Tachyons

Classically, a particle cannot be accelerated to a speed faster than c no matter how hard you push it. But in quantum mechanics, it's theoretically possible to create a particle with initial velocity faster than light, and then it continues to move faster than light until it's eventually absorbed in some quantum process. Such a hypothetical particle is called a *tachyon**; its very existence would break the relativistic causality, but that's OK for a theoretical exercise.

Semiclassically, the relation between a tachyon's velocity, energy, and momentum is similar to an ordinary relativistic particle with a rest mass analytically continued to an imaginary value $m = i\mu$. Thus, in $c = 1$ units

$$\mathbf{p} \;=\; \frac{\mu\mathbf{v}}{\sqrt{\mathbf{v}^2 - 1}}\,, \quad E \;=\; \frac{\mu}{\sqrt{\mathbf{v}^2 - 1}}\,, \quad \mathbf{p}^2 \;-\; E^2 \;=\; \mu^2 \;>\; 0. \tag{1}$$

Note that the tachyon's momentum decreases with its velocity, so trying to accelerate it with a forward-directed force would actually decrease its velocity; although it would always remain faster than light. On the other hand, a backward force on a tachyon makes it move faster and faster, until it eventually reaches infinity, and then the tachyon starts moving backward in time!

The quantum mechanics of a tachyon is also weird, but it's perfectly consistent if you don't mind the broken causality. But the quantum field theory of a tachyon field does not work at all as it suffers from the vacuum instability.

Free Tachyon Field

Later in class we shall learn that a tachyon — if it exists at all — must have zero spin, so a would-be tachyon field is simply a scalar field $\Phi(x)$ with a negative mass$^2 = -\mu^2$. So let's start with a free scalar field of this kind; it's classical Lagrangian is

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\nu\Phi\partial^\nu\Phi \;+\; \frac{\mu^2}{2}\Phi^2, \tag{2}$$

$\star$ The name 'tachyon' was given by Gerald Feinberg in 1967, but several physicists considered such particles before him, in particular Bulaniuk, Deshpande, and Sudarshan in 1962.

while the quantum Hamiltonian is

$$\hat{H} = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\nabla\hat{\Phi})^2 - \frac{m\mu^2}{2}\hat{\Phi}^2 \right). \tag{3}$$

Or after expanding the quantum fields into the $\hat{\Phi}_{\mathbf{k}}$ and $\hat{\Pi}_{\mathbf{k}}$ modes in the big-box normalization,

$$\hat{H} = \sum_{\mathbf{k}} \left(\tfrac{1}{2}\hat{\Pi}^\dagger_{\mathbf{k}}\hat{\Pi}_{\mathbf{k}} + \tfrac{1}{2}(\mathbf{k}^2 - \mu^2)\Phi^\dagger_{\mathbf{k}}\Phi_{\mathbf{k}} \right). \tag{4}$$

The trouble with this formula is the range over which we sum over the momenta $\mathbf{k}$. Indeed, for a classical tachyon moving at any superluminal velocity $\mathbf{v}$ the momentum has a lower bound $|\mathbf{p}| \geq \mu$. In the quantum mechanics of a single tachyon, we may likewise restrict the spectrum of the momentum operator to $|\mathbf{p}| \geq \mu$; this would look godawful in the coordinate basis, but in the momentum basis that's OK. But in the quantum field theory, the fields $\hat{\Phi}(\mathbf{x})$ and $\hat{\Pi}(\mathbf{x})$ are general functions of the coordinates $(x, y, z) = \mathbf{x}$, so *their Fourier decomposition*

$$\begin{aligned}
\hat{\Phi}(\mathbf{x}) &= \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \times \hat{\Phi}_{\mathbf{k}}, & \hat{\Phi}_{\mathbf{k}} &= \int d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \times \hat{\Phi}(\mathbf{x}), \\
\hat{\Pi}(\mathbf{x}) &= \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \times \hat{\Pi}_{\mathbf{k}}, & \hat{\Pi}_{\mathbf{k}} &= \int d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \times \hat{\Pi}(\mathbf{x}),
\end{aligned} \tag{5}$$

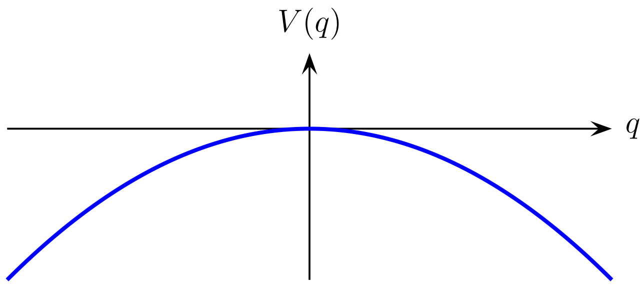
must include all the momentum modes $\mathbf{k}$, *including the modes with* $|\mathbf{k}| < \mu$. Thus, the Hamiltonian (4) is a sum over all the momentum modes, including the modes with

$$\omega^2_{\mathbf{k}} = \mathbf{k}^2 - \mu^2 < 0. \tag{6}$$

While the modes with $\omega^2_{\mathbf{k}} > 0$ act as harmonic oscillators, the modes with $\omega^2_{\mathbf{k}} < 0$ act as a particle on a potential hill rather than in a potential well. Indeed, a single-mode Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}(\text{negative } \omega^2) \times \hat{q}^2 \tag{7}$$

describes the particle in the potential



$$V(q) \;=\; \tfrac{1}{2}(\text{negative } \omega^2) \times q^2. \tag{8}$$

Classically, a particle in this potential runs away to infinity instead of oscillating,

$$q(t) \;\sim\; q_0 \times \exp(+\sqrt{-\omega^2} \times t) \;\to\; \pm\infty \quad \text{for } t \to \infty. \tag{9}$$

The quantum version of this runaway is the Hamiltonian $\hat{H}$ being unbounded from below and hence not having a ground state. Also, the expectation values of powers of $\hat{q}$ exponentially increase with time, in particular

$$\left\langle \hat{q}^2 \right\rangle(t) \;\sim\; \text{const} \times \exp(+2\sqrt{-\omega^2} \times t) \;\to\; \pm\infty \quad \text{for } t \to \infty, \tag{10}$$

and for a generic initial state we also have

$$\langle \hat{q} \rangle(t) \;\sim\; \text{const} \times \exp(+\sqrt{-\omega^2} \times t) \;\to\; \pm\infty \quad \text{for } t \to \infty, \tag{11}$$

In the QFT of the tachyon field, we have similar runaways for all the modes with $|\mathbf{k}| < \mu$,

$$\left\langle \hat{\Phi}_{\mathbf{k}} \right\rangle \;\propto\; \exp(+\sqrt{\mu^2 - \mathbf{k}^2} \times t) \quad \text{for } t \to \infty. \tag{12}$$

Note that the expectation value of the $\mathbf{k} = 0$ mode runs away faster than all the other $|\mathbf{k}| < \mu$ modes, so at late times

$$\left\langle \hat{\Phi}(\mathbf{x}, t) \right\rangle \text{ runs to } \infty \text{ as } t \to \infty \text{ approximately uniformly in } \mathbf{x}. \tag{13}$$

Thus, *in QFT a would-be tachyon field does not have a stable vacuum while the field's expectation value runs away to infinity.*

Self–Interacting Tachyon Field

Now consider a self-interacting tachyon field $\Phi(x)$ with the classical Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\nu\Phi\partial^\nu\Phi \;-\; V(\Phi) \tag{14}$$

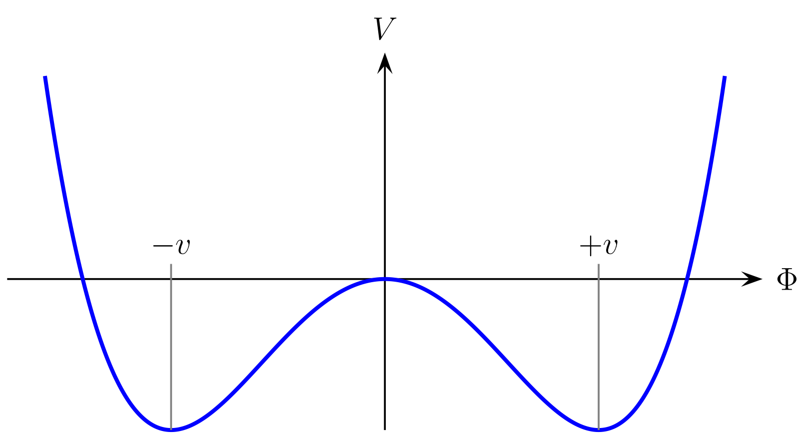
where

$$V(\Phi) \;=\; -\frac{\mu^2}{2}\Phi^2 \;+\; \frac{\lambda}{4}\Phi^4 \tag{15}$$

for a negative $m^2 = -\mu^2$ but positive λ (so that the potential V is bounded from below). This potential — which we may rewrite as

$$V(\Phi) \;=\; \text{const} \;+\; \frac{\lambda}{4}\bigl(\Phi^2 - v^2\bigr)^2 \quad \text{for} \quad v^2 \;=\; \frac{\mu^2}{\lambda} \;>\; 0, \tag{16}$$

— has a double-well form with a local maximum at $\Phi = 0$ and two local minima at $\phi = \pm v$,



(17)

A particle in a similarly shaped potential $V(q)$ would roll off the hill at $q = 0$ towards a minimum — either $q = +v$ or $q = -v$ depending on the initial condition, — then oscillate around that minimum. Assuming some kind of friction, the oscillations would diminish in amplitude, and eventually the particle would settle at the minimum.

Likewise for the classical field $\Phi(\mathbf{x}, t)$ in the potential (16): The $\Phi(\mathbf{x}) \equiv 0$ configuration is unstable, and a tiniest fluctuation would make the field roll away from the local maximum of V. But instead of rolling away to infinity, the field value would roll to a minimum — either $\Phi = +v$ or $\Phi = -v$, — and then oscillate around that minimum. The self interactions of the oscillating fields would lead to emission of quanta, which would take the energy away from the oscillations, hence the oscillations would be dumped down, and eventually the field configuration would settle at a ground state with either $\Phi(\mathbf{x}) \equiv +v$ or $\Phi(\mathbf{x}) \equiv -v$.

Similar to the classical theory, the ground state of the quantum theory with the Hamiltonian

$$\hat{H} = \int d^3\mathbf{x} \Big(\tfrac{1}{2}\hat{\Pi}^2 + \tfrac{1}{2}(\nabla\hat{\Phi})^2 + V(\hat{\Phi}) \Big) \tag{18}$$

also seeks the minimum of the effective potential

$$V_{\text{eff}}(\Phi) = V_{\text{classical}}(\Phi) + \text{quantum corrections.} \tag{19}$$

That is, the expectation value of the field in the ground state is the minimum of the effective potential,

$$\langle\text{ground}|\,\hat{\Phi}(\mathbf{x})\,|\text{ground}\rangle = \Phi_{\min}\,, \quad \text{same } \forall\mathbf{x}. \tag{20}$$

In a relativistic QFT, the ground state is the physical vacuum, so if the minimum of $V_{\text{eff}}(\Phi)$ lies at $\Phi \neq 0$, then the field has a non-zero Vacuum Expectation Value (VEV),

$$\langle\Phi\rangle \stackrel{\text{def}}{=} \langle\text{vac}|\,\hat{\Phi}\,|\text{vac}\rangle = \Phi_{\min} \neq 0. \tag{21}$$

Note: if the effective potential has more than one minimum, then the QFT has multiple ground states. For example, the tachyonic theory at hand has $V(\Phi)$ with two degenerate minima at $\Phi = \pm v$, so the quantum theory has two distinct vacua, one with $\langle\Phi\rangle = +v$ and the other with $\langle\Phi\rangle = -v$. Such multiple physical vacua lead to *spontaneous symmetry breaking*, but let me postpone the explanation of *that* subject until November.

A non-zero VEV of a scalar field may drastically change the particle spectrum of the quantum theory. Indeed, before we expand the quantum fields into the creation and the annihilation operators, let us *shift the scalar field by its VEV*,

$$\hat{\Phi}(x) = \langle\Phi\rangle + \delta\hat{\Phi}(x). \tag{22}$$

(Note that $\langle\Phi\rangle$ is the same for all x.) Now let's expand $V(\Phi)$ into powers of the $\delta\Phi$:

$$V(\Phi) \;=\; V(\langle\Phi\rangle) \;+\; \frac{dV}{d\Phi}\bigg|_{\langle\Phi\rangle} \times \delta\Phi \;+\; \frac{1}{2}\,\frac{d^2V}{d\Phi^2}\bigg|_{\langle\Phi\rangle} \times \delta\Phi^2 \;+\; \frac{1}{6}\,\frac{d^3V}{d\Phi^3}\bigg|_{\langle\Phi\rangle} \times \delta\Phi^3 \;+\; \cdots. \tag{23}$$

Since we are expanding around the minimum of V, we have

$$\frac{dV}{d\Phi}\bigg|_{\langle\Phi\rangle} \;=\; 0 \quad\text{while}\quad \frac{d^2V}{d\Phi^2}\bigg|_{\langle\Phi\rangle} \;>\; 0, \tag{24}$$

hence

$$\begin{aligned} V(\delta\Phi) \;&=\; \text{const} \;+\; (\text{positive}) \times \tfrac{1}{2}\delta\Phi^2 \;+\; (\text{something}) \times \tfrac{1}{6}\delta\Phi^3 \;+\; \cdots \\ &=\; \text{const} \;+\; \frac{M_{\text{eff}}^2}{2} \times \delta\Phi^2 \;+\; V_{\text{int}}(\delta\Phi) \end{aligned} \tag{25}$$

where $M_{\text{eff}}^2 > 0$ and $V_{\text{eff}}(\delta\Phi)$ comprises terms of third and higher powers in $\delta\Phi$.

Consequently, in the quantum theory we have

$$\begin{aligned} \hat{H} \;&=\; \int d^3\mathbf{x}\Big(\tfrac{1}{2}\hat{\Pi}^2 \;+\; \tfrac{1}{2}(\nabla\hat{\Phi})^2 \;+\; V(\hat{\Phi})\Big) \\ &=\; \int d^3\mathbf{x}\Big(\tfrac{1}{2}\hat{\Pi}^2 \;+\; \tfrac{1}{2}(\nabla\delta\hat{\Phi})^2 \;+\; V(\delta\hat{\Phi})\Big) \\ &=\; \text{const} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{int}} \end{aligned} \tag{26}$$

where

$$\hat{H}_{\text{free}} \;=\; \int d^3\mathbf{x}\Big(\tfrac{1}{2}\hat{\Pi}^2 \;+\; \tfrac{1}{2}(\nabla\delta\hat{\Phi})^2 \;+\; \tfrac{1}{2}M_{\text{eff}}^2\delta\Phi^2\Big), \tag{27}$$

$$\hat{H}_{\text{int}} \;=\; \int d^3\mathbf{x}\, V_{\text{int}}(\delta\Phi). \tag{28}$$

Physically, the free part (27) of this Hamiltonian describes a field whose quanta have a positive mass^2

$$M_{\text{eff}}^2 \;=\; \frac{d^2V}{d\Phi^2}\bigg|_{\langle\Phi\rangle} \;>\; 0. \tag{29}$$

Indeed, we may expand the shifted field $\delta\hat{\Phi}(x)$ and its canonical conjugate $\hat{\Phi}(x)$ (which does not need any shifting) into the creation and annihilation operators in the usual way

$$\delta\hat{\Phi}(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\Big(e^{-ikx}\hat{a}_{\mathbf{k}} \;+\; e^{+ikx}\hat{a}_{\mathbf{k}}^{\dagger}\Big) \tag{30}$$

for

$$k^0 \;=\; +\omega_{\mathbf{k}} \;=\; +\sqrt{M_{\text{eff}}^2+\mathbf{k}^2} \tag{31}$$

while the free Hamiltonian becomes

$$\hat{H}_{\text{free}} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\omega_{\mathbf{k}}\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}. \tag{32}$$

Note that since $M_{\text{eff}}^2>0$, there are no tachyons!

General Rule: Quantum field theories with stable vacua do not have tachyons. If your theory *seems* to have a tachyon, you are expanding around a wrong vacuum state!

For example, let's go back the theory of a scalar field with the potential

$$\begin{aligned} V(\Phi) \;&=\; -\frac{\mu^2}{2}\Phi^2 \;+\; \frac{\lambda}{4}\Phi^4 \quad \text{with } m^2=-\mu^2<0 \text{ but } \lambda>0 \\ &=\; \frac{\lambda}{4}\big(\Phi^2-v^2\big)^2 \;+\; \text{const} \quad \text{for } v^2=\mu^2/\lambda. \end{aligned} \tag{33}$$

Expanding around $\Phi=0$ — which is a maximum of the potential — leads to a tachyon with mass$^2=-\mu^2<0$. But that's a wrong would-be vacuum state to expand around! Instead, we should expand around a minimum of $V(\Phi)$ at $\Phi=\pm v$. Thus, let

$$\Phi(x) \;=\; \pm v \;+\; \delta\Phi(x), \tag{34}$$

hence

$$\Phi^2 \;-\; v^2 \;=\; \pm 2v\times\delta\Phi \;+\; \delta\Phi^2, \tag{35}$$

$$\begin{aligned} V \;&=\; \frac{\lambda}{4}\big(\pm 2v\delta\Phi+\delta\Phi^2\big)^2 \qquad \langle\!\langle \text{ up to a constant } \rangle\!\rangle \\ &=\; \lambda v^2\times\delta\Phi^2 \;+\; \lambda v\times\delta\Phi^3 \;+\; \frac{\lambda}{4}\times\delta\Phi^4, \end{aligned} \tag{36}$$

where the quadratic terms gives us the particles' mass^2:

$$M_{\text{eff}}^2 \;=\; 2\lambda v^2 \;=\; +2\mu^2 \;>\; 0. \tag{37}$$

Thus, expanding around the right vacuum state yields particles of positive mass^2 rather than tachyons.