mathbf{p}_{\text{red}}, s_1, s_2) = -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2) = -(-1)^L \times \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \quad (\text{S.25})$$

and hence definite P-parity

$$P = -(-1)^L, \quad (\text{S.26})$$

Quod erat demonstrandum.

Problem 2(c):

Finally, consider the positronium atom decaying into photons. Since the EM interactions respect the charge conjugation symmetry, the EM processes such as $e^- + e^+ \rightarrow$ photons conserve C-parity. A photon of any momentum or polarization has $C = -1$, so the net C-parity of an n -photon final state is $(-1)^n$. Consequently, if the initial electron and positron are in a bound state with $C = +1$ they must annihilate into an even number of photons, $e^- + e^+ \rightarrow 2\gamma, 4\gamma, 6\gamma, \dots$. But if the bound state has $C = -1$, the electron and the positron must annihilate into an odd number of photons, $e^- + e^+ \rightarrow 3\gamma, 5\gamma, \dots$ (Annihilation into a single photon is forbidden because of $\mathbf{p}_{\text{net}}^2 < E_{\text{net}}^2$.)

The ground state of a hydrogen-like positronium ‘atom’ is 1S, meaning $n_{\text{rad}} = 1$ and $L = 0$. Due to spins, there are actually 4 almost-degenerate 1S states; the hyperfine structure splits them into a 1S₃ triplet and a 1S₁ singlet of the net spin. According to eq. (S.21), the triplet states have $C = (-1)^L(-1)^S = (-1)^0(-1)^1 = -1$ while the singlet state has $C = (-1)^L(-1)^S = (-1)^0(-1)^0 = +1$. Consequently, the singlet $S = 0$ state decays into an even number of photons,

$$(e^- + e^+)@1S_1 \rightarrow 2\gamma, 4\gamma, \dots, \quad (\text{S.27})$$

while the triplet $S = 1$ states decay into odd numbers of photons,

$$(e^- + e^+)@1S_3 \rightarrow 3\gamma, 5\gamma, \dots \quad (\text{S.28})$$

This difference affects the net decay rate of each state because QED (Quantum ElectroDy-

namics) has a rather small coupling constant $\alpha = (e^2/4\pi) \approx 1/137$. For each photon in the final state, the decay amplitude carries a factor of e , so the decay rate of a positronium atom into n photons $\Gamma(e^-e^+ \rightarrow n\gamma)$ is $O(\alpha^n)$. Consequently, the $S = 0$ positronium state usually decays into just 2 photons while decays into 4, 6, or more photons are allowed but much less common. Likewise, the $S = 1$ positronium states usually decays into 3 photons while decays into 5 or more photons are allowed but rare. More over, the decay rate into 3 photons is much slower than the decay rate into just 2 photons,

$$\frac{\Gamma((e^- + e^+)@1S_3 \rightarrow 3\gamma)}{\Gamma((e^- + e^+)@1S_1 \rightarrow 2\gamma)} = \frac{O(\alpha^3)}{O(\alpha^2)} = O(\alpha), \quad (\text{S.29})$$

hence the net decay rate of an $S = 1$ state into anything it can decay to — *i.e.*, into any odd number of photons — is much slower than the net decay rate of the $S = 0$ state,

$$\frac{\Gamma((e^- + e^+)@1S_3 \rightarrow \text{anything})}{\Gamma((e^- + e^+)@1S_1 \rightarrow \text{anything})} = O(\alpha) \ll 1. \quad (\text{S.30})$$

And that's why the $S = 1$ states have much longer lifetimes than the $S = 0$ state.

Problem 3(a):

Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus $(\Psi_\alpha^\dagger \Psi_\beta)^\dagger = +\Psi_\beta^\dagger \Psi_\alpha$. Consequently, for any 4×4 matrix Γ , $(\Psi^\dagger \Gamma \Psi)^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi$, and hence $(\bar{\Psi} \Gamma \Psi)^\dagger = \bar{\Psi} \bar{\Gamma} \Psi$ where $\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$ is the Dirac conjugate of Γ .

Now consider the 16 matrices which appear in the bilinears (7). Obviously $\bar{1} = +1$ and this gives us $S^\dagger = +S$. We saw in class that $\overline{\gamma^\mu} = +\gamma^\mu$ for all $\mu = 0, 1, 2, 3$ (*cf. my notes on Dirac spinor fields*), and this gives us $(V^\mu)^\dagger = +V^\mu$. We also saw that $\overline{\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}} = -\frac{i}{2}\gamma^{[\nu}\gamma^{\mu]} = +\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$, and this gives us $(T^{\mu\nu})^\dagger = +T^{\mu\nu}$. As to the γ^5 matrix, we saw in the [last homework#7](#) (problem 1(d-e)) that it's Hermitian and anticommutes with all the γ^μ . Hence $\overline{\gamma^5} = \gamma^0(\gamma^5)^\dagger\gamma^0 = +\gamma^0\gamma^5\gamma^0 = -\gamma^5 \implies i\overline{\gamma^5} = +i\gamma^5$, which gives us $P^\dagger = +P$. Finally, $\overline{\gamma^5\gamma^\mu} = \overline{\gamma^\mu}\overline{\gamma^5} = -\gamma^\mu\gamma^5 = +\gamma^5\gamma^\mu$, which gives us $(A^\mu)^\dagger = +A^\mu$. Thus, by inspection, all the bilinears (7) are Hermitian. *Quod erat demonstrandum.*

Problem 3(b):

Under a continuous Lorentz symmetry $x \mapsto x' = Lx$, the Dirac spinor field and its conjugate transform according to

$$\Psi'(x') = M(L)\Psi(x = L^{-1}x'), \quad \bar{\Psi}'(x') = \bar{\Psi}(x = L^{-1}x')M^{-1}(L), \quad (\text{S.31})$$

hence any bilinear $\bar{\Psi}\Gamma\Psi$ transforms according to

$$\bar{\Psi}'(x')\Gamma\Psi(x') = \bar{\Psi}(x)\Gamma'\Psi(x) \quad (\text{S.32})$$

where

$$\Gamma' = M^{-1}(L)\Gamma M(L). \quad (\text{S.33})$$

So the Lorentz transformation properties of the Dirac bilinears (7) follow from this transformation rule for the 16 Γ matrices in question.

Obviously for $\Gamma = 1$, $\Gamma' = M^{-1}M = 1$, which makes S a Lorentz scalar.

For $\Gamma = \gamma^\mu$, we saw in class that $\Gamma' = M^{-1}\gamma^\mu M = L^\mu_\nu \gamma^\nu$ — see [my notes on Dirac spinors](#), eq. (22). Consequently $V'^\mu = L^\mu_\nu V^\nu$, which makes V^μ a Lorentz vector.

For $\Gamma = \gamma^\mu \gamma^\nu$, $M^{-1}\gamma^\mu \gamma^\nu M = (M^{-1}\gamma^\mu M)(M^{-1}\gamma^\nu M) = L^\mu_\kappa \gamma^\kappa \times L^\nu_\lambda \gamma^\lambda$. Similar transformation works for $\Gamma = \frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$: $\Gamma' = L^\mu_\kappa L^\nu_\lambda \times \frac{i}{2}\gamma^{[\kappa}\gamma^{\lambda]}$. This makes $T^{\mu\nu}$ a Lorentz tensor (with two antisymmetric indices).

Next, the γ^5 commutes with even products of the γ^μ matrices and hence with $M(L) = \exp(\frac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu)$. Consequently, $M^{-1}\gamma^5 M = \gamma^5$, which makes P a Lorentz scalar.

Finally, $M^{-1}(\gamma^\mu \gamma^5)M = (M^{-1}\gamma^\mu M)\gamma^5 = L^\mu_\nu \gamma^\nu \gamma^5$, which makes A^μ a Lorentz vector.

Problem 3(c):

In problem (1) we saw that the Dirac fields transform under parity as

$$\Psi'(x') = \pm \gamma^0 \Psi(x), \quad \bar{\Psi}'(x') = \pm \bar{\Psi}(x) \gamma^0. \quad (\text{S.34})$$

Consequently, the Dirac bilinears transform as

$$\mathcal{P} : \bar{\Psi} \Gamma \Psi \Big|_x \mapsto \bar{\Psi}' \Gamma \Psi' \Big|_{x'} = \bar{\Psi} (\gamma^0 \Gamma \gamma^0) \Psi \Big|_x. \quad (\text{S.35})$$

By inspection, out of 16 possible Γ matrices, 1 , γ^0 , $\gamma^{[i} \gamma^{j]}$, and $\gamma^5 \gamma^i$ commute with the γ^0 , while γ^i , $\gamma^0 \gamma^i$, $\gamma^5 \gamma^0$, and γ^5 anticommute with the γ^0 . Therefore,

- the bilinears S , V^0 , T^{ij} , and A^i are P-even, *i.e.* remain invariant under parity, while
- the bilinears V^i , T^{0i} , A^0 , and P are P-odd — the parity flips their signs.

From the 3D point of view, this means that S and V^0 are true scalars, P and A^0 are pseudo-scalars, $\mathbf{V}$ is a true or polar vector, $\mathbf{A}$ is a pseudo-vector or axial vector, and the tensor T contains one true vector T^{0i} and one axial vector $\frac{1}{2} \epsilon^{ijk} T^{jk}$. In space-time terms, we call S a true (Lorentz) scalar, P a (Lorentz) pseudoscalar, V^μ a true (Lorentz) vector, and A^μ an axial (Lorentz) vector. Finally, the tensor $T^{\mu\nu}$ is a true Lorentz tensor. However, a physically equivalent tensor $\tilde{T}^{\kappa\lambda} = \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} T_{\mu\nu}$ — for which $\tilde{T}^{0i} = -\frac{1}{2} \epsilon^{ijk} T^{jk}$ is an axial 3-vector while $\frac{1}{2} \epsilon^{ijk} \tilde{T}^{jk} = +T^{0i}$ is a polar 3-vector — is a Lorentz pseudo-tensor.

Problem 3(d):

In class we saw that in the Weyl convention, the charge conjugation symmetry acts on Dirac fields as

$$\begin{aligned} \mathbf{C} : \Psi(x) &\rightarrow \Psi'(x) = \gamma^2 \Psi^*(x) = \gamma^2 (\Psi^\dagger(x))^\top, \\ \mathbf{C} : \bar{\Psi}(x) &\rightarrow \bar{\Psi}'(x) = \bar{\Psi}^*(x) \gamma^2 = \Psi^\top(x) \gamma^0 \gamma^2 = -\Psi^\top(x) \gamma^2 \gamma^0. \end{aligned} \quad (\text{S.36})$$

Consequently, for any Dirac bilinear $\bar{\Psi} \Gamma \Psi$,

$$\bar{\Psi}' \Gamma \Psi' = -\Psi^\top \gamma^2 \gamma^0 \Gamma \gamma^2 (\Psi^\dagger)^\top = +\Psi^\dagger (\gamma^2 \gamma^0 \Gamma \gamma^2)^\top \Psi = +\bar{\Psi} \gamma^0 \gamma^2 \Gamma^\top \gamma^0 \gamma^2 \Psi \equiv \bar{\Psi} \Gamma^c \Psi. \quad (\text{S.37})$$

The second equality here follows by transposition of the Dirac “sandwich” $\Psi^\top \dots (\Psi^\dagger)^\top$, which carries an extra minus sign because the fermionic fields Ψ and Ψ^* anticommute with

each other (in the classical limit). The third equality follows from $(\gamma^0)^\top = +\gamma^0$, $(\gamma^2)^\top = +\gamma^2$, and $\Psi^\dagger = \bar{\Psi}\gamma^0$.

Problem 3(e):

By inspection, $\mathbf{1}^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +\mathbf{1}$. The γ_5 matrix is symmetric and commutes with the $\gamma^0\gamma^2$, hence $\gamma_5^c = +\gamma_5$. Among the four γ_μ matrices, the γ_1 and γ_3 are anti-symmetric and commute with the $\gamma^0\gamma^2$ while the γ_0 and γ_2 are symmetric but anti-commute with the $\gamma^0\gamma^2$; hence, for all four γ_μ , $\gamma_\mu^c = -\gamma_\mu$. Finally, because of the transposition involved, $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$, hence $(\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]})^c = +\frac{i}{2}\gamma^{[\nu}\gamma^{\mu]} = -\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$. Likewise, $(\gamma^5\gamma^\mu)^c = (\gamma^\mu)^c(\gamma^5)^c = -\gamma^\mu\gamma^5 = +\gamma^5\gamma^\mu$.

Therefore, according to eq. (S.37), the scalar S , the pseudoscalar P , and the axial vector A_μ are C-even, while the vector V_μ and the tensor $T_{\mu\nu}$ are C-odd.