Problem **2**(a):

For the ultra-relativistic electrons and positrons, the spinors $u(p,s)$ and $v(p,s)$ become chiral. Indeed, by inspection of eqs. (2), u has chirality matching the electron's helicity — left for $\lambda = -\frac{1}{2}$ and right for $\lambda = +\frac{1}{2}$ — while v has chirality opposite to the positron's helicity — left for $\lambda = +\frac{1}{2}$ and right for $\lambda = -\frac{1}{2}$. At the same time, the amplitude (1) depends on the electron's and positron's spin states via the 'Dirac sandwich' $\bar{v}(e^+)\gamma_\nu u(e^-)$ which does not mix helicities. Indeed,

$$\bar{v}\gamma_\nu u \;=\; v^\dagger\gamma^0\gamma_\nu u \;=\; v^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} u \;=\; v_L^\dagger \bar{\sigma}_\nu u_L \;+\; v_R^\dagger \sigma_\nu u_R\,, \tag{S.1}$$

so if u and v are chiral, then they should have the same chirality — both left or both right — or else $\bar{v}\gamma_\nu u = 0$. In terms of helicities, $u(e^-)$ and $v(e^+)$ being both left-handed means $\lambda(e^-) = -\frac{1}{2}$ while $\lambda(e^+) = +\frac{1}{2}$; likewise, $u(e^-)$ and $v(e^+)$ being both right-handed means $\lambda(e^-) = +\frac{1}{2}$ while $\lambda(e^+) = -\frac{1}{2}$. Thus, a non-zero amplitude (3) requires the electron and the positron to have opposite helicities, $\lambda(e^+) = -\lambda(e^-)$; for similar helicities of the two initial particles, the amplitude vanishes. *Quod erat demonstrandum.*

For future reference, let me calculate the explicit Dirac sandwiches (S.1) for the ultra-relativistic electron and positron spinors (2):

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_L) \;=\; -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} \;=\; 0, \tag{3}$$

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_R) \;=\; -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} \;=\; -2E \times \eta_L^\dagger \sigma_\nu \xi_R\,, \tag{S.2}$$

$$\bar{v}(e^+_R)\gamma_\nu u(e^-_L) \;=\; +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} \;=\; +2E \times \eta_R^\dagger \bar{\sigma}_\nu \xi_L\,, \tag{S.3}$$

$$\bar{v}(e^+_R)\gamma_\nu u(e^-_R) \;=\; +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} \;=\; 0. \tag{3}$$

As promised, the sandwich — and hence the amplitude (1) vanishes for the same helicities.

Problem **2**(b):

For the ultra-relativistic muons, the $u(\mu-)$ and $v(\mu^+)$ are chiral, and the chiralities behave exactly similar to the electron and the positron in part (a): the Dirac sandwich $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ vanishes unless u and v have the same chirality, which requires the μ^- and the μ^+ to have opposite helicities. Specifically,

$$\bar{u}(\mu_L^-)\gamma_\nu v(\mu_L^+) \;=\; -2E \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} \;=\; 0, \tag{4}$$

$$\bar{u}(\mu_L^-)\gamma_\nu v(\mu_R^+) \;=\; -2E \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} \;=\; +2E \times \xi_L^\dagger \bar{\sigma}_\nu \eta_R \,, \tag{S.4}$$

$$\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) \;=\; -2E \begin{pmatrix} 0 \\ \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} \;=\; -2E \times \xi_R^\dagger \sigma_\nu \eta_L \,, \tag{S.5}$$

$$\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) \;=\; -2E \begin{pmatrix} 0 \\ \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} \;=\; 0. \tag{4}$$

Eqs. (4) — and similar formulae for the other fermion-antifermion pairs produced with ultra-relativistic speeds in electron-positron collisions — assure that the fermion and the antifermion always have opposite helicities. Experimentally, this means that if for some event we are able to determine the helicity of one final particle, then we may infer the second final particle's helicity without any further experimental effort.

Problem **2**(c):

The electron moves in the positive z direction, so its helicity λ is the same as its S_z — the z component of its spin. Hence, the ξ spinors corresponding to the 2 helicities are

$$\xi(e_L^-) \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi(e_R^-) \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{S.6}$$

The positron moves in the negative z direction, so its helicity is opposite from S_z, hence

$$\xi(e_L^+) \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi(e_R^+) \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{S.7}$$

However, the η spinors in eqs. (2) for the positrons have opposite spins from these ξ spinors,

specifically $\eta = \sigma_2 \xi^*$, thus

$$\eta(e_L^+) \;=\; \begin{pmatrix} 0 \\ +i \end{pmatrix} \quad \text{and} \quad \eta(e_R^+) \;=\; \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{S.8}$$

Substituting these 2–component spinors into eqs. (S.2) and (S.3), we obtain

$$\begin{aligned} \bar{v}(e_L^+)\gamma_\nu u(e_R^-) \;&=\; -2E \times \begin{pmatrix} 0 & -i \end{pmatrix} \sigma_\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &=\; +2iE \times (\sigma_\nu)_{21} \\ &=\; 2E \times (0, -i, +1, 0)_\nu, \\ \bar{v}(e_R^+)\gamma_\nu u(e_L^-) \;&=\; +2E \times \begin{pmatrix} +i & 0 \end{pmatrix} \bar{\sigma}_\nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &=\; +2iE \times (\bar{\sigma}_\nu)_{12} \\ &=\; 2E \times (0, +i, +1, 0)_\nu. \end{aligned} \tag{5}$$

When taking the 21 and 12 matrix elements of the σ_ν and $\bar{\sigma}_\nu$ matrices, please remember that $\sigma_\nu = (1, \sigma^x, \sigma^y, \sigma^z)$ while $\bar{\sigma}_\nu = (1, -\sigma^x, -\sigma^y, -\sigma^z)$.

Problem **2**(d):

Suppose for a moment $\theta = 0$ and the $\mu^\mp$ move in the same directions as $e^\mp$. Then the muons have exactly the same spinors u and v as the $e^\mp$ of the same charge and helicity, and similarly to eqs. (5) we have

$$\bar{v}(\mu_L^+)\gamma_\nu u(\mu_R^-) \;=\; 2E \times (0, -i, +1, 0)^\nu \quad \text{and} \quad \bar{v}(\mu_R^+)\gamma_\nu u(\mu_L^-) \;=\; 2E \times (0, +i, +1, 0)^\nu. \tag{S.9}$$

For the muons, the amplitude (1) involves the $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ Dirac sandwich rather than the $\bar{v}(\mu^+)\gamma_\nu u(\mu^-)$, but these two sandwiches are related by complex conjugation

$$\big(\bar{v}\gamma^\nu u\big)^* \;=\; \bar{u}\overline{\gamma^\nu}v = \; \bar{u}\gamma^\nu v, \tag{S.10}$$

as well as raising the index ν, hence

$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) \;&=\; 2E \times \big[(0, -i, +1, 0)_\nu = (0, +i, -1, 0)^\nu\big]^* \;=\; 2E \times (0, -i, -1, 0)^\nu, \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) \;&=\; 2E \times \big[(0, +i, +1, 0)_\nu = (0, -i, -1, 0)^\nu\big]^* \;=\; 2E \times (0, +i, -1, 0)^\nu. \end{aligned} \tag{S.11}$$

Eqs. (S.11) apply for $\theta = 0$. For other muon directions, we may simply rotate the

4–vectors (S.11) through angle θ in the xz plane, thus

$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) &= 2E \times (0, -i\cos\theta, -1, +i\sin\theta)^\nu, \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) &= 2E \times (0, +i\cos\theta, -1, -i\sin\theta)^\nu. \end{aligned} \tag{7}$$

Problem **2**(e):

Substituting the Dirac sandwiches (5) and (7) into the pair production amplitude (1), we obtain

$$\begin{aligned} \big\langle \mu_L^-, \mu_R^+ \big| \mathcal{M} \big| e_L^-, e_R^+ \big\rangle &= \big\langle \mu_R^-, \mu_L^+ \big| \mathcal{M} \big| e_R^-, e_L^+ \big\rangle = -e^2 \times (1 + \cos\theta), \\ \big\langle \mu_R^-, \mu_L^+ \big| \mathcal{M} \big| e_L^-, e_R^+ \big\rangle &= \big\langle \mu_L^-, \mu_R^+ \big| \mathcal{M} \big| e_R^-, e_L^+ \big\rangle = -e^2 \times (1 - \cos\theta), \end{aligned} \tag{S.12}$$

while all the other polarized amplitudes vanish by eqs. (3) and (4):

$$\begin{aligned} \big\langle \mu_{\text{any}}^-, \mu_{\text{any}}^+ \big| \mathcal{M} \big| e_L^-, e_L^+ \big\rangle &= \big\langle \mu_{\text{any}}^-, \mu_{\text{any}}^+ \big| \mathcal{M} \big| e_R^-, e_R^+ \big\rangle = 0, \\ \big\langle \mu_L^-, \mu_L^+ \big| \mathcal{M} \big| e_{\text{any}}^-, e_{\text{any}}^+ \big\rangle &= \big\langle \mu_R^-, \mu_R^+ \big| \mathcal{M} \big| e_{\text{any}}^-, e_{\text{any}}^+ \big\rangle = 0. \end{aligned} \tag{S.13}$$

The partial cross-sections (8) follow from these amplitudes according to

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \left(\frac{|\mathbf{p}'|}{|\mathbf{p}|} \approx 1 \right). \tag{S.14}$$

Problem **2**(f):

Summing the polarized cross-sections (8) over the muons' helicities, we get

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} \\ &= \frac{\alpha^2}{4s} \times (1 + \cos\theta)^2 \;+\; \frac{\alpha^2}{4s} \times (1 - \cos\theta)^2 \;+\; 0 \;+\; 0 \\ &= \frac{\alpha^2}{2s} \times (1 + \cos^2\theta) \end{aligned} \tag{S.15}$$

while

$$\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0. \tag{S.16}$$

Averaging these cross-sections over the electron's and positron's helicities gives

$$\begin{aligned}\frac{d\sigma(e^-_{\text{avg}} + e^+_{\text{avg}} \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} &= \frac{1}{4}\left(\frac{\alpha^2}{2s} \times (1+\cos^2\theta) + \frac{\alpha^2}{2s} \times (1+\cos^2\theta) + 0 + 0\right) \\ &= \frac{\alpha^2}{4s} \times (1+\cos^2\theta),\end{aligned} \tag{S.17}$$

which is exactly what we found in class for the un-polarized cross-section when $E \gg M_\mu$.

Problem **3**(a):

In the first diagram (9), the virtual photon has momentum $q = p_1' - p_1 = p_2 - p_2'$, hence $q^2 = t$. In the second diagram, the virtual photon's momentum is $\tilde{q} = p_1 + p_2 = p_1' + p_2'$, hence $\tilde{q}^2 = s$. Accordingly, the two diagrams are called the t-channel diagram and the s-channel diagram.

The t-channel diagram evaluates to

$$\begin{aligned}i\mathcal{M}_1 &= -\Big(\bar{v}(e^+)(ie\gamma_\mu)v(e^{+\prime})\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)u(e^-)\Big) \times \frac{-ig^{\mu\nu}}{q^2} \\ &= \frac{-ie^2}{t} \times \bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu u(e^-)\end{aligned} \tag{S.18}$$

where the overall minus sign is due to the positron-out to positron-in fermionic line. And the s-channel diagram evaluates to

$$\begin{aligned}i\mathcal{M}_2 &= +\Big(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)v(e^{+\prime}\Big) \times \frac{-ig^{\mu\nu}}{\tilde{q}^2} \\ &= \frac{+ie^2}{s} \times \bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\end{aligned} \tag{S.19}$$

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

Problem **3**(b):

Summing /averaging the $|\mathcal{M}_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \Big[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-)\gamma_\nu v(e^+)\Big] \times \Big[\bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma^\nu u(e^{-\prime})\Big] \\
&= \left(\frac{e^2}{s}\right)^2 \text{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] \times \text{tr}\left[(\not{p}'_1 - m)\gamma^\mu(\not{p}'_2 - m)\gamma^\nu\right] \\
&\quad \langle\!\langle \text{ neglecting the mass relative to the momenta } \rangle\!\rangle \\
&\approx \left(\frac{e^2}{s}\right)^2 \text{tr}\left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right] \times \text{tr}\left[\not{p}'_1\gamma^\mu \not{p}'_2\gamma^\nu\right] \qquad \text{(S.20)} \\
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1)\right] \times 4\left[p_2^{\prime\mu}p_1^{\prime\nu} + p_2^{\prime\nu}p_1^{\prime\mu} - g^{\mu\nu}(p'_2p'_1)\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(p'_2p_2)(p'_1p_1) + 2(p'_2p_1)(p'_1p_2). \\
&\qquad\qquad - 2(p'_2p'_1)(p_2p_1) - 2(p'_2p'_1)(p_2p_1) + 4(p'_2p'_1)(p_2p_1)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(p'_2p_2)(p'_1p_1) + (p'_2p_1)(p'_1p_2)\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 + u^2\right] \qquad \text{(S.20)}
\end{aligned}$$

where the last equality follows from the kinematic relations (12). Altogether,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (13)$$

Problem **3**(c):

The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to each other via analytic continuation of particle's momenta. In terms of the spin-summed $|\mathcal{M}|^2$ and Mandelstam's variables,

$$\sum_{\text{spins}} |\mathcal{M}_1(s,t,u)|^2 = \sum_{\text{spins}} |\mathcal{M}_2(t,s,u)|^2, \qquad \text{(S.21)}$$

hence eq. (13) for the second amplitude implies a similar equation for the first amplitude,

but with s and t exchanged with each other — *i.e.*, eq. (14).

Alternatively, we may sum the $|\mathcal{M}_1|^2$ over all the spins in the same way as we summed the $|\mathcal{M}_2|^2$ in part (b):

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^{-\prime})\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime})\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\right] \\
&= \left(\frac{e^2}{t}\right)^2 \text{tr}\left[(\not{p}_1' + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right] \times \text{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_2' - m)\gamma_\nu\right] \\
&\approx \left(\frac{e^2}{t}\right)^2 \text{tr}\left[\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\right] \times \text{tr}\left[\not{p}_2\gamma_\mu \not{p}_2'\gamma_\nu\right] \qquad \text{(S.22)} \\
&= \left(\frac{e^2}{t}\right)^2 \times 4\left[p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu - g^{\mu\nu}(p_1'p_1)\right] \times 4\left[p_{2\mu}' p_{2\nu} + p_{2\nu}' p_{2\mu} - g_{\mu\nu}(p_2'p_2)\right] \\
&= 16\left(\frac{e^2}{t}\right)^2 \Big[2(p_1'p_2')(p_1p_2) + 2(p_1'p_2)(p_1p_2') \\
&\qquad\qquad - 2(p_1'p_1)(p_2'p_2) - 2(p_1'p_1)(p_2'p_2) + 4(p_1'p_1)(p_2'p_2)\Big] \\
&= 32\left(\frac{e^2}{t}\right)^2 \left[(p_1'p_2')(p_1p_2) + (p_1'p_2)(p_1p_2')\right] \\
&= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right] \qquad \text{(S.22)}
\end{aligned}$$

and hence

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2+u^2}{t^2}\,. \qquad (14)$$

Problem **3**(d):

The interference term between the two diagrams is more complicated:

$$\begin{aligned}
\mathcal{M}_1^* \times \mathcal{M}_2 &= -\frac{e^2}{t}\Big(\bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\Big) \times \\
&\qquad \times \frac{e^2}{s}\Big(\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\Big) \\
&= -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+) \times \bar{v}(e^+)\gamma_\mu u(e^-)
\end{aligned} \qquad \text{(S.23)}$$

where on the last line I have re-ordered the factors so that each $\bar{u}$ is followed by u of the

same electron and each $\bar{v}$ is followed by v for the same positron. After summing over all the spins, each $u \times \bar{u}$ becomes $(\not{p} + m)$, each $v \times \bar{v}$ becomes $(\not{p} - m)$, and the whole product becomes a single big trace rather than a product of two traces,

$$\begin{aligned} \sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 &= -\frac{e^4}{st} \times \text{tr}\Big[(\not{p}_1 + m)\gamma^\nu(\not{p}'_1 + m)\gamma^\mu(\not{p}'_2 - m)\gamma_\nu(\not{p}_2 - m)\gamma_\mu\Big] \\ &\approx -\frac{e^4}{st} \times \text{tr}\Big[\not{p}_1\gamma^\nu\, \not{p}'_1\gamma^\mu\, \not{p}'_2\gamma_\nu\, \not{p}_2\gamma_\mu\Big]. \end{aligned} \tag{S.24}$$

This trace looks more complicated than it is, and we may drastically simplify it by summing over ν and μ before taking the trace. Back in homework#7 we saw that

$$\gamma^\alpha \not{a}\not{b}\not{c}\gamma_\alpha = -2\,\not{c}\not{b}\not{a} \quad \text{and} \quad \gamma^\alpha \not{a}\not{b}\gamma_\alpha = 4(ab). \tag{S.25}$$

For the problem at hand, this gives us $\gamma^\nu \not{p}'_1\gamma^\mu\, \not{p}'_2\gamma_\nu = -2\,\not{p}'_2\gamma^\mu\, \not{p}'_1$ and hence

$$\begin{aligned} \text{tr}\Big[\not{p}_1 \times \gamma^\nu \not{p}'_1\gamma^\mu\, \not{p}'_2\gamma_\nu \times \not{p}_2\gamma_\mu\Big] &= -2\,\text{tr}\Big[\not{p}_1 \times \not{p}'_2\gamma^\mu\, \not{p}'_1 \times \not{p}_2\gamma_\mu\Big] = -2\,\text{tr}\Big[\not{p}_1\, \not{p}'_2 \times \gamma^\mu\, \not{p}'_1\, \not{p}_2\gamma_\mu\Big] \\ &= -2\,\text{tr}\Big[\not{p}_1\, \not{p}'_2 \times 4(p'_1p_2)\Big] = -8(p'_1p_2) \times \text{tr}\Big[\not{p}_1\, \not{p}'_2\Big] \\ &= -8(p'_1p_2) \times 4(p_1p'_2) \\ &= -8u^2. \end{aligned} \tag{S.26}$$

Plugging this trace back into eq. (S.23), we arrive at

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = +2e^4 \times \frac{u^2}{st}\,. \tag{15}$$

Problem **3**(e):

Assembling the spin sums / averages (13–15) together according to eq. (11), we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\overset{\text{def}}{=} \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1 + \mathcal{M}_2|^2 \\
&= \frac{1}{4}\sum_{\text{spins}} \Big(|\mathcal{M}_1|^2 \,+\, |\mathcal{M}_2|^2 \,+\, 2\,\text{Re}\,\mathcal{M}_1^* \mathcal{M}_2 \Big) \\
&= 2e^4 \times \frac{s^2+u^2}{t^2} \;+\; 2e^4 \times \frac{t^2+u^2}{s^2} \;+\; 4e^4 \times \frac{u^2}{st} \\
&= 2e^4 \left(\frac{s^2}{t^2} \,+\, \frac{t^2}{s^2} \,+\, \frac{u^2}{s^2 t^2} \times \Big(s^2 + t^2 + 2st \,=\, (s+t)^2 \,=\, u^2 \Big) \right) \\
&= 2e^4 \times \frac{s^4+t^4+u^4}{s^2 \times t^2} \,.
\end{aligned} \tag{S.27}$$

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2\, s} \;=\; \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2 \times t^2} \,. \tag{16.a}$$

To complete the problem, let's work out the kinematics in the center of mass frame:

$$\begin{aligned}
s &\;=\; 4E^2 \;\approx\; 4\mathbf{p}^2, \\
t &\;=\; -(\mathbf{p}_1' - \mathbf{p}_1)^2 \;=\; -2\mathbf{p}^2(1-\cos\theta), \\
u &\;=\; -(\mathbf{p}_2' - \mathbf{p}_1)^2 \;=\; -2\mathbf{p}^2(1+\cos\theta),
\end{aligned} \tag{S.28}$$

hence

$$\begin{aligned}
\frac{s^4+t^4+u^4}{s^2\,t^2} &\;=\; \frac{(4\mathbf{p}^2)^4 \,+\, (2\mathbf{p}^2)^4 \times (1-\cos\theta)^4 \,+\, (2\mathbf{p}^2)^4 \times (1+\cos\theta)^4}{(4\mathbf{p}^2)^2 \times (2\mathbf{p}^2)^2 (1-\cos\theta)^2} \\
&\;=\; \frac{16 + (1-\cos\theta)^4 + (1+\cos\theta)^4}{4 \times (1-\cos\theta)^2} \;=\; \frac{18 + 12\cos^2\theta + 2\cos^4\theta}{4 \times (1-\cos\theta)^2} \\
&\;=\; \frac{(3+\cos^2\theta)^2}{2(1-\cos\theta)^2} \,.
\end{aligned} \tag{S.29}$$

Plugging this formula into eq. (16.a), we finally obtain

$$\frac{d\sigma^{\text{Bhabha}}}{d\Omega_{\text{cm}}} \;=\; \frac{\alpha^2}{4s} \times \frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2} \,. \tag{16.b}$$

Quod erat demonstrandum.

Problem **4**, preamble:

A point of notation: In the solutions to this problem, the indices μ, e, $\nu \equiv \nu_\mu$, and $\bar{\nu} \equiv \bar{\nu}_e$ denote the particles. For the Lorentz indices, I shall use α, β, γ, δ, κ, λ, σ, ρ, but never μ or ν. Thus, $p_{\mu\alpha}$ denotes the α component of the muon's 4–momentum, *etc.*, *etc.*

Problem **4**(a):

Since the muon is so much lighter than the $W^\pm$ bosons, we have $q^2 \ll M_W^2$ which justifies using Fermi's effective low-energy theory instead of the full Glashow–Weinberg–Salam theory of the weak interactions. In terms of the diagram (17), this means approximating the W propagator as $ig_{\kappa\lambda}/M_W^2$, hence combining this propagator with the two vertices and the external line factors of the diagram, we get

$$\begin{aligned} i\mathcal{M} &= \frac{ig_{\kappa\lambda}}{M_W^2} \times \bar{u}(\nu_\mu) \left(\frac{-ig_2}{\sqrt{2}} \gamma_\kappa \frac{1-\gamma_5}{2} \right) u(\mu) \times \bar{u}(e) \left(\frac{-ig_2}{\sqrt{2}} \gamma_\lambda \frac{1-\gamma_5}{2} \right) v(\bar{\nu}_e) \\ &= -\frac{ig_2^2}{8M_W^2} \times \bar{u}(\nu_\mu) \gamma^\lambda (1-\gamma_5) u(\mu) \times \bar{u}(e) \gamma_\lambda (1-\gamma_5) v(\bar{\nu}_e). \end{aligned} \tag{S.30}$$

Eq. (20) obtains from this formula by identifying the overall factor here as the Fermi constant, or rather

$$\frac{G_F}{\sqrt{2}} \stackrel{\text{def}}{=} \frac{g_2^2}{8M_W^2}\,. \tag{21}$$

Problem **4**(b):

Let's start with the muon decay amplitude

$$\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = -\frac{G_F}{\sqrt{2}} \Big[\bar{u}(\nu_\mu) \gamma^\alpha (1-\gamma^5) u(\mu^-) \Big] \times \Big[\bar{u}(e^-) \gamma_\alpha (1-\gamma^5) v(\bar{\nu}_e) \Big]\,. \tag{18}$$

Since the Dirac conjugate of the $\gamma^\alpha(1-\gamma^5)$ matrix is the same

$$\overline{\gamma^\alpha(1-\gamma^5)} = (1-\bar{\gamma}^5)\bar{\gamma}^\alpha = (1+\gamma^5)\gamma^\alpha = \gamma^\alpha(1-\gamma^5), \tag{S.31}$$

the complex conjugate of the amplitude (20) is

$$\mathcal{M}^* = \frac{G_F}{\sqrt{2}} \Big[\bar{u}(\mu^-) \gamma^\beta (1-\gamma^5) u(\nu_\mu) \Big] \times \Big[\bar{v}(\bar{\nu}_e) \gamma_\beta (1-\gamma^5) u(e^-) \Big]\,. \tag{S.32}$$

Note: I changed the summed-over Lorentz index here from α to β, so that in the product

$\mathcal{M} \times \mathcal{M}^*$ below I can separately sum over α and β. Thus,

$$\begin{aligned}
|\mathcal{M}|^2 \;&=\; \tfrac{1}{2}G_F^2 \times \Big[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\Big] \times \Big[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e)\Big] \times \\
&\qquad \times \Big[\bar{u}(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\Big] \times \Big[\bar{v}(\bar{\nu}_e)\gamma_\beta(1-\gamma^5)u(e^-)\Big] \\
&=\; \tfrac{1}{2}G_F^2 \times \Big[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-) \times \bar{u}(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\Big] \times \\
&\qquad \times \Big[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e) \times \bar{v}(\bar{\nu}_e)\gamma_\beta(1-\gamma^5)u(e^-)\Big] .
\end{aligned}$$

Consequently, when this $|\mathcal{M}|^2$ is summed over the final fermions spins and averaged over the spin of the initial muon, it becomes a product of two traces,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} \;\overset{\text{def}}{=}\; \tfrac{1}{2}\sum_{\substack{\text{all}\\ \text{spins}}} |\mathcal{M}|^2 \;&=\; \tfrac{1}{4}G_F^2 \times \sum_{s_\mu, s_\nu} \Big[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-) \times \bar{u}(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\Big] \times \\
&\qquad \times \sum_{s_e, s_{\bar{\nu}}} \Big[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e) \times \bar{v}(\bar{\nu}_e)\gamma_\beta(1-\gamma^5)u(e^-)\Big] \\
&=\; \tfrac{1}{4}G_F^2 \times \mathrm{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu + M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu + m_\nu)\Big) \\
&\qquad \times \mathrm{tr}\Big(\gamma_\alpha(1-\gamma^5)(\not{p}_{\bar{\nu}} - m_{\bar{\nu}})\gamma_\beta(1-\gamma^5)(\not{p}_e + m_e)\Big).
\end{aligned} \tag{S.33}$$

Problem **4**(c):

Our next task is to evaluate the traces in eq. (S.33). For the first trace, we have

$$\begin{aligned}
&\mathrm{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu + M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu + m_\nu)\Big) \;= \\
&\qquad =\; \mathrm{tr}\Big(\gamma^\alpha(1-\gamma^5)\,\not{p}_\mu\gamma^\beta(1-\gamma^5)\,\not{p}_\nu\Big) \;+\; M_\mu m_\nu \times \mathrm{tr}\Big(\gamma^\alpha(1-\gamma^5)\gamma^\beta(1-\gamma^5)\Big) \\
&\qquad\quad +\; \text{vanishing traces of odd numbers of } \gamma^\lambda \text{ matrices} \\
&\qquad\quad \langle\!\langle\, \text{next, move the } (1-\gamma^5) \text{ factors left using } \gamma^\lambda(1\mp\gamma^5) = (1\pm\gamma^5)\gamma^\lambda \,\rangle\!\rangle \\
&\qquad =\; \mathrm{tr}\Big((1+\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}(1-\gamma^5)p_\nu\Big) \;+\; M_\mu m_\nu \times \mathrm{tr}\Big((1+\gamma^5)\gamma^\alpha\gamma^\beta(1-\gamma^5)\Big) \\
&\qquad =\; \mathrm{tr}\Big((1+\gamma^5)^2\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; M_\mu m_e \times \mathrm{tr}\Big((1+\gamma^5)(1-\gamma^5)\gamma^\alpha\gamma^\beta\Big) \\
&\qquad\quad \langle\!\langle\, \text{use } (1+\gamma^5)^2 = 2(1+\gamma^5) \text{ while } (1+\gamma^5)(1-\gamma^5) = 0 \,\rangle\!\rangle \\
&\qquad =\; 2\,\mathrm{tr}\Big((1+\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; 0 \\
&\qquad =\; 2\,\mathrm{tr}\Big(\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big) \;+\; 2\,\mathrm{tr}\Big(\gamma^5\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big) .
\end{aligned} \tag{S.34}$$

Furthermore, the traces on the last line here were explicitly evaluated in my notes on Dirac traces:

$$\begin{aligned} \mathrm{tr}\left(\gamma^\alpha \not{p}_\mu \gamma^\beta \not{p}_\nu\right) &= 4p_\mu^\alpha p_\nu^\beta + 4p_\mu^\beta p_\nu^\alpha - 4g^{\alpha\beta}(p_\mu \cdot p_\nu), \\ \mathrm{tr}\left(\gamma^5 \gamma^\alpha \not{p}_\mu \gamma^\beta \not{p}_\nu\right) &= -4i\epsilon^{\alpha\gamma\beta\delta} p_{\mu\gamma} p_{\nu\delta}\,, \end{aligned} \tag{S.35}$$

hence

$$\begin{aligned} \mathrm{tr}\left(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu + M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu + m_\nu)\right) &= \\ &= 8\left[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu \cdot p_\nu)\right] - 8i\epsilon^{\alpha\gamma\beta\delta} p_{\mu\gamma} p_{\nu\delta}. \end{aligned} \tag{S.36}$$

In exactly the same way, the second big trace in eq. (S.33) evaluates to

$$\begin{aligned} \mathrm{tr}\left(\gamma_\alpha(1-\gamma^5)(\not{p}_e + m_e)\gamma_\beta(1-\gamma^5)(\not{p}_{\bar\nu} - m_{\bar\nu}\right) &= \\ &= 8\left[(p_{e\alpha} p_{\bar\nu\beta} + p_{e\beta} p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e \cdot p_{\bar\nu})\right] - 8i\epsilon_{\alpha\rho\beta\sigma} p_{\bar\nu}^\rho p_e^\sigma\,. \end{aligned} \tag{S.37}$$

Problem **4**(d):

Let's plug the traces (S.36) and (S.37) back into eq. (S.33) and contract the Lorentz indices α and β:

$$\begin{aligned} \overline{|\mathcal{M}|^2} = 16G_F^2 \times &\left(\left[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu \cdot p_\nu)\right] - i\epsilon^{\alpha\gamma\beta\delta} p_{\mu\gamma} p_{\nu\delta}\right) \times \\ &\times \left(\left[p_{e\alpha} p_{\bar\nu\beta} + p_{e\beta} p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e \cdot p_{\bar\nu})\right] - i\epsilon_{\alpha\rho\beta\sigma} p_{\bar\nu}^\rho p_e^\sigma\right). \end{aligned} \tag{S.38}$$

Note that inside each pair of () here, the first term is symmetric WRT $\alpha \leftrightarrow \beta$ while the second term is antisymmetric, so when we multiply the two factors together and contract the indices, only the symmetric $\times$ symmetric and antisymmetric $\times$ antisymmetric products contribute to the

$$\begin{aligned} \overline{|\mathcal{M}|^2} = 16G_F^2 \Bigg(&\left[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu \cdot p_\nu)\right] \times \left[p_{e\alpha} p_{\bar\nu\beta} + p_{e\beta} p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e \cdot p_{\bar\nu})\right] \\ &- i0 - i0 - \epsilon^{\alpha\gamma\beta\delta} p_{\mu\gamma} p_{\nu\delta} \times \epsilon_{\alpha\rho\beta\sigma} p_{\bar\nu}^\rho p_e^\sigma \Bigg). \end{aligned} \tag{S.39}$$

Moreover, when we open the brackets on the top line here, several terms cancel each other:

$$\begin{aligned}\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \times \Big[p_{e\alpha} p_{\bar\nu\beta} \,+\, p_{e\beta} p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \;=\;& \\ = 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)& \\ \cancel{-\; 2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})} \;\cancel{-\; 2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ \cancel{+\; (g^{\alpha\beta}g_{\alpha\beta}=4)\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ = 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e).&\end{aligned} \tag{S.40}$$

As to the second line of eq. (S.39), contracting the two Levi–Civita tensors gives us

$$\epsilon^{\alpha\gamma\beta\delta}\times\epsilon_{\alpha\rho\beta\sigma} \;=\; -2\delta^\gamma_\rho\delta^\delta_\sigma \;+\; 2\delta^\gamma_\sigma\delta^\delta_\rho\,, \tag{S.41}$$

hence

$$-\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p^\rho_{\bar\nu}p^\sigma_e \;=\; +2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \;-\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}). \tag{S.42}$$

Finally, combining the two lines of eq. (S.39) gives us one more cancellation, thus

$$\overline{|\mathcal{M}|^2} \;=\; 16G_F^2\left(\begin{array}{cc} \cancel{2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})} & +\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \\ +\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) & \cancel{-\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})}\end{array}\right) \;=\; 64G_F^2\times(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e). \tag{23}$$

Quod erat demonstrandum.