ABRAHAM–LORENTZ RADIATION REACTION FORCE

When the motion of a charged point particle involves acceleration, the particle radiates EM energy at the rate

$$P = \frac{q^2\mu_0}{6\pi c}\,\mathbf{a}^2 \tag{1}$$

(assuming motion at speeds $v \ll c$). This EM energy comes at the expense of the particle's own energy, so the particle feels an effective *radiation reaction force* $\mathbf{F}_{\text{rad}}$ that tries to slow it down. To find this force, we start with the work-energy theorem for the particle:

$$\Delta U = -\int P\,dt = -\frac{q^2\mu_0}{6\pi c}\int \mathbf{a}^2\,dt$$

where

$$\begin{aligned} \mathbf{a}^2\,dt &= \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\,dt = \mathbf{a}\cdot d\mathbf{v} = d(\mathbf{a}\cdot\mathbf{v}) - (d\mathbf{a})\cdot\mathbf{v} \\ &= d(\mathbf{a}\cdot\mathbf{v}) - (\dot{\mathbf{a}}\,dt)\cdot\mathbf{v} = d(\mathbf{a}\cdot\mathbf{v}) - \dot{\mathbf{a}}\cdot(\mathbf{v}\,dt) = d(\mathbf{a}\cdot\mathbf{v}) - \dot{\mathbf{a}}\cdot d\mathbf{r}, \end{aligned} \tag{2}$$

hence

$$\Delta U = -\frac{q^2\mu_0}{6\pi c}\,\Delta(\mathbf{a}\cdot\mathbf{v}) + \frac{q^2\mu_0}{6\pi c}\int \dot{\mathbf{a}}\cdot d\mathbf{r}. \tag{3}$$

For a periodic motion of the particle, $\Delta(\mathbf{a}\cdot\mathbf{v}) = 0$ when we integrate over the period of motion. And when we integrate over a long time — even if this time is not a multiple of the period — we may neglect the $\Delta(\mathbf{a}\cdot\mathbf{v})$ term because it does not grow with time but oscillates with a small amplitude. Consequently, we get

$$\Delta U \approx \frac{q^2\mu_0}{6\pi c}\int \dot{\mathbf{a}}\cdot d\mathbf{r}, \tag{4}$$

which has the form of

$$\Delta U = \int \mathbf{F}_{\text{rad}}\cdot d\mathbf{r} = \int (\mathbf{F}_{\text{rad}}\cdot\mathbf{v})dt \tag{5}$$

for

$$\mathbf{F}_{\text{rad}} = \frac{q^2\mu_0}{6\pi c}\,\dot{\mathbf{a}}. \tag{6}$$

This is *the Abraham–Lorentz formula for the radiation reaction force.* By the way, the $\dot{\mathbf{a}}$ in this formula — the time derivative of the particle's acceleration, *i.e.* the third time derivative $\dddot{\mathbf{r}}$ of its position — is called the *jerk*.

Note: we have derived the Abraham–Lorentz formula under the assumption of a periodic motion, so when we try to apply it to a non-periodic motion, we get all kinds of un-physical results. For example, consider the motion of a free electron — which is subject to no external forces but only to the Abraham–Lorentz force that is always there. In this case, the Second Law of Newton becomes

$$m_e\mathbf{a} \;=\; \mathbf{F}_{\text{rad}} \;=\; +\frac{e^2\mu_0}{6\pi c}\,\dot{\mathbf{a}}, \tag{7}$$

hence

$$\frac{d\mathbf{a}}{dt} \;=\; +\frac{\mathbf{a}(t)}{\tau} \tag{8}$$

where

$$\tau \;=\; \frac{e^2\mu_0}{6\pi c m_e} \;\approx\; 6.27\cdot 10^{-24}\ \text{s}. \tag{9}$$

Note the + sign in eq. (8), so its general solution is a runaway acceleration

$$\mathbf{a}(t) \;=\; \mathbf{a}_0\exp(+t/\tau) \tag{10}$$

on a very short time scale τ. Clearly, this is an un-physical behavior, so we must set $\mathbf{a}_0 = 0$ by hand.

Similar problem with exponential runaway solutions plague other physical systems, for example an electron in a harmonic potential, *cf.* homework set#9, problem 4. The harmonic potential provides for an external force $F_{\text{ext}}(x) = -m\omega_0^2 \times x$, hence Newton equation

$$ma \;=\; F_{\text{ext}} \;+\; F_{\text{rad}} \;=\; -m\omega_0^2 \times x \;+\; m\tau \times \dot{a}. \tag{11}$$

or in terms of $x(t)$ and its explicit time derivatives,

$$\tau\,\dddot{x} \;-\; \ddot{x} \;-\; \omega_0^2\,x \;=\; 0. \tag{12}$$

This is a third-order linear differential equation, so it has 3 independent solutions. In your homework, you should see that one of the solutions is an exponential runaway while the other two solutions describe dumped oscillations. Physically, only the damped oscillations make sense for this system, so one has to impose a constraint on the solutions of eq. (12) to avoid the unphysical exponential runaway solution.

Worse, sometime avoiding the exponential runaway leads to other un-physical effects. For example, suppose the external force on an electron amounts to two very brief impulses in opposite directions, one at time $t = 0$ and the other at a later time $t = T$:

$$F_{\text{ext}} \;=\; I\delta(t) \;-\; I\delta(t-T). \tag{13}$$

Plugging this external force into the Newton equation, we get

$$\tau\dot{a} \;-\; a \;=\; -\frac{F_{\text{ext}}(t)}{m} \;=\; \frac{I}{m}\Big(-\delta(t)+\delta(t-T)\Big). \tag{14}$$

Since the RHS here vanishes at all times except $t = 0$ and $t = T$, the general solution has form

$$a(t) \;=\; \exp(+t/\tau)\times\begin{cases} A & \text{for } t<0,\\ B & \text{for } 0<t<T,\\ C & \text{for } t>T,\end{cases} \tag{15}$$

for some constant coefficients A, B, C related by the boundary conditions at $t = 0$ and $t = T$. Indeed, integrating eq. (14) over a very short time interval from $t = -\epsilon$ to $t = +\epsilon$, we get

$$\tau\Delta a \;-\; \Delta v \;=\; -\frac{I}{m}\,. \tag{16}$$

The finite Δa here means $a(t)$ remains finite (albeit discontinuous), hence $v(t)$ is continuous and $\Delta v = 0$. Thus, at $t = 0$ $v(t)$ is continuous while $a(t)$ jumps by $-I/m$; in terms of the coefficients A, B, C this means

$$B \;-\; A \;=\; -\frac{I}{m}\,. \tag{17}$$

Likewise, at $t = T$ the velocity is continuous while $a(t)$ jumps by $+I/m$; in terms of the A, B, C this means

$$Ce^{T/\tau} \;-\; Be^{T/\tau} \;=\; +\frac{I}{A}\,. \tag{18}$$

Altogether, the three coefficients A, B, C are related by two constraints, which leaves us with one free parameter, and we should choose it so as to keep the solution physical. In particular, we want to avoid the rapid exponential runaway at late times $t > T$, which calls for $C = 0$. We also want to keep the solution causal, which forbids pre-acceleration — the acceleration starting before any external force has acted on the particle, — so we need $A = 0$. But alas, having both $A = 0$ and $C = 0$ is incompatible with the boundary conditions (18) and (17): indeed

$$\begin{aligned} A &= 0 \quad\Longrightarrow\quad B = -\frac{I}{m}\,, \\ \text{while} C &= 0 \quad\Longrightarrow\quad B = -\frac{I}{m}\times e^{-T/\tau} \neq -\frac{I}{m}\,. \end{aligned} \tag{19}$$

Thus, any particular solution of the equation of motion (14) may avoid pre-acceleration or exponential runaway but not both, which means there are no completely physical solutions at all!

A good way to avoid all these problems with the Abraham–Lorentz formula is to treat the radiation reaction force as a small perturbation of the external force. This way, we first calculate the acceleration to the zeroth order from the external force alone,

$$m\mathbf{a}(t) = \mathbf{F}_{\text{ext}}(t), \tag{20}$$

and then plug this zeroth-order acceleration into the Abraham–Lorentz force

$$\mathbf{F}_{\text{rad}} = \frac{q^2\mu_0}{6\pi c}\,\frac{d}{dt}\,\frac{\mathbf{F}_{\text{ext}}(t)}{m} = \tau\,\frac{d\mathbf{F}_{\text{ext}}}{dt}\,. \tag{21}$$

Consequently, the Newton equation becomes

$$m\mathbf{a}(t) = \mathbf{F}_{\text{ext}}(t) + \tau\,\frac{d\mathbf{F}_{\text{ext}}}{dt}\,. \tag{22}$$

From the $\mathbf{r}(t)$ point of view, this is a second-order — rather than a third order — differential equation, and it does not have the spurious unphysical solutions. Indeed, you should see in your homework set#9 (problem 5) that the solutions of eq. (22) have neither pre-acceleration nor exponential runaways.

Radiation Reaction Force as Internal Force

To understand the physical origin of the Abraham–Lorentz force, let's replace a point charge with a particle of a very small but finite size and consider the internal forces between different parts of that particle. As you saw in homework set#7 (problem 5), the electrodynamic forces between two moving charges do not obey Newton's Third Law because some momentum is transferred from the two particles to the EM fields or vice verse, hence

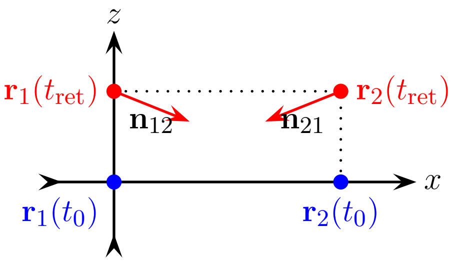
$$\mathbf{F}_{12}^{\rm net}\ =\ \mathbf{F}_{1\,\rm on\,2}\ +\ \mathbf{F}_{2\,\rm on\,1}\ \neq\ 0. \tag{23}$$

Let me show you that it is this net internal force between different parts of the charged particle that lead to the Abraham–Lorentz radiation reaction force.

The complete analysis of the net internal force on a 3D particle is done in §16.5 of the *Jackson's textbook*, but it's too complicated for this undergraduate class. Instead, let's work with a simplified toy model of the particle: just two point charges, q_1 and q_2 (such that $q_1 + q_2 = q$) separated by a short distance ℓ in the $\hat{\mathbf{x}}$ direction. The two charges move in lockstep, and for simplicity let me assume that they move only in the $\hat{\mathbf{z}}$ direction, $\perp$ to the separation between the charges, thus

$$\mathbf{r}_1(t)\ =\ z(t)\hat{\mathbf{z}}\quad\text{while}\quad \mathbf{r}_2(t)\ =\ z(t)\hat{\mathbf{z}}\ +\ \ell\hat{\mathbf{x}}. \tag{24}$$

Let $z(t_0) = 0$ at some *observer time* t_0, so the locations of the two charges are indicated by the blue discs on the diagram below:



(25)

However, the two point charges do not see each other at their current locations. Instead, they see each other where they were at the retarded time $t_{\rm ret} = t_0 - \mathcal{R}/c$, as shown in red at the above diagram: same values of x, but a different value of $z(t_{\rm ret}) \neq z(t_0)$. It is this apparent displacement that causes $\mathbf{F}_{12}^{\rm net} \neq 0$; indeed, at the level of the naive Coulomb forces,

$$\mathbf{F}_{12}^{\rm net} \;=\; \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{\mathbf{n}_{12}}{\mathcal{R}^2}+\frac{\mathbf{n}_{21}}{\mathcal{R}^2}\right) \;=\; \frac{q_1 q_2}{4\pi\epsilon_0} * \frac{2(z(t_0)-z(t_{\rm ret}))\hat{\mathbf{z}}}{\mathcal{R}^3} \;\neq\; 0. \tag{26}$$

However, to properly account for the forces on each charge, we need the generalized Coulomb fields, the acceleration fields, and the magnetic fields, so the actual formula for the net force is a lot more complicated. Nevertheless, the $x \to \ell - x$ symmetry of the system tells us that the forces in $\hat{\mathbf{x}}$ direction are going to cancel out between the $F_{1\,\rm on\,2}$ and $F_{2\,\rm on\,1}$, so let's focus on just the $\hat{\mathbf{z}}$ components of the forces.

For simplicity, let's assume that at the time t_0 the velocity dz/dt of the charges happen to vanish. Consequently, at that moment there are no magnetic forces on the charges but only the electric forces. In particular,

$$\mathbf{F}_{1\,\rm on\,2} \;=\; q_2\mathbf{E}_1(\mathbf{r}_2) \;=\; \frac{q_2 q_1}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret}, \tag{27}$$

and our next task is to evaluate the expression in the big brackets — or rather, its $\hat{\mathbf{z}}$ component — as a power series in $t - t_{\rm ret} = \mathcal{R}/c$. First of all, since the particle moves only in the $\hat{\mathbf{z}}$ direction, we have $\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$, hence

$$\Big[\cdots\Big]^z \;=\; \left[\frac{(1-\beta^2)(n_z-\beta)}{\mathcal{R}^2(1-\beta n_z)^3} - \frac{(1-n_z^2)a}{c^2\mathcal{R}(1-\beta n_z)^3}\right]_{\rm ret}. \tag{28}$$

Second, let's expand $z(t)$ in the power series in $(t - t_0)$. Since at $t = t_0$ the particle is at $z_0 = 0$ and has zero velocity, the expansion starts with the acceleration term:

$$z(t) \;=\; \frac{a(t_0)}{2}\,(t-t_0)^2 \;+\; \frac{\dot{a}(t_0)}{6}\,(t-t_0)^3 \;+\; \frac{\ddot{a}(t_0)}{24}\,(t-t_0)^4 \;+\; \cdots, \tag{29}$$

hence at the retarded time $t_{\rm ret} = t_0 - \mathcal{R}/c$,

$$z(t_{\rm ret}) \;=\; \frac{a(t_0)}{2}\,(\mathcal{R}/c)^2 - \frac{\dot{a}(t_0)}{6}\,(\mathcal{R}/c)^3 + \frac{\ddot{a}(t_0)}{24}\,(\mathcal{R}/c)^4 \;+\; \cdots. \tag{30}$$

Likewise, the velocity and the acceleration at the retarded time are

$$\begin{aligned}
\beta(t_{\rm ret}) \;&=\; \frac{v(t_{\rm ret})}{c} \;=\; \frac{a(t_0)}{c}\,(t_{\rm ret}-t_0) \;+\; \frac{\dot{a}(t_0)}{2c}\,(t_{\rm ret}-t_0)^2 \;+\; \frac{\ddot{a}(t_0)}{6c}\,(t-t_0)^3 \;+\; \cdots \\
&\;=\; -\frac{a(t_0)\mathcal{R}}{c^2} + \frac{\dot{a}(t_0)\mathcal{R}^2}{2c^3} - \frac{\ddot{a}(t_0)\mathcal{R}^3}{6c^4} \;+\; \cdots, \\
\frac{a(t_{\rm ret})}{c^2} \;&=\; \frac{a(t_0)}{c^2} \;+\; \frac{\dot{a}(t_0)}{c^2}\,(t_{\rm ret}-t_0) \;+\; \frac{\ddot{a}(t_0)}{2c^2}\,(t_{\rm ret}-t_0)^2 \;+\; \cdots \\
&\;=\; \frac{a(t_0)}{c^2} - \frac{\dot{a}(t_0)\mathcal{R}}{c^3} + \frac{\ddot{a}(t_0)\mathcal{R}^2}{2c^4} \;+\; \cdots.
\end{aligned} \tag{31}$$

Next,

$$\begin{aligned}
n_z(t_{\rm ret}) \;&=\; \frac{z(t_0)-z(t_{\rm ret})}{\mathcal{R}} \;=\; -\frac{z(t_{\rm ret})}{\mathcal{R}} \\
&=\; -\frac{a(t_0)\mathcal{R}}{2c^2} \;+\; \frac{\dot{a}(t_0)\mathcal{R}^2}{6c^3} \;-\; \frac{\ddot{a}(t_0)\mathcal{R}^3}{24c^4} \;+\; O(\mathcal{R}^4)
\end{aligned} \tag{32}$$

and consequently

$$\left[\frac{n_z-\beta}{\mathcal{R}^2}\right]_{\rm ret} \;=\; +\frac{a(t_0)}{2c^2\mathcal{R}} \;-\; \frac{\dot{a}(t_0)}{3c^3} \;+\; \frac{\ddot{a}(t_0)\mathcal{R}}{8c^4} \;+\; O(\mathcal{R}^2) \tag{33}$$

while

$$\left[\frac{-a}{c^2\mathcal{R}}\right]_{\rm ret} \;=\; -\frac{a(t_0)}{c^2\mathcal{R}} \;+\; \frac{\dot{a}(t_0)}{c^3} \;-\; \frac{\ddot{a}(t_0)\mathcal{R}}{2c^4} \;+\; O(\mathcal{R}^2). \tag{34}$$

At the same time,

$$\begin{aligned}
(1-\beta^2)_{\rm ret} \;&=\; 1 \;-\; \frac{a^2(t_0)\mathcal{R}^2}{c^4} \;+\; O(\mathcal{R}^3), \\
(1-n_z^2)_{\rm ret} \;&=\; 1 \;-\; \frac{a^2(t_0)\mathcal{R}^2}{4c^4} \;+\; O(\mathcal{R}^3), \\
(1-\beta n_z)_{\rm ret}^{-3} \;&=\; 1 \;+\; \frac{3a^2(t_0)\mathcal{R}^2}{2c^4} \;+\; O(\mathcal{R}^3),
\end{aligned} \tag{35}$$

hence

$$\left[\frac{(1-\beta^2)}{(1-\beta n_z)^3}\times\frac{(n_z-\beta)}{\mathcal{R}^2}\right]_{\rm ret} \;=\; +\frac{a(t_0)}{2c^2\mathcal{R}} \;-\; \frac{\dot{a}(t_0)}{3c^3} \;+\; \left(\frac{\ddot{a}(t_0)}{8c^4} \;+\; \frac{a^3(t_0)}{4c^6}\right)\times\mathcal{R} \;+\; O(\mathcal{R}^2), \tag{36}$$

$$\left[\frac{(1-n_z^2)}{(1-\beta n_z)^3}\times\frac{-a}{c^2\mathcal{R}}\right]_{\rm ret} = -\frac{a(t_0)}{c^2\mathcal{R}} + \frac{\dot{a}(t_0)}{c^3} - \left(\frac{\ddot{a}(t_0)}{2c^4} + \frac{5a^3(t_0)}{4c^6}\right)\times\mathcal{R} + O(\mathcal{R}^2), \quad (37)$$

$$\text{altogether} = -\frac{a(t_0)}{2c^2\mathcal{R}} + \frac{2\dot{a}(t_0)}{3c^3} - \left(\frac{3\ddot{a}(t_0)}{8c^4} + \frac{a^3(t_0)}{c^6}\right)\times\mathcal{R} + O(\mathcal{R}^2), \quad (38)$$

and therefore

$$F^z_{1\,\rm on\,2} = \frac{q_2q_1}{4\pi\epsilon_0}\left[-\frac{a(t_0)}{2c^2\mathcal{R}} + \frac{2\dot{a}(t_0)}{3c^3} - \left(\frac{3\ddot{a}(t_0)}{8c^4} + \frac{a^3(t_0)}{c^6}\right)\times\mathcal{R} + O(\mathcal{R}^2)\right]. \quad (39)$$

Now let's re-express this force in terms of the simultaneous distance ℓ between the charges instead of the retarded distance $\mathcal{R} = \sqrt{\ell^2 + z^2(t_{\rm ret})}$. In terms of the retarded-time $\mathcal{R}$ and $\mathbf{n}$,

$$\ell = \mathcal{R}\times|n_x| = \mathcal{R}\times\sqrt{1-n_z^2} = \mathcal{R} - \frac{a^2(t_0)}{8c^4}\times\mathcal{R}^3 + O(\mathcal{R}^4), \quad (40)$$

so inverting this power series we get

$$\mathcal{R} = \ell + \frac{a^2(t_0)}{8c^4}\times\ell^3 + O(\ell^4). \quad (41)$$

Plugging this expansion into eq. (39), we arrive at

$$F^z_{1\,\rm on\,2} = \frac{q_2q_1}{4\pi\epsilon_0}\left[-\frac{a}{2c^2}\times\frac{1}{\ell} + \frac{2\dot{a}}{3c^3} - \left(\frac{3\ddot{a}}{8c^4} + \frac{15a^3}{16c^6}\right)\times\ell + O(\ell^2)\right] \quad (42)$$

where the acceleration and all its time derivatives are evaluated at the observer time t_0 rather than the retarded time. Finally, by the symmetry between the two charges,

$$F^z_{2\,\rm on\,1} = F^z_{1\,\rm on\,2}\,, \quad (43)$$

so the net internal force on the two-charge system is

$$F^z_{\rm net} = 2\times F^z_{1\,\rm on\,2} = -\frac{q_1q_2}{4\pi\epsilon_0}\times\frac{a}{\ell c^2} + \frac{q_1q_2}{3\pi\epsilon_0}\times\frac{\dot{a}}{c^3} - \frac{q_1q_2}{32\pi\epsilon_0}\times\left(\frac{6\ddot{a}}{c^4} + \frac{15a^3}{c^6}\right)\times\ell + O(\ell^2). \quad (44)$$

Now let's take a closer look at the leading term in this power series,

$$-\frac{q_1 q_2}{4\pi\epsilon_0}\times\frac{a}{\ell c^2} \;=\; -\frac{a}{c^2}\times\frac{q_1 q_2}{4\pi\epsilon_0 \ell} \;=\; -\frac{a}{c^2}\times U_C \tag{45}$$

where

$$U_C \;=\; \frac{q_1 q_2}{4\pi\epsilon_0 \ell} \tag{46}$$

is the electrostatic potential energy of the two charges. Consequently, we may rewrite the Newton equation for the force (44) as

$$(m_1+m_2)a \;=\; F_{\text{net}} \;=\; -\frac{a}{c^2}\times U_C \;+\; \text{other terms} \tag{47}$$

as

$$\left(m_1 \;+\; m_2 \;+\; \frac{U_C}{c^2}\right)\times a \;=\; \text{other terms.} \tag{48}$$

In this context, the leading term in eq. (44) acts not as a force but rather as a *correction to the net inertial mass of the particle*,

$$m_{\text{net}} \;=\; m_1 \;+\; m_2 \;+\; \frac{U_C}{c^2}\,. \tag{49}$$

This is a basic relativistic effect: the energy affects the inertial mass according to the Einstein's famous formula $E = mc^2$. I shall explain it in detail sometimes in April. (For the impatient, here are my notes on the subject for the graduate class I taught back in 2019).

Apart from the mass-correcting leading terms in the expansion (44), the remaining terms amount to the true internal force

$$F_{\text{int}} \;=\; \frac{q_1 q_2}{3\pi\epsilon_0}\times\frac{\dot{a}}{c^3} \;+\; O(\ell). \tag{50}$$

So in the limit of a very compact particle of width $\ell \to 0$, we end up with the net internal force proportional to the jerk $\dot{a}$, specifically

$$F_{\text{int}} = \frac{q_1 q_2}{3\pi\epsilon_0 c^3} \times \dot{a} = \frac{q_1 q_2 \mu_0}{3\pi c} \times \dot{a}. \tag{51}$$

This force looks just like the Abraham–Lorentz radiation reaction force

$$F_{\text{AL}} = \frac{q^2 \mu_0}{6\pi c} \times \dot{a}, \tag{52}$$

except for the overall coefficient, which depends on how we split the net charge q into two point charges $q_1 + q_2$. In particular, for the even split $q_1 = q_2 = \frac{1}{2}q$ we get

$$F_{\text{int}} = \frac{q^2 \mu_0}{12\pi c} \times \dot{a} = \frac{1}{2} F_{\text{AL}}\,. \tag{53}$$

So here is the bottom line: *in our toy model of two point charges, the internal forces between these 2 point charges explain $\frac{1}{2}$ of the Abraham–Lorentz radiation reaction force.*

To explain the other half of the Abraham–Lorentz force, we need to split each of the two point charges into more parts and look at the internal forces between those parts. For example, we could split each of q_1 and q_2 into halves, then each half into two quarters, *etc.*, *etc.*, and then sum up the resulting series in the internal forces between all these fragments. But instead, let me go all the way from a few discrete point charges to a continuous line charge. That is, replace the original particle with a 1D bar of length L and uniform charge density $\lambda = Q/L$ stretching along the x axis, $\perp$ to the particle's motion in $\hat{\mathbf{z}}$ direction. Then, the net internal force between all pairs of infinitesimal stretches of this bar is

$$F^z_{\text{net}} = \int_0^L dx_1 \int_0^L dx_2\, F^z_{12}(q_1 = \lambda dx_1; q_2 = \lambda dx_2; \ell = |x_1 - x_2|) \tag{54}$$

where $F^z_{12}(q_1, q_2, \ell)$ is exactly as in eq. (42). (But without adding the F^z_{21} as in eq. (44) since we already allow for both $x_1 < x_2$ and $x_1 > x_2$.) Thus,

$$F_z^{\text{net}} = \iint_0^L dx_1\, dx_2\, \frac{\lambda^2}{4\pi\epsilon_0} \left[\begin{array}{c} -\dfrac{a}{2c^2} \times \dfrac{1}{|x_1 - x_2|} + \dfrac{2\dot{a}}{3c^3} \\ -\left(\dfrac{3\ddot{a}}{8c^4} + \dfrac{15a^3}{16c^6}\right) \times |x_1 - x_2| + O(|x_1 - x_2|^2) \end{array} \right] \tag{55}$$

$$\begin{aligned}
&= -\frac{a}{c^2}\times\iint\limits_0^L \frac{\lambda^2\,dx_1\,dx_2}{8\pi\epsilon_0\,|x_1-x_2|}\\
&\quad +\ \frac{\lambda^2\dot{a}}{6\pi\epsilon_0 c^3}\times\iint\limits_0^L dx_1\,dx_2\\
&\quad -\ \frac{\lambda^2}{64\pi\epsilon_0}\times\left(\frac{6\ddot{a}}{c^4}+\frac{15a^3}{c^6}\right)\times\iint\limits_0^L dx_1\,dx_2\,|x_1-x_2|\\
&\quad +\ \lambda^2\times\iint\limits_0^L dx_1\,dx_2\,O(|x_1-x_2|^2)\\
&= -\frac{a}{c^2}\times U_C[\text{bar}]\ +\ \frac{\lambda^2\dot{a}}{6\pi\epsilon_0 c^3}\times L^2\ -\ \frac{\lambda^2}{64\pi\epsilon_0}\times\left(\frac{6\ddot{a}}{c^4}+\frac{15a^3}{c^6}\right)\times\frac{L^3}{3}\ +\ \lambda^2\times O(L^4)\\
F_z^{\text{net}} &= -a\times\frac{U_C[\text{bar}]}{c^2}\ +\ \frac{Q^2\dot{a}}{6\pi\epsilon_0 c^3}\ -\ \frac{Q^2 L}{64\pi\epsilon_0}\left(\frac{2\ddot{a}}{c^4}+\frac{5a^3}{c^6}\right)\ +\ Q^2\times O(L^2). \qquad (55)
\end{aligned}$$

Similar to the 2-point-charges model, the first term of this series amounts to the relativistic correction to the bar's inertial mass,

$$m[\text{bar}]\ =\ m_{\text{naive}}[\text{bar}]\ +\ \frac{U_C[\text{bar}]}{c^2}\,, \qquad (56)$$

while the remaining terms give the radiation reaction force,

$$\begin{aligned}
F^z_{\text{rad}} &= \frac{Q^2\dot{a}}{6\pi\epsilon_0 c^3}\ -\ \frac{Q^2 L}{64\pi\epsilon_0}\left(\frac{2\ddot{a}}{c^4}+\frac{5a^3}{c^6}\right)\ +\ Q^2\times O(L^2).\\
&\xrightarrow[L\to 0]{}\ \frac{Q^2\dot{a}}{6\pi\epsilon_0 c^3}\ =\ \frac{Q^2\mu_0}{6\pi c}\times\dot{a},
\end{aligned} \qquad (57)$$

in perfect agreement with the Abraham–Lorentz formula.