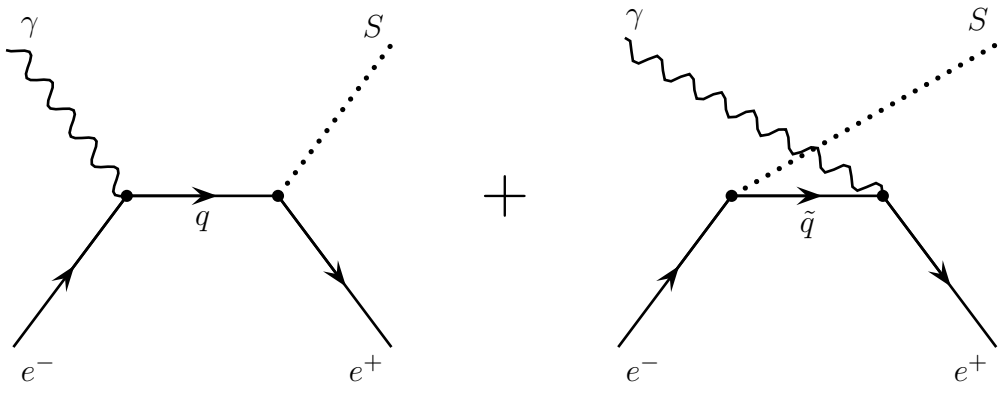
Problem **2**(a):

There are two tree diagrams for the $e^-e^+ \to S\gamma$ process, namely



(S.1)

These two diagrams are related by the $t \leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial e^- and e^+). The net tree-level amplitude is

$$\mathcal{M}_{\rm tree} \;=\; \mathcal{E}^{*\mu}_{\mathbf{k},\lambda}(\gamma) \times \mathcal{M}_\mu \,, \tag{S.2.a}$$

$$\mathcal{M}^\mu \;=\; \mathcal{M}^\mu_1 \;+\; \mathcal{M}^\mu_2 \,, \tag{S.2.b}$$

$$\begin{aligned} i\mathcal{M}^\mu_1 \;&=\; -\bar v(e^+)(-ig)\frac{i}{\not q - m_e}\,(ie\gamma^\mu)\,u(e^-) \\ &=\; \frac{ieg}{t-m_e^2} \times \bar v(\not q + m_e)\gamma^\mu u \,, \end{aligned} \tag{S.2.c}$$

$$\begin{aligned} i\mathcal{M}^\mu_2 \;&=\; \bar v(e^+)\,(ie\gamma^\mu)\,\frac{i}{\not{\tilde q} - m_e}(-ig)u(e^-) \\ &=\; \frac{ieg}{u-m_e^2} \times \bar v\gamma^\mu(\not{\tilde q} + m_e)u \,, \end{aligned} \tag{S.2.d}$$

where

$$\begin{aligned} q \;&=\; p_- \,-\, k_\gamma \;=\; k_s \,-\, p_+ \,, \quad q^2 \;=\; t, \\ \text{and}\quad \tilde q \;&=\; p_- \,-\, k_s \;=\; k_\gamma \,-\, p_+ \,, \quad \tilde q^2 \;=\; u. \end{aligned} \tag{S.3}$$

Problem **2**(b):

The Ward identity for the one-photon amplitude (S.2.a) says $k_\gamma^\mu \times \mathcal{M}_\mu = 0$. To verify it, let's start with the first diagram:

$$\begin{aligned} k_\gamma^\mu \times \bar{v}(\not{q}_1 + m_e)\gamma_\mu u &= \bar{v}(\not{q}_1 + m_e)\, \not{k}_\gamma u \\ &= \bar{v}(\not{p}_- - \not{k}_\gamma + m_e)\, \not{k}_\gamma u \\ &= \bar{v}(\not{p}_- + m_e)\, \not{k}_\gamma u \qquad \langle\langle \text{ because } \not{k}_\gamma \not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\rangle \\ &= \bar{v}\Big(2(p_- k_\gamma) - \not{k}_\gamma(\not{p}_- - m_e)\Big) u \quad \langle\langle \text{ anticommuting } \not{p}_- \text{ and } \not{k}_\gamma \,\rangle\rangle \\ &= 2(p_- k_\gamma) \times \bar{v}u \;-\; 0 \qquad \langle\langle \text{ because } (\not{p}_- - m_e) \times u(e^-) = 0 \,\rangle\rangle \\ &= (m_e^2 - t) \times \bar{v}u\,, \end{aligned} \tag{S.4}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{1\mu} \;=\; -eg \times \bar{v}u\,. \tag{S.5}$$

We see that *by itself*, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram's contribution

$$\begin{aligned} k_\gamma^\mu \times \bar{v}\gamma_\mu(\not{q}_1 + m_e)u &= \bar{v}\, \not{k}_\gamma(\not{q}_1 + m_e)u \\ &= \bar{v}\, \not{k}_\gamma(\not{k}_\gamma - \not{p}_+ + m_e)u \\ &= \bar{v}\, \not{k}_\gamma(-\not{p}_+ + m_e)u \qquad \langle\langle \text{ because } \not{k}_\gamma \not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\rangle \\ &= \bar{v}\Big(-2(p_+ k_\gamma) \;+\; (\not{p}_+ + m_e)\, \not{k}_\gamma\Big) u \quad \langle\langle \text{ anticommuting } \not{p}_+ \text{ and } \not{k}_\gamma \,\rangle\rangle \\ &= -2(p_+ k_\gamma) \times \bar{v}u \;+\; 0 \qquad \langle\langle \text{ because } \bar{v}(e^+) \times (\not{p}_+ + m_e) = 0 \,\rangle\rangle \\ &= (u - m_e^2) \times \bar{v}u\,, \end{aligned} \tag{S.6}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{2\mu} \;=\; +eg \times \bar{v}u\,. \tag{S.7}$$

Again, the second diagram does not satisfy the Ward identity *by itself*, but the net amplitude does:

$$k_\gamma^\mu \times (\mathcal{M}_\mu \;=\; \mathcal{M}_{1\mu} \;+\; \mathcal{M}_{2\mu}) \;=\; 0. \tag{S.8}$$

Problem **2**(c):

Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon's polarizations is easy:

$$\begin{aligned}
\sum_\lambda |\mathcal{M}|^2 &= -\mathcal{M}^\mu \mathcal{M}^*_\mu \qquad \langle\!\langle \text{see my notes on Ward identities} \rangle\!\rangle \\
&= -\mathcal{M}_1^\mu \mathcal{M}^*_{1\mu} \;-\; \mathcal{M}_2^\mu \mathcal{M}^*_{2\mu} \;-\; 2\,\mathrm{Re}\left(\mathcal{M}_1^\mu \mathcal{M}^*_{2\mu}\right) \\
&= -\frac{e^2 g^2}{(t-m_e^2)^2} \times \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m_e)v \\
&\quad - \frac{e^2 g^2}{(u-m_e^2)^2} \times \bar{v}\gamma^\mu(\not{\tilde{q}}+m_e)u \times \bar{u}(\not{\tilde{q}}+m_e)\gamma_\mu v \\
&\quad - \frac{2e^2 g^2}{(t-m_e^2)(u-m_e^2)} \times \mathrm{Re}\Big(\bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}(\not{\tilde{q}}+m_e)\gamma_\mu v\Big).
\end{aligned} \tag{S.9}$$

Consequently, averaging this formula over the electron's and the positron's spins yields

$$\overline{|\mathcal{M}|^2} \;=\; \frac{1}{4}\sum_{s_-,s_+}\sum_\lambda |\mathcal{M}|^2 \;=\; e^2 g^2 \left(\frac{A_{11}}{(t-m_2^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{10}$$

where

$$\begin{aligned}
A_{11} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m_e)v, \\
A_{22} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}\gamma^\mu(\not{\tilde{q}}+m_e)u \times \bar{u}(\not{\tilde{q}}+m_e)\gamma_\mu v, \\
A_{12} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}(\not{\tilde{q}}+m_e)\gamma_\mu v.
\end{aligned} \tag{S.10}$$

At this point, we use the spin sums

$$\sum_{s_-} u\times\bar{u} \;=\; (\not{p}_- + m_e), \quad \sum_{s_+} v\times\bar{v} \;=\; (\not{p}_+ - m_e) \tag{S.11}$$

to convert eqs. (S.10) to Dirac traces (3):

$$\begin{aligned}
A_{11} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)\gamma_\mu(\not{q}+m_e)\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q}+m_e)\Big),
\end{aligned} \tag{S.12}$$

and likewise

$$\begin{aligned} A_{22} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)\gamma^\mu(\not{q}+m_e)\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\not{q}_1+m_e)\gamma_\mu\right) \\ &= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\not{q}+m_e)(\not{p}_- + m_e)(\not{q}_1+m_e)\gamma_\mu\Big), \qquad \text{(S.13)} \\ A_{12} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\not{q}_1+m_e)\gamma_\mu\right) \\ &= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)(\not{q}_1+m_e)\gamma_\mu\Big). \qquad \text{(S.14)} \end{aligned}$$

Quod erat demonstrandum.

Problem **2**(d):

Evaluating the Dirac traces (3) is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron's mass. In that limit, the first trace becomes

$$\begin{aligned} A_{11} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\not{q}\gamma^\mu\not{p}_-\gamma_\mu\not{q}) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\not{q}\not{p}_-\not{q}) \qquad \langle\!\langle\ \text{using}\ \gamma^\mu\not{p}_-\gamma_\mu = -2\not{p}_-\ \rangle\!\rangle \\ &= 2(p_+q)(p_-q)\times 2\ -\ 2(p_+p_-)\,q^2 \qquad \text{(S.15)} \\ &\approx (M_s^2-t)\times t\ -\ s\times t\ =\ (M_s^2-t-s)\times t \\ &\approx u\times t\,, \end{aligned}$$

where the last two lines follow from

$$\begin{aligned} q^2 &= t, \\ 2(p_+p_-) &= (p_-+p_+)^2\ -\ (p_-^2+p_+^2)\ =\ s\ -\ 2m_e^2\ \approx\ s, \\ 2(p_-q) &= 2p_-(p_- - k_\gamma)\ =\ (p_- - k_\gamma)^2\ +\ p_-^2\ -\ k_\gamma^2\ =\ t\ +\ m_e^2\ -\ 0\ \approx\ t, \\ 2(p_+q) &= 2p_+(k_s - p_+)\ =\ -(k_s-p_+)^2\ -\ p_+^2\ +\ k_s^2\ =\ -t\ -\ m_e^2\ +\ M_s^2\ \approx\ M_s^2\ -\ t, \\ s\ +\ t\ +\ u &=\ M_s^2\ +\ 2m_e^2\ \approx\ M_s^2\,. \end{aligned} \qquad \text{(S.16)}$$

Likewise, the second trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\gamma^\mu\,\not{\tilde{q}}\not{p}_-\not{\tilde{q}}\gamma_\mu) \\ &= -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\mu\not{p}_+\gamma^\mu\not{\tilde{q}}\not{p}_-\not{\tilde{q}}) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\not{\tilde{q}}\not{p}_-\not{\tilde{q}}) \qquad \langle\!\langle\ \text{using}\ \gamma_\mu\not{p}_+\gamma^\mu = -2\not{p}_+\ \rangle\!\rangle \\ &= 2(p_+\tilde{q})(p_-\tilde{q})\times 2 - \ 2(p_+p_-)\,\tilde{q}^2 \\ &\approx (M_s^2-u)\times u \ - \ s\times u \ = \ (M_s^2-u-s)\times u \\ &\approx t\times u\,, \end{aligned} \tag{S.17}$$

where the last two lines follow from (S.16) and

$$\begin{aligned} \tilde{q}^2 &= u, \\ 2(p_+\tilde{q}) &= 2p_+(k_\gamma-p_+) \ = \ -(k_\gamma-p_+)^2 \ - \ p_+^2 \ - \ +k_\gamma^2 \ = \ -u \ - \ m_e^2 \ + \ 0 \ \approx \ -u\,, \\ 2(p_-\tilde{q}) &= 2p_-(p_--k_s) \ = \ (p_--k_s)^2 \ - \ k_s^2 \ + \ p_-^2 \ = \ u \ - \ M_s^2 \ + \ m_e^2 \ \approx \ u \ - \ M_s^2\,. \end{aligned} \tag{S.18}$$

Finally, the third trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\not{q}\gamma^\mu\not{p}_-\not{\tilde{q}}\gamma_\mu) \\ &= -(p_-\tilde{q})\times\mathrm{Tr}(\not{p}_+\not{q}) \qquad \langle\!\langle\ \text{using}\ \gamma^\mu\not{p}_-\not{\tilde{q}}\gamma_\mu = +4(p_-\tilde{q})\ \rangle\!\rangle \\ &= -4(p_-\tilde{q})(p_+q) \\ &\approx +(u-M_s^2)(t-M_s^2). \end{aligned} \tag{S.19}$$

Quad erat demonstrandum.

Problem **2**(e):

Now let's evaluate eq. (2) for the spin summed/averaged $\overline{|\mathcal{M}|^2}$. Neglecting the m_e^2 terms in

the denominators and plugging in eqs. (4) for the A_{11}, A_{22}, and A_{12}, we have

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= e^2g^2\left(\frac{tu}{t^2} + \frac{ut}{u^2} + \frac{2(t-M_s^2)(u-M_s^2)}{tu}\right) \\
&= \frac{e^2g^2}{tu}\times\Big(u^2 + t^2 + 2(t-M_s^2)(u-M_s^2)\Big) \\
&= \frac{e^2g^2}{tu}\times\Big((t+u-M_s^2)^2 + M_s^4\Big) \\
&= e^2g^2\times\frac{s^2+M_s^4}{tu}\,.
\end{aligned} \tag{S.20}$$

Now let's work out the kinematics in the center of mass frame. The initial electron and positron have 4–momenta $p^\mu_\mp = (E_e, \pm\mathbf{p})$ where $E_e \approx |\mathbf{p}|$. But since the scalar and the photon produced in the collision have different masses, they have equal and opposite 3-momenta (in the CM frame) but different energies: $k^\mu_\gamma = (\omega, +\mathbf{k})$ while $k^\mu_S = (E_s, -\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{\mathbf{k}^2 + M_s^2}$. By energy conservation

$$\omega + E_s = 2E_e = \sqrt{s}\,. \tag{S.21}$$

To solve this equation, we rewrite it as

$$\omega^2 + M_s^2 = E_s^2 = (\sqrt{s}-\omega)^2 = s - 2\sqrt{s}\times\omega + \omega^2, \tag{S.22}$$

which gives us

$$\omega = \frac{s-M_s^2}{2\sqrt{s}} \quad\Longrightarrow\quad E_s = \frac{s+M_s^2}{2\sqrt{s}}\,. \tag{S.23}$$

Given all these momenta, Mandelstam's t and u obtain as

$$\begin{aligned}
t &\approx -2(p_-k_\gamma) = -2E_e\omega + 2\mathbf{p}\cdot\mathbf{k} \approx -2E_e\omega\times(1-\cos\theta) \\
&= -\tfrac{1}{2}(s-M_s^2)\times(1-\cos\theta),
\end{aligned} \tag{S.24}$$

$$u \approx -2(p_+k_\gamma) = -2E_e\omega - 2\mathbf{p}\cdot\mathbf{k} \approx -2E_e\omega \times (1+\cos\theta)$$
$$= -\tfrac{1}{2}(s-M_s^2)\times(1+\cos\theta). \tag{S.25}$$

Hence, plugging these values into eq. (S.20) gives us

$$\overline{|\mathcal{M}|^2} = 4e^2g^2 \times \frac{s^2+M_S^4}{(s-M_S^2)^2} \times \frac{1}{\sin^2\theta}\,. \tag{S.26}$$

Finally, the partial cross-section for a 2 *particles* $\to$ 2 *particles* inelastic scattering in the CM frame is given by

$$\frac{d\sigma}{d\Omega_{\rm cm}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \frac{|\mathbf{p}'|}{|\mathbf{p}|}\,. \tag{S.27}$$

For the problem at hand, the inelasticity factor $|\mathbf{p}'|/|\mathbf{p}|$ is

$$\frac{|\mathbf{k}|}{|\mathbf{p}|} \approx \frac{\omega}{E_e} = \frac{s-M_s^2}{s}\,. \tag{S.28}$$

Combining this factor with eq. (S.26), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} = \frac{\alpha g^2}{4\pi} \times \frac{s^2+M_s^4}{s^2(s-M_s^2)} \times \frac{1}{\sin^2\theta}\,. \tag{S.29}$$

Note the forward-backward symmetry $\theta \leftrightarrow \pi-\theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.29) has divergent peaks in forward and backward directions, $\theta \to 0$ or $\theta \to \pi$. The divergence here is an artefact of the $m_e^2 = 0$ approximation, which becomes inaccurate at very small angles $\theta \lesssim (m_e/E)$ (or $\pi-\theta \lesssim (m_e/E)$).

A more careful analysis — which was not a required part of this homework — leads to

$$\text{for } \theta \lesssim \gamma^{-1}, \quad \frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} \approx \frac{\alpha g^2}{4\pi s} \times \left(\frac{s-M_s^2}{s} \times \frac{1}{\theta^2+\gamma^{-2}} + \frac{M_s^2}{s-M_s^2} \times \frac{2\theta^2}{(\theta^2+\gamma^{-2})^2} \right) \tag{S.30}$$

— where $\gamma^{-1} = m_e/E \ll 1$ — instead of eq. (S.29). Consequently, the total cross-section

turns out to be finite rather then divergent, namely

$$\sigma_{\rm tot}(e^-e^+ \to \gamma S) \;=\; \alpha g^2 \times \frac{(s^2+M_s^4)}{s^2(s-M_s^2)} \left(\log\frac{2E_e}{m_e} \;-\; \frac{sM_s^2}{s^2+M_s^4} + \; O\left(\frac{m_e^2}{E_e^2}\right)\right). \qquad \text{(S.31)}$$

Problem **3**(a):

The scalar potential part of the linear sigma model's Lagrangian (5) is

$$V(\phi) \;=\; \frac{\lambda}{8}\left(\sum\nolimits_i \phi_i^2 \;-\; f^2\right)^2 \;-\; \beta\lambda f^2 \times \phi_{N+1}\,, \qquad \text{(S.32)}$$

where the last term explicitly breaks the $O(N+1)$ symmetry of the first term down to the $O(N)$. To find the minimum of this potential, let's first find the stationary points where all the first derivatives $\partial V/\partial\phi_i$ are zero:

$$\text{for } i = 1,\ldots,N, \quad \frac{\partial V}{\partial\phi_i} \;=\; \frac{\lambda}{2}\left(\sum\nolimits_j \phi_j^2 \;-\; f^2\right) \times \phi_i \;=\; 0, \qquad \text{(S.33)}$$

$$\text{and} \quad \frac{\partial V}{\partial\phi_{N+1}} \;=\; \frac{\lambda}{2}\left(\sum\nolimits_j \phi_j^2 \;-\; f^2\right) \times \phi_{N+1} \;-\; \beta\lambda f^2 \;=\; 0. \qquad \text{(S.34)}$$

From eq. (S.34) we immediately see that at any stationary point $\left(\sum \phi^2 - f^2\right) \neq 0$, hence eqs. (S.33) tell us that $\phi_1 = \cdots = \phi_N = 0$. In other words, all the stationary points lie on the ϕ_{N+1} axis in the $(N+1)$ dimensional space of the scalar field values. And in this space, eq. (S.34) becomes a simple cubic equation

$$\phi_{N+1}^3 \;-\; f^2 \times \Phi_{N+1} \;-\; 2\beta f^2 \;=\; 0. \qquad \text{(S.35)}$$

For small $\beta \ll f$, this cubic equation has 3 real solutions, approximately

$$\langle\phi_{N+1}\rangle_1 \;\approx\; -2\beta, \qquad \langle\phi_{N+1}\rangle_2 \;\approx\; -f \;+\; \beta, \qquad \langle\phi_{N+1}\rangle_3 \;\approx\; +f \;+\; \beta. \qquad \text{(S.36)}$$

Now let's find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.32). Along the ϕ_{N+1} axis

in the field space, the second derivatives amount to

$$\frac{\partial^2 V}{\partial\phi_i\partial\phi_j} \;=\; \frac{\lambda}{2}\times\begin{cases}\big(3\phi_{N+1}^2-f^2\big) & \text{for } i=j=N+1,\\ 0 & \text{for } i\le N,\ j=N+1 \text{ or } j\le N,\ i=N+1,\\ \big(\phi_{N+1}^2-f^2\big)\times\delta_{ij} & \text{for } i,j\le N.\end{cases} \tag{S.37}$$

Evaluating these derivatives for the 3 stationary points (S.36) — while assuming small $\beta>0$ — gives us

$$\begin{aligned}
&@\ \langle\phi_{N+1}\rangle_1: && \frac{\partial^2 V}{(\partial\phi_{N+1})^2} \;<\; 0 \quad\text{while other}\quad \frac{\partial^2 V}{(\partial\phi_i)^2} \;<\; 0 && \Longrightarrow\ \text{maximum},\\
&@\ \langle\phi_{N+1}\rangle_2: && \frac{\partial^2 V}{(\partial\phi_{N+1})^2} \;>\; 0 \quad\text{while other}\quad \frac{\partial^2 V}{(\partial\phi_i)^2} \;<\; 0 && \Longrightarrow\ \text{saddle point},\\
&@\ \langle\phi_{N+1}\rangle_3: && \frac{\partial^2 V}{(\partial\phi_{N+1})^2} \;>\; 0 \quad\text{while other}\quad \frac{\partial^2 V}{(\partial\phi_i)^2} \;>\; 0 && \Longrightarrow\ \text{minimum}.
\end{aligned} \tag{S.38}$$

Thus, the potential (S.32) has a unique minimum at

$$\langle\phi_1\rangle \;=\; \cdots \;=\; \langle\phi_N\rangle \;=\; 0, \quad \langle\phi_{N+1}\rangle \;=\; +f \;+\; \beta \;+\; O(\beta^2/f). \tag{6}$$

Quod erat demonstrandum.

Problem **3**(b–c):

Let's shift the fields as in eq. (7). In terms of the shifted fields,

$$T \;\stackrel{\text{def}}{=}\; \sum\nolimits_i \phi_i^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \big(\sigma+\langle\phi_{N+1}\rangle\big)^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \sigma^2 + 2\,\langle\phi_{N+1}\rangle\times\sigma + \big(\langle\phi_{N+1}\rangle^2 - f^2\big), \tag{S.39}$$

where $\underset{\sim}{\pi}$ is a short-hand for N-vector $(\pi^1,\ldots,\pi^N)$ of the pion fields, thus $\underset{\sim}{\pi}^2 = (\pi^1)^2+\cdots+(\pi^N)^2$. Therefore, expanding the scalar potential (S.32) into powers of the shifted fields, we obtain

$$\begin{aligned}
V \;&=\; \frac{\lambda}{8}\times T^2 \;-\; \beta\lambda f^2\times\big(\sigma+\langle\phi_{N+1}\rangle\big)\\
&=\; \frac{\lambda}{8}\times\big(\underset{\sim}{\pi}^2+\sigma^2\big)^2 \;+\; \frac{\lambda\,\langle\phi_{N+1}\rangle}{2}\times\big(\sigma\underset{\sim}{\pi}^2+\sigma^3\big)\\
&\quad+\; \frac{\lambda\,\langle\phi_{N+1}\rangle^2}{2}\times\sigma^2 \;+\; \frac{\lambda(\langle\phi_{N+1}\rangle^2-f^2)}{4}\times\big(\underset{\sim}{\pi}^2+\sigma^2\big)\\
&\quad+\; \left(\frac{\lambda\,\langle\phi_{N+1}\rangle}{2}\times\big(\langle\phi_{N+1}\rangle^2-f^2\big) \;-\; \beta\lambda f^2\right)\times\sigma \;+\; \text{const}.
\end{aligned} \tag{S.40}$$

On the last line here, the coefficient of σ vanishes thanks to $\langle\phi_{N+1}\rangle$ obeying the cubic equation (S.35). For the same reason, the coefficient of $(\underset{\sim}{\pi}^2+\sigma^2)$ on the line before the last may be simplified as

$$\frac{\lambda(\langle\phi_{N+1}\rangle^2-f^2)}{4} \;=\; \frac{\beta\lambda f^2}{2\,\langle\phi_{N+1}\rangle}\,. \tag{S.41}$$

Altogether, we have

$$V(\sigma,\underset{\sim}{\pi}) \;=\; \frac{\lambda}{8}\times\big(\underset{\sim}{\pi}^2+\sigma^2\big)^2 \;+\; \frac{\kappa}{2}\times\big(\sigma^3+\sigma\underset{\sim}{\pi}^2\big) \;+\; \frac{M_\sigma^2}{2}\times\sigma^2 \;+\; \frac{M_\pi^2}{2}\times\underset{\sim}{\pi}^2 \;+\; \text{const}, \tag{S.42}$$

where

$$\begin{aligned}
\text{quartic coupling}\quad \lambda &= \lambda,\\
\text{cubic coupling}\quad \kappa &= \lambda\times\langle\phi_{N+1}\rangle \;\approx\; \lambda(f+\beta),\\
\text{pion mass}^2\quad M_\pi^2 &= \frac{\beta\lambda f^2}{\langle\phi_{N+1}\rangle} \;\approx\; \beta\lambda f,\\
\text{sigma mass}^2\quad M_\sigma^2 &= M_\pi^2 \;+\; \lambda\,\langle\phi_{N+1}\rangle^2 \;\approx\; \lambda f(f+3\beta).
\end{aligned} \tag{S.43}$$

Note that

$$\kappa^2 \;=\; \lambda^2\,\langle\phi_{N+1}\rangle^2 \;=\; \lambda\times\lambda\,\langle\phi_{N+1}\rangle^2 \;=\; \lambda\times(M_\sigma^2-M_\pi^2), \tag{S.44}$$

precisely as in eq. (8).

Finally, let's take a closer look at the pion's mass^2, $M_\pi^2 \approx\ \beta\times\lambda f$. In the $\beta=0$ limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for $\beta=0$ the sigma model's Lagrangian has an exact $SO(N+1)$ symmetry which is spontaneously broken down to an $SO(N)$ subgroup; there are N spontaneously broken generators, so there should be N massless Goldstone bosons. But for $\beta\neq 0$, the $SO(N+1)$ symmetry of the Lagrangian is only approximate, and its *explicit* breaking by the $\beta\lambda f^2\times\phi_{N+1}$ term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in β, their mass^2 should be proportional to β. And indeed, in the linear sigma model $M_\pi^2\approx\ \beta\times\lambda f$.

Still, for $\beta \ll f$, the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.43),

$$\frac{M_\pi^2}{M_\sigma^2} \approx \frac{\beta\lambda f}{\lambda f^2} = \frac{\beta}{f} \ll 1. \tag{S.45}$$

Problem **3**(d):

Back in homework#9 (problem 4), we wrote the Lagrangian (HW9.3) for the fields $\sigma(x)$ and $\pi^i(x)$ without explaining where it came from. Now we see that it came from the shifted fields of the linear sigma model with $\beta = 0$ and hence exact $O(N+1)$ symmetry spontaneously broken down to $O(N)$. In the present set up for $\beta \neq 0$, the pions are not exactly massless but are merely much lighter than the sigma. Also, eq. (8) relates the cubic and the quartic couplings to the difference $M_\sigma^2 - M_\pi^2$ rather than just the M_σ^2 as we had in the homework#9.

Consequently, when we calculate the tree-level pion-pion scattering amplitudes in the present setup, we get exactly the same formula in terms of λ, κ, and M_σ as in eq. (S9.33) from the solutions to homework#9, namely

$$\begin{aligned}\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) = &- \delta^{jk}\delta^{\ell m}\left(\lambda + \frac{\kappa^2}{s - M_\sigma^2}\right) \\ &- \delta^{j\ell}\delta^{km}\left(\lambda + \frac{\kappa^2}{t - M_\sigma^2}\right) \\ &- \delta^{jm}\delta^{k\ell}\left(\lambda + \frac{\kappa^2}{u - M_\sigma^2}\right).\end{aligned} \tag{S.46}$$

Furthermore, in light of eq. (8),

$$\lambda + \frac{\kappa^2}{s - M_\sigma^2} = \frac{\lambda s - \lambda M_\sigma^2 + \kappa^2}{s - M_\sigma^2} = \frac{\lambda(s - M_\pi^2)}{s - M_\sigma^2}, \tag{S.47}$$

and likewise

$$\lambda + \frac{\kappa^2}{t - M_\sigma^2} = \frac{\lambda(t - M_\pi^2)}{t - M_\sigma^2}, \qquad \lambda + \frac{\kappa^2}{u - M_\sigma^2} = \frac{\lambda(u - M_\pi^2)}{u - M_\sigma^2}, \tag{S.48}$$

so the amplitude (S.46) becomes

$$\begin{aligned}\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;=\; -\frac{\lambda(s-M_\pi^2)}{s-M_\sigma^2}\times\delta^{jk}\delta^{\ell m} \;-\; \frac{\lambda(t-M_\pi^2)}{t-M_\sigma^2}\times\delta^{j\ell}\delta^{km} \\ -\;\frac{\lambda(u-M_\pi^2)}{u-M_\sigma^2}\times\delta^{jm}\delta^{k\ell}.\end{aligned} \tag{S.49}$$

Moreover, when the pions' energies become low compared to M_σ — or in Lorentz-invariant terms, when $s,t,u \ll M_\sigma^2$ — we may simplify the amplitude (S.49) by approximating all the denominators as $-M_\sigma^2$, thus

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;\approx\; \left(\frac{\lambda}{M_\sigma^2}\;\approx\;\frac{1}{f^2}\right)\times\left(\begin{array}{r}(s-M_\pi^2)\times\delta^{jk}\delta^{\ell m} \;+\; (t-M_\pi^2)\times\delta^{j\ell}\delta^{km} \\ +\;(u-M_\pi^2)\times\delta^{jm}\delta^{k\ell}\end{array}\right), \tag{S.50}$$

exactly as in eq. (9).

Note: in the $\beta\to 0$ and hence $M_\pi^2\to 0$ limit, the amplitude (9) becomes exactly as in homework#9. But for $\beta\neq 0$ we have extra M_π^2 terms in the numerators, and these M_π^2 terms significantly change the very low-energy limit $s,t,u\sim M_\pi^2$ of pion scattering.

Also, since the pions become massive for $\beta\neq 0$, we cannot take all 4 components of a pion's p^μ to zero. The best we can do is to take $\mathbf{p}\to 0$ while $p^0\to m$, which is the *non-relativistic limit*. However, if only one pion is non-relativistic while the other 3 pions have $E\gg M_\pi$ (but $E\ll M_\sigma$), we generally have $s,t,u = O(E\times M_\pi)\gg M_\pi^2$ (albeit $s,t,u\ll M_\sigma^2$), and the scattering amplitude becomes

$$\mathcal{M} \;=\; O\left(\frac{E\times M_\pi}{f^2}\right) \;\not\to\; 0. \tag{S.51}$$

The strongest low-energy limit we an take for massive pions is to make all four pions non-relativistic. In this limit, $s = E_{\rm cm}^2\approx 4M_\pi^2$ while $u,t = O(\mathbf{p}^2)\ll M_\pi^2$, so the scattering amplitude (17) becomes

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;\approx\; \left(\frac{\lambda M_\pi^2}{M_\sigma^2}\approx\frac{\beta\lambda}{f}\right)\times\Big(3\delta^{jk}\delta^{\ell m} \;-\; \delta^{j\ell}\delta^{km} \;-\; \delta^{jm}\delta^{k\ell}\Big). \tag{18}$$

This amplitude is suppressed by the factor β/f, but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic

limit,

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1)\;\approx\;\frac{\lambda\beta}{f}\;\neq\;0, \tag{S.52}$$

unlike what we had back in homework#9.

Problem **4**(a):

$$\begin{aligned}
\text{given}\quad \Phi\;&\to\; e^{+i\theta}U_L\Phi U_R^\dagger\,, && (13)\\
\text{we have}\quad \Phi^\dagger\;&\to\; e^{-i\theta}U_R\Phi^\dagger U_L^\dagger\,, && (S.53)\\
\text{hence}\quad \Phi^\dagger\Phi\;&\to\; U_R(\Phi^\dagger\Phi)U_R^\dagger\,, && (S.54)\\
\big(\Phi^\dagger\Phi\big)^2\;&\to\; U_R\Phi^\dagger\Phi U_R^\dagger\,U_R\Phi^\dagger\Phi U_R^\dagger\;=\;U_R\big(\Phi^\dagger\Phi\big)^2U_R^\dagger\,, && (S.55)\\
\text{likewise}\quad \big(\Phi^\dagger\Phi\big)^n\;&\to\; U_R\big(\Phi^\dagger\Phi\big)^nU_R^\dagger\quad\forall n=1,2,3,\ldots, && (S.56)
\end{aligned}$$

and therefore

$$\text{all traces}\quad \mathrm{tr}\Big(\big(\Phi^\dagger\Phi\big)^n\Big)\quad\text{are invariant under symmetries (13),} \tag{S.57}$$

thanks to the cyclic invariance rule for traces, $\mathrm{tr}\big(U_RXU_R^\dagger\big)=\mathrm{tr}\big(XU_R^\dagger U_R\big)=\mathrm{tr}\big(X\big)$ for any $X=\big(\Phi^\dagger\Phi\big)^n$. Consequently, the scalar potential (12) is invariant under the symmetries (13).

For the global symmetries where $e^{i\theta}$, U_L, and U_R do not depend on x, the kinetic term in the Lagrangian (11) is also invariant. Indeed,

$$\begin{aligned}
&\text{for constant } e^{i\theta},\ U_L,\ U_R,\\
&\qquad\partial_\mu\Phi\;\to\; e^{+i\theta}U_L(\partial_\mu\Phi)U_R^\dagger\,,\\
&\qquad\partial_\mu\Phi^\dagger\;\to\; e^{-i\theta}U_R(\partial_\mu\Phi^\dagger)U_L^\dagger\,, && (S.58)\\
&\qquad\partial^\mu\Phi^\dagger\,\partial_\mu\Phi\;\to\; U_R(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi)U_R^\dagger\,,\\
&\text{and}\quad \mathrm{tr}\big(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi\big)\quad\text{is invariant.}
\end{aligned}$$

Altogether, the whole Lagrangian (11) is invariant, $\quad\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Problem **4**(b):

Given the eigenvalues $(\kappa_1, \ldots, \kappa_N)$ of the $\Phi^\dagger\Phi$ matrix, the invariant traces (S.57) obtain as

$$\mathrm{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right) \;=\; \sum_{i=1}^{N} \kappa_i^n \,. \tag{S.59}$$

Consequently, the scalar potential is

$$V \;=\; \frac{\alpha}{2}\sum_i \kappa_i^2 \;+\; \frac{\beta}{2}\left(\sum_i \kappa_i\right)^2 \;+\; m^2 \sum_i \kappa_i \,. \tag{S.60}$$

Now let's minimize this potential. Since the matrix $\Phi^\dagger\Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1, \ldots, \kappa_N)$ *under constraints* $\kappa_i \geq 0$. This requires

$$\forall i = 1, \ldots, N, \quad \textbf{either } \kappa_i \geq 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;=\; 0, \quad \textbf{or else } \kappa_i = 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;>\; 0, \tag{S.61}$$

where

$$\frac{\partial V}{\partial \kappa_i} \;=\; \alpha\kappa_i \;+\; m^2 \;+\; \beta\sum_j \kappa_j \,. \tag{S.62}$$

These derivatives are linear functions of the eigenvalues κ_i, so all the non-zero eigenvalues must obey the same linear equation

$$\alpha \times \kappa_i \;=\; -m^2 \;-\; \beta \times \sum_j \kappa_j \,, \quad \text{same for all } \kappa_i \neq 0,$$

which means that all non-zero κ_i have the same value. Thus, up to a permutation of eigenvalues,

$$\kappa_1 \;=\; \cdots \;=\; \kappa_k \;=\; C^2, \quad \kappa_{k+1} \;=\; \cdots \;=\; \kappa_N \;=\; 0, \tag{S.63}$$

for some $k = 0, 1, 2, \ldots, N$, and C^2 obtains from

$$\alpha \times C^2 \;+\; m^2 \;+\; \beta \times kC^2 \;=\; 0 \quad\longrightarrow\quad C^2 \;=\; \frac{-m^2}{\alpha + k\beta} \,. \tag{S.64}$$

To make sure that the solution (S.63) is a minimum rather that a maximum or a saddle

point, we need

$$\begin{aligned} C^2 &= \frac{-m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = 0, \\ m^2 + \beta k C^2 &= \frac{\alpha m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = N. \end{aligned} \tag{S.65}$$

Depending on the signs of α, β and m^2 parameters, this limits the solutions to the following:

- For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \cdots \kappa_N = 0$ and hence $\langle \Phi \rangle = 0$.
- For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

$$\kappa_1 = \cdots = \kappa_N = C^2 = \frac{-m^2}{\alpha + N\beta} > 0, \tag{14}$$

  and hence $\langle \Phi \rangle = C \times$ a unitary matrix as in eq. (15). We shall focus on this regime through the rest of this problem.

- For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:
  - When $\alpha + \beta < 0$ or $\alpha + N\beta < 0$, the scalar potential (12) is unbounded from below and the theory is sick.
  - When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the κ_i are as in eq. (14).
  - When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle \Phi \rangle = 0$, but for $m^2 < 0$ all the solutions (S.63) with $k = 1, 2, \ldots, N$ are good local minima.

    To find the global minimum, we compare the potentials at the local minima,

$$\begin{aligned} V(\text{minimum}\#k) &= \frac{\alpha}{2} \times kC^4 + \frac{\beta}{2} \times (kC^2)^2 + m^2 \times kC^2 \\ &= \frac{k\alpha + k^2\beta}{2} \times \frac{m^4}{(\alpha + k\beta)^2} + km^2 \times \frac{-m^2}{(\alpha + k\beta)} \\ &= -\frac{m^4}{2} \times \frac{k}{k\beta + \alpha}\,. \end{aligned} \tag{S.66}$$

Since $\alpha < 0$ but $\alpha + \beta > 0$, the deepest minimum obtains for $k = 1$, thus

$$\kappa_1 \;=\; \frac{-m^2}{\alpha+\beta}\,, \quad \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; 0. \tag{S.67}$$

Problem **4**(c):

Let's act with some $SU(N)_L \times SU(N)_R \times U(1)$ symmetry (13) on the vacuum expectation values (15):

$$\langle\Phi\rangle \;=\; C \times \mathbf{1}_{N\times N} \;\to\; e^{i\theta} U_L \,\langle\Phi\rangle\, U_R^\dagger \;=\; C \times e^{i\theta} U_L U_R^\dagger\,. \tag{S.68}$$

Clearly, to keep the VEVs $\langle\Phi\rangle$ invariant, we need

$$e^{i\theta} U_L U_R^\dagger \;=\; \mathbf{1}_{N\times N} \tag{S.69}$$

and hence

$$U_R \;=\; e^{i\theta} \times U_L\,. \tag{S.70}$$

Moreover, since the U_L and U_R matrices have unit determinants, this requires

$$\det\Big(e^{i\theta} \times \mathbf{1}_{N\times N}\Big) \;=\; 1 \quad\Longrightarrow\quad N \times \theta \;=\; 0 \pmod{2\pi}. \tag{S.71}$$

Such a phase can be absorbed into the $U_L \in SU(N)$, so without loss of generality we need

$$e^{i\theta} \;=\; 1 \quad\text{and}\quad U_L \;=\; U_R \;\in\; SU(N). \tag{S.72}$$

In other words, the *unbroken* symmetry group is $SU(N)_V$ which acts on the scalar fields as

$$\Phi(x) \;\to\; U\Phi(x)\,U^\dagger\,, \quad U \in SU(N). \tag{S.73}$$

Problem **4**(d):

In terms of the shifted fields $\delta\Phi = \Phi - C \times \mathbf{1}_{N\times N}$ and $\delta\Phi^\dagger = \Phi^\dagger - C \times \mathbf{1}_{N\times N}$,

$$\mathrm{tr}(\Phi^\dagger\Phi) \;=\; C^2 \times N \;+\; C\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \;+\; \mathrm{tr}(\delta\Phi^\dagger\delta\Phi), \tag{S.74}$$

$$\begin{aligned}\mathrm{tr}^2(\Phi^\dagger\Phi) \;=\;& N^2C^4 \;+\; 2NC^3\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \\ &+\; 2NC^2\,\mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \;+\; C^2\,\mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi) \\ &+\; 2C\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \times \mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \;+\; \mathrm{tr}^2(\delta\Phi^\dagger\delta\Phi),\end{aligned} \tag{S.75}$$

$$\begin{aligned}\mathrm{tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi) \;=\;& C^4 \times N \;+\; 2C^3\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \\ &+\; C^2\,\mathrm{tr}\big((\delta\Phi)^2 + (\delta\Phi^\dagger)^2 + 4(\delta\Phi^\dagger)(\delta\Phi)\big) \\ &+\; 2C\,\mathrm{tr}\big((\delta\Phi^\dagger)^2(\delta\Phi) + (\delta\Phi^\dagger)(\delta\Phi)^2\big) \\ &+\; \mathrm{tr}\big((\delta\Phi^\dagger)(\delta\Phi)(\delta\Phi^\dagger)(\delta\Phi)\big).\end{aligned} \tag{S.76}$$

Plugging these formulae into the potential (12) and arranging by the powers of $\delta\Phi$ and $\delta\Phi^\dagger$ according to eq. (16), we get

$$V_1 \;=\; \big(\alpha C^3 + \beta N C^3 + m^2 C\big) \times \mathrm{tr}(\delta\Phi^\dagger + \delta\Phi), \tag{S.77}$$

$$\begin{aligned}V_2 \;=\;& \big(2\alpha C^2 + \beta N C^2 + m^2\big) \times \mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \\ &+\; \frac{\alpha C^2}{2} \times \mathrm{tr}\big((\delta\Phi)^2 + (\delta\Phi^\dagger)^2\big) + \frac{\beta C^2}{2} \times \mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi),\end{aligned} \tag{S.78}$$

$$\begin{aligned}V_3 \;=\;& \alpha C \times \mathrm{tr}\big((\delta\Phi^\dagger)^2(\delta\Phi) + (\delta\Phi^\dagger)(\delta\Phi)^2\big) \\ &+\; \beta C \times \mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \times \mathrm{tr}(\delta\Phi^\dagger\delta\Phi),\end{aligned} \tag{S.79}$$

$$\begin{aligned}V_4 \;=\;& \frac{\alpha}{2} \times \mathrm{tr}\big((\delta\Phi^\dagger)(\delta\Phi)(\delta\Phi^\dagger)(\delta\Phi)\big) \\ &+\; \frac{\beta}{2} \times \mathrm{tr}^2(\delta\Phi^\dagger\delta\Phi).\end{aligned} \tag{S.80}$$

Furthermore, the specific value of C^2 in eq. (14) leads to

$$\alpha C^2 \;+\; \beta N C^2 \;+\; m^2 \;=\; 0 \tag{S.81}$$

and hence $V_1 = 0$. Also, eq. (S.78) for the V_2 simplifies to

$$\begin{aligned} V_2 &= \alpha C^2 \times \mathrm{tr}(\delta\Phi^\dagger \delta\Phi) \;+\; \frac{\alpha C^2}{2} \times \mathrm{tr}\big((\delta\Phi)^2 + (\delta\Phi^\dagger)^2\big) \;+\; \frac{\beta C^2}{2} \times \mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi) \\ &= \frac{\alpha C^2}{2} \times \mathrm{tr}\big((\delta\Phi^\dagger + \delta\Phi)^2\big) \;+\; \frac{\beta C^2}{2} \times \mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi), \end{aligned} \tag{S.82}$$

exactly as in eq. (17).

Problem **4**(e):

First, let's check that the fields χ_1, χ_2, φ_1, and φ_2 have properly normalized kinetic terms. Since the φ_1 and φ_2 matrices are traceless, we immediately have

$$\begin{aligned} \mathcal{L}_{\rm kin} &= \mathrm{tr}\big(\partial_\mu \Phi^\dagger \times \partial^\mu \Phi\big) \;=\; \mathrm{tr}\big(\partial_\mu \delta\Phi^\dagger \times \partial^\mu \delta\Phi\big) \\ &= \frac{1}{2N} \partial_\mu(\chi_1 - i\chi_2)\partial^\mu(\chi_1 + i\chi^2) \times \mathrm{tr}(\mathbf{1}_{N\times N}) \;+\; \frac{1}{2}\,\mathrm{tr}\big(\partial_\mu(\varphi_1 - i\varphi_2)\partial^\mu(\varphi_1 + i\varphi^2\big) \\ &= \frac{1}{2}\big((\partial_\mu\chi_1)^2 \,+\, (\partial_\mu\chi_2)^2\big) \;+\; \frac{1}{2}\,\mathrm{tr}\big((\partial_\mu\varphi_1)^2 \,+\, (\partial_\mu\varphi_2)^2\big), \end{aligned} \tag{S.83}$$

so all the kinetic terms are indeed properly normalized.

Now consider the mass terms V_2 in the scalar potential. Both terms in eq. (17) involve

$$\delta\Phi^\dagger \;+\; \delta\Phi \;=\; \sqrt{\frac{2}{N}}\,\chi_1 \times \mathbf{1}_{N\times N} \;+\; \sqrt{2}\varphi_1\,, \tag{S.84}$$

hence

$$\begin{aligned} V_2 &= \alpha C^2 \times \mathrm{tr}\left(\left(\varphi_1 \,+\, \frac{\chi_1}{\sqrt{N}} \times \mathbf{1}_{N\times N}\right)^2\right) \;+\; \beta C^2 \times \mathrm{tr}^2\left(\varphi_1 \,+\, \frac{\chi_1}{\sqrt{N}} \times \mathbf{1}_{N\times N}\right) \\ &= \alpha C^2 \times \Big(\mathrm{tr}\big(\varphi_1^2\big) \,+\, \chi_1^2\Big) + \;\beta C^2 \times \left(\frac{\chi_1}{\sqrt{N}} \times N\right)^2 \\ &= \alpha C^2 \times \mathrm{tr}\big(\varphi_1^2\big) \;+\; (\alpha C^2 + \beta N C^2) \times \chi_1^2. \end{aligned} \tag{S.85}$$

Therefore:

- The χ_1 field has mass $2(\alpha + N\beta)C^2$.
- All components of the traceless matrix φ_1 have mass $2\alpha C^2$.
- The χ_2 field and all components of the φ_2 matrix are massless.

Problem **4**(f):

The unbroken $SU(N)_V$ symmetry acts on the scalar fields according to

$$\Phi(x) \;\to\; U \times \Phi(x) \times U^\dagger. \tag{S.73}$$

and since the VEV (15) is invariant, the shifted fields $\delta\Phi(x) = \Phi(x) - \langle\Phi\rangle$ also transform according to

$$\delta\Phi(x) \;\to\; U \times \delta\Phi(x) \times U^\dagger. \tag{S.86}$$

Moreover, the unitarity transforms like this preserve the decomposition (18) of $\delta\Phi$ into its hermitian and antihermitian parts and also into the traces $\chi_{1,2}$ and the traceless parts $\varphi_{1,2}$. In particular, the trace parts $\chi_1(x)$ and $\chi_2(x)$ remain invariant — which makes them singlets of the $SU(N)_V$ symmetry, — while the traceless matrices $\varphi_1(x)$ and $\varphi_2(x)$ transform as

$$\varphi_1(x) \;\to\; U\varphi_1(x)\,U^\dagger, \qquad \varphi_2(x) \;\to\; U\varphi_2(x)\,U^\dagger, \tag{S.87}$$

which makes them *adjoint multiplets* of the $SU(N)_V$.

Clearly, this multiplet structure agrees with the masses we have obtained in part (e): All N^2-1 members of the adjoint multiplet φ_1 have the same mass $M = 2\alpha C^2$, while all N^2-1 members of the adjoint multiplet φ_2 have $M = 0$. On the other hand, the singlet χ_1 has a different mass from the adjoint fields φ_2, but that's OK since they belong to different multiplets.

However, the second singlet χ_2 has the same zero mass as the second adjoint multiplet φ_2, but that's because they are both Goldstone bosons of the spontaneously broken continuous symmetries

$$G/H \;=\; \Big(SU(N)_L \times SU(N)_R \times U(1)\Big) \Big/ SU(N)\,. \tag{S.88}$$

Specifically, the singlet χ_2 is the Goldstone boson of the broken $U(1)$ symmetry. Indeed, the $U(1)$'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators T_L^a of the $SU(N)_L$ form an adjoint multiplet of the $SU(N)_L$, but are invariant under the $SU(N)_R$. Likewise, generators T_R^a of the $SU(N)_R$ form an adjoint multiplet of the $SU(N)_R$, but are invariant under the $SU(N)_L$. In other words, under an $(U_L, U_R) \in SU(N)_L \times SU(N)_R$ they transform as

$$T_L^a \;\to\; U_L T_L^a U_L^\dagger\,, \qquad T_R^a \;\to\; U_R T_R^a U_R^\dagger\,. \tag{S.89}$$

When the $SU(N)_L \times SU(N)_R$ is broken down to a single $SU(N)$ spanning $U_L = U_R = U$, both T_L^a and T_R^a transform as

$$T_L^a \;\to\; U T_L^a U^\dagger, \qquad T_R^a \;\to\; U T_R^a U^\dagger, \tag{S.90}$$

which puts them into *two adjoint multiplets* of the unbroken $SU(N)$. Equivalently, we may form two adjoint multiplets out of

$$T_V^a \;=\; T_L^a \,+\, T_R^a \qquad \text{and} \qquad T_A^a \;=\; T_L^a \,-\, T_R^a\,, \tag{S.91}$$

which act on the scalar fields according to

$$T_V^a \Phi \;=\; \tfrac{i}{2}[\lambda^a, \Phi], \qquad T_A^a \Phi \;=\; \tfrac{i}{2}\{\lambda^a, \Phi\}. \tag{S.92}$$

The T_V^a generate the unbroken $SU(N)_V$ symmetry, *cf.* eq. (S.73). The T_A^a generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the φ_2.