RELATIVISTIC ENERGY AND MOMENTUM

Overview

Non-relativistically, the momentum and the energy of a free particle are related to its velocity $\mathbf{v}$ as

$$\mathbf{p} = m\mathbf{v}, \quad E = \text{const} + \tfrac{1}{2}m\mathbf{v}^2, \tag{1}$$

where m is the particle's mass. In special relativity, the relations are similar for particles moving much slower than light, but for fast particles there are more complicated formulae

$$\mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1-(\mathbf{v}/c)^2}} = m\mathbf{v}\left(1 + \frac{\mathbf{v}^2}{2c^2} + O\left(\frac{v^4}{c^4}\right)\right) \tag{2}$$

and

$$E = \gamma mc^2 = mc^2 + \tfrac{1}{2}m\mathbf{v}^2 + \frac{3mv^4}{8c^2} + O\left(\frac{mv^6}{c^4}\right). \tag{3}$$

The m in these formulae is the *rest mass*, *i.e.* the mass of the particle in its rest frame. Nowadays, when we say *mass* we mean the rest mass, but in the early days of the relativity theory the name mass was commonly used for the *relativistic inertia* or *relativistic mass*

$$\mathcal{M}(v) = \gamma(v) \times m; \tag{4}$$

in terms of this relativistic mass

$$\mathbf{p} = \mathcal{M}(v)\mathbf{v} \quad \text{and} \quad E = \mathcal{M}(v)c^2. \tag{5}$$

In particular, Einstein's famous equation $E = \mathcal{M}c^2$ was written in terms of the relativistic mass $\mathcal{M}(v)$ rather than the rest mass m!

The relativistic energy and momentum form a 4–vector. For a single particle moving with proper velocity u^μ,

$$p^\mu = \left(\frac{E}{c}; p^x, p^y, p^z\right) = mu^\mu, \tag{6}$$

hence

$$(p \cdot p) = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2(u \cdot u) = m^2c^2 \tag{7}$$

regardless of the particle's 3-velocity $\mathbf{v}$. Consequently, the particle's energy as a function of its momentum is

$$E = \sqrt{c^2\mathbf{p}^2 + c^4m^2}\,. \tag{8}$$

For multiple particles, their energy-momenta 4–vectors p^μ add up:

$$P^\mu_{\text{net}} = \sum_{a=1}^{N} p^\mu_a, \tag{9}$$

which in 3D terms means

$$E_{\text{net}} = \sum_{a=1}^{N} E_a\,, \tag{10}$$

$$\mathbf{P}_{\text{net}} = \sum_{a=1}^{N} \mathbf{p}_a\,. \tag{11}$$

For any closed system — that is, the system not interacting with anything else — **these net energy and net momentum are absolutely conserved**. That is, the particles may exchange energy and momentum with each other, may split and rejoin into new particles, but the net energy and the net momentum of the whole system stay constant.

On the other hand, *the net mass of all the particles in the system is not conserved.* Instead, the mass can be converted to or from the kinetic energy and hence to/from other forms of energy. This fact used to be esoteric knowledge of a few physicists, but became painfully known to the whole world in August 1945, when 0.7 grams of uranium, and then 1 gram of plutonium were converted to energy, with devastated consequences for the cities of Hiroshima and Nagasaki.

Note the distinction between an *invariant* quantity — which has the same value in all inertial frames of reference — and a *conserved* quantity — which has the same value before and after some process:

- Mass is invariant but not conserved.
- Energy is conserved but not invariant.
- Electric charge is both conserved and invariant.
- The velocity is neither invariant nor conserved.

Derivation of relativistic momentum and energy.

Before we explore the consequences of the relativistic formulae (2) for the momentum and the energy, let's derive them from the conservation laws. In any collision of particles, the net momentum is conserved,

$$\sum_i^{\text{particles}} \mathbf{p}_i^{\text{after}} = \sum_i^{\text{particles}} \mathbf{p}_i^{\text{before}}, \tag{12}$$

and in an elastic collision the net kinetic energy is also conserved,

$$\sum_i^{\text{particles}} K_i^{\text{after}} = \sum_i^{\text{particles}} K_i^{\text{before}}, \tag{13}$$

Moreover, these conservation laws must work in any inertial frame of reference! Alas, plugging non-relativistic formulae (1) for the particles' energies and momenta into these conservation laws make them invariant under the Galilean boosts $\mathbf{v}' = \mathbf{u} + \mathbf{v}$, but not under the Lorentz boosts which act non-linearly on the velocities.

To repair the law of momentum conservation, we need to change Newton's formula $\mathbf{p} = m\mathbf{v}$ to

$$\mathbf{p} = \mathcal{M}(v)\mathbf{v} \tag{14}$$

with some velocity-dependent inertia $\mathcal{M}(v)$, although by the rotational symmetry of the 3-space it should depend only on the speed $v = |\mathbf{v}|$ but not on the velocity's direction. For the moment, all we know is that $\mathcal{M}$ is some analytic function of $(\mathbf{v}/c)^2$; we shall determine its exact form from the Lorentz invariance of the momentum conservation in collisions.

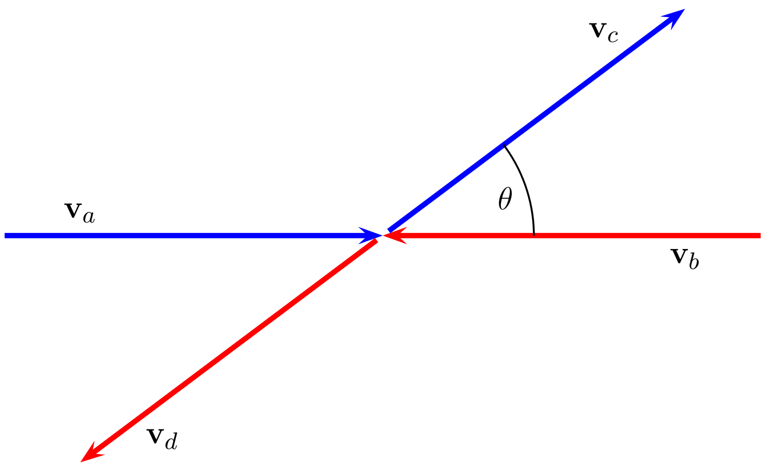
Indeed, consider an elastic collision of two similar particles in the center-of-mass frame. Relativistically, this frame is defined as the frame where the net momentum is zero before the collision and hence also after the collision. Thus, in the CM frame, the two particles collide head-on — so their momenta is equal in magnitude and opposite in direction, — and after the collision they also fly in opposite directions. In terms of velocities,

$$\begin{aligned} \mathcal{M}(v_a)\mathbf{v}_a \;+\; \mathcal{M}(v_b)\mathbf{v}_b \;&=\; 0 \quad \text{before the collision,} \\ \mathcal{M}(v_c)\mathbf{v}_c \;+\; \mathcal{M}(v_d)\mathbf{v}_d \;&=\; 0 \quad \text{after the collision,} \end{aligned} \tag{15}$$

and since $\mathcal{M}(-\mathbf{v}) = \mathcal{M}(+\mathbf{v})$ it follows that

$$\mathbf{v}_b \;=\; -\mathbf{v}_a\,, \quad \mathbf{v}_c \;=\; -\mathbf{v}_d\,, \tag{16}$$

or graphically



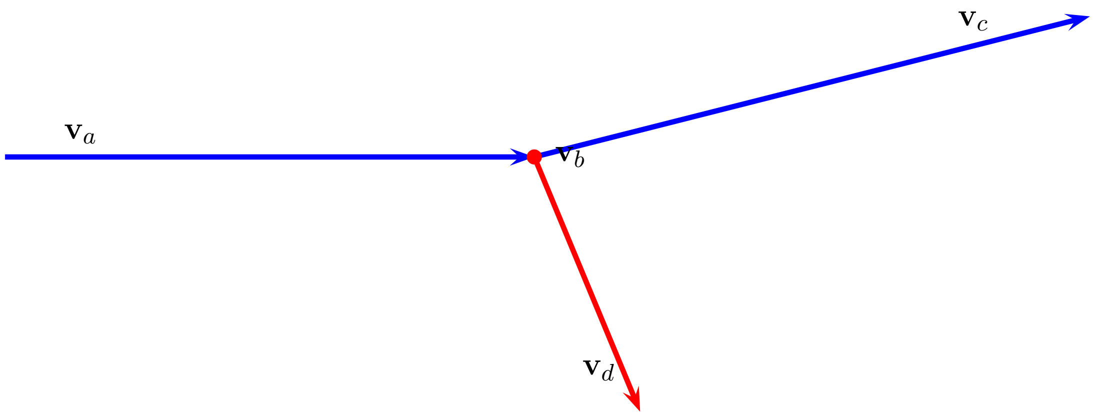
Also, assuming the kinetic energy $K(\mathbf{v})$ is some monotonically increasing function of the speed $|\mathbf{v}|$ which does not depend on the velocity's direction, conservation of energy in an elastic collision makes the particle's speeds after the collision equal to their speeds before the collision, thus

$$|\mathbf{v}_a| \;=\; |\mathbf{v}_b| \;=\; |\mathbf{v}_c| \;=\; |\mathbf{v}_d| \;=\; v. \tag{17}$$

Now let's consider the same collision in the lab frame where one of the particles was at rest before the collisions. Lorentz-boosting all the velocities by $\mathbf{u} = -\mathbf{v}_b = +\mathbf{v}_a$, we find (in the coordinates where x axis points in the direction of $\mathbf{v}_a$)

$$\begin{aligned}
v_{a,x}^{\rm lab} &= \frac{2v}{1+\beta^2}\,, \\
v_{a,y}^{\rm lab} &= v_{b,x}^{\rm lab} \;=\; v_{b,y}^{\rm lab} \;=\; 0, \\
v_{c,x}^{\rm lab} &= \frac{v(1+\cos\theta)}{1+\beta^2\cos\theta}\,, \\
v_{c,y}^{\rm lab} &= \frac{v\sin\theta}{\gamma(1+\beta^2\cos\theta)}\,, \\
v_{d,x}^{\rm lab} &= \frac{v(1-\cos\theta)}{1-\beta^2\cos\theta}\,, \\
v_{d,y}^{\rm lab} &= \frac{-v\sin\theta}{\gamma(1-\beta^2\cos\theta)}\,,
\end{aligned} \tag{18}$$

or graphically



Note that after the collision $v_{d,y}^{\rm lab} \neq -v_{c,y}^{\rm lab}$, so to assure momentum conservation in the y direction we must have velocity dependent relativistic inertia $\mathcal{M}(v)$; specifically, we need

$$\mathcal{M}(v_d^{\rm lab}) \times \frac{-v\sin\theta}{\gamma(1-\beta^2\cos\theta)} \;+\; \mathcal{M}(v_c^{\rm lab}) \times \frac{v\sin\theta}{\gamma(1+\beta^2\cos\theta)} \;=\; 0 \tag{19}$$

and hence

$$\frac{\mathcal{M}(v_c^{\rm lab})}{\mathcal{M}(v_d^{\rm lab})} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2\cos\theta}\,. \tag{20}$$

In particular, consider a grazing collision with a very small scattering angle θ. In the limit of $\theta \to 0$, we have

$$\mathbf{v}_d^{\text{lab}} \;\to\; 0 \quad \text{while} \quad \mathbf{v}_d^{\text{lab}} \;\to\; \mathbf{v}_a^{\text{lab}}, \tag{21}$$

so eq. (20) becomes

$$\frac{\mathcal{M}(v_a^{\text{lab}})}{\mathcal{M}(0)} \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{22}$$

Moreover, the expression on the RHS here is nothing but $\gamma(v_a^{\text{lab}})$; indeed,

$$\frac{1}{\gamma^2(v_a^{\text{lab}})} \;=\; 1 \;-\; (v_a^{\text{lab}}/c)^2 \;=\; 1 \;-\; \left(\frac{2\beta}{1+\beta^2}\right)^2 \;=\; \left(\frac{1-\beta^2}{1+\beta^2}\right)^2 \quad\Longrightarrow\quad \gamma(v_a^{\text{lab}}) \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{23}$$

Thus, momentum conservation requires

$$\frac{\mathcal{M}(v_a^{\text{lab}})}{\mathcal{M}(0)} \;=\; \gamma(v_a^{\text{lab}}) \tag{24}$$

and hence for any other speed v'

$$\mathcal{M}(v') \;=\; \gamma(v') \times \mathcal{M}(0). \tag{25}$$

Given this formula for the relativistic inertia $\mathcal{M}(v)$ — and hence the momentum

$$\mathbf{p} \;=\; \mathcal{M}(v)\mathbf{v} \;=\; \gamma(v)m_{\text{rest}}\mathbf{v} \tag{26}$$

— the Second Law of Newton becomes

$$\mathbf{F} \;=\; \frac{d\mathbf{p}}{dt} \;=\; m_{\text{rest}}\,\frac{d(\gamma\mathbf{v})}{dt} \tag{27}$$

In terms of the acceleration $\mathbf{a} = d\mathbf{v}/dt$,

$$\frac{d(\gamma\mathbf{v})}{dt} \;=\; \gamma\mathbf{a} \;+\; \left(\frac{d\gamma}{dt} \;=\; \gamma^3 \times \frac{(\mathbf{v}\cdot\mathbf{a})}{c^2}\right)\mathbf{v} \;=\; \gamma\mathbf{a}_\perp \;+\; (\gamma + \gamma^3\beta^2 \;=\; \gamma^3)\mathbf{a}_\parallel\,, \tag{28}$$

hence

$$\mathbf{a}_\perp \;=\; \frac{\mathbf{F}_\perp}{\gamma m_{\text{rest}}} \quad \text{but} \quad \mathbf{a}_\parallel \;=\; \frac{\mathbf{F}_\parallel}{\gamma^3 m_{\text{rest}}}\,. \tag{29}$$

In the early days of the special relativity theory, the γm_{rest} was called the *transverse relativistic mass* while $\gamma^3 m_{\text{rest}}$ was called the *longitudinal relativistic mass*. This proliferation of different things called some kind of mass was rather confusing, so eventually the name *mass* was reserved for the rest mass $m = \mathcal{M}(0)$.

Now consider the relativistic kinetic energy $K(v)$. Since it depends only on the speed $|\mathbf{v}|$, we may write it as a function of $\gamma(v)$,

$$K(v) \;=\; f(\gamma(v)) \tag{30}$$

for example, the non-relativistic kinetic energy can be written as

$$K_{\text{non-rel}}(v) \;=\; \frac{mc^2}{2}\left(1 - \frac{1}{\gamma^2(v)}\right). \tag{31}$$

In an elastic collision, the net kinetic energy is conserved, $K_c + K_d = K_a + K_b$, so in the lab frame we should have

$$f(\gamma_c^{\text{lab}}) \;+\; f(\gamma_d^{\text{lab}})) \;=\; f(\gamma_a^{\text{lab}}) \;+\; f(\gamma_b^{\text{lab}}) \tag{32}$$

But in the lab frame $\gamma_b^{\text{lab}} = 1$ and we have already calculated

$$\gamma_a^{\text{lab}} \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{33}$$

In a similar fashion

$$\gamma_c^{\text{lab}} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2}\,, \quad \gamma_d^{\text{lab}} \;=\; \frac{1-\beta^2\cos\theta}{1-\beta^2}\,; \tag{34}$$

indeed

$$\begin{aligned} \frac{1}{\gamma^2(v_c^{\text{lab}})} \;&=\; 1 \;-\; \frac{(v_{c,x}^{\text{lab}})^2 + (v_{c,y}^{\text{lab}})^2}{c^2} \;=\; 1 \;-\; \frac{\beta^2(1+\cos\theta)^2}{(1+\beta^2\cos\theta)^2} \;-\; \frac{\beta^2\sin^2\theta}{\gamma^2(1+\beta^2\cos\theta)^2} \\ &=\; \frac{(1+\beta^2\cos\theta)^2 \;-\; \beta^2(1+\cos\theta)^2 \;-\; \beta^2(1-\beta^2)\sin^2\theta}{(1+\beta^2\cos\theta)^2} \end{aligned} \tag{35}$$

where

$$
\begin{aligned}
\text{the numerator} \;&=\; (1+\beta^2\cos\theta)^2 \;-\; \beta^2(1+\cos\theta)^2 \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;+\; 2\beta^2\cos\theta \;+\; \beta^4\cos^4\theta \\
&\qquad -\; \beta^2 \;-\; 2\beta^2\cos\theta \;-\; \beta^2\cos^2\theta \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;-\; \beta^2 \;-\; \beta^2(1-\beta^2)\cos^2\theta \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;-\; \beta^2 \;-\; \beta^2(1-\beta^2) \;=\; (1-\beta^2)^2
\end{aligned}
\tag{36}
$$

,

thus

$$
\frac{1}{\gamma^2(v_c^{\text{lab}})} \;=\; \left(\frac{1-\beta^2}{1+\beta^2\cos\theta}\right)^2 \quad\Longrightarrow\quad \gamma_c^{\text{lab}} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2}\,, \tag{37}
$$

and likewise

$$
\gamma_d^{\text{lab}} \;=\; \frac{1-\beta^2\cos\theta}{1-\beta^2}\,. \tag{38}
$$

With these formulae for the $\gamma_{a,b,c,d}$ in the lab frame, the kinetic energy conservation in the elastic collision requires

$$
f\left(\frac{1+\beta^2\cos\theta}{1-\beta^2}\right) \;+\; f\left(\frac{1-\beta^2\cos\theta}{1-\beta^2}\right) \;=\; f\left(\frac{1+\beta^2}{1-\beta^2}\right) \;+\; f(1), \tag{39}
$$

and this equality must hold for any angle θ and any $\beta \leq 1$. Mathematically, the only analytic function which obeys this requirements is the linear function

$$
f(\gamma) \;=\; A\times\gamma \;+\; B \tag{40}
$$

for some constants A and B. In terms of the kinetic energy, this means

$$
K(v) \;=\; A\times\gamma(v) \;+\; B, \tag{41}
$$

and in order to agree with the non-relativistic limit of the kinetic energy, we need $A = mc^2$

and $B = -A$, thus

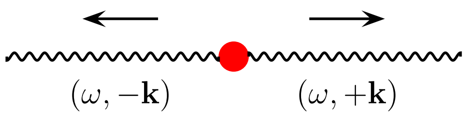
$$K(v) \;=\; mc^2 \times (\gamma(v) - 1) \;=\; \tfrac{1}{2}mv^2 \;+\; O\left(\frac{mv^4}{c^2}\right). \tag{42}$$

Or in terms of the total energy of the particle,

$$E \;=\; mc^2 \times \gamma(v) \;+\; \text{const} \;=\; \mathcal{M}(v)c^2 \;+\; \text{const}. \tag{43}$$

To find the constant term here, we need to consider an inelastic process in which the net energy of all kinds is conserved but the kinetic energy is not. And as Einstein found out, the mass is also not conserved!

To see how this works consider a nucleus emitting two photons of equal frequencies in opposite directions,



The two photons have opposite momenta $\pm\mathbf{k}$, so their net momentum is zero and there is no recoil — the nucleus initially at rest remains at rest. However, its internal energy U drops by the energy of the two photons,

$$\Delta U \;=\; U_1 \;-\; U_2 \;=\; 2\hbar\omega \tag{44}$$

Now consider the same process in a different frame where the nucleus moves with velocity $\mathbf{v}$ in the direction of one of the photons. In this frame, the photons have different frequencies due to Doppler effect,

$$\omega_1 \;=\; \gamma(1+\beta) \times \omega, \quad \omega_2 \;=\; \gamma(1-\beta) \times \omega, \tag{45}$$

and hence larger net energy

$$\hbar\omega_1 \;+\; \hbar\omega_2 \;=\; 2\hbar\omega \times \gamma. \tag{46}$$

This energy comes from the net kinetic + internal energy of the nucleus $E = K + U$, hence

$$\Delta E \;=\; E_1 \;-\; E_2 \;=\; \Delta K \;+\; \Delta U \;=\; \gamma \times 2\hbar\omega. \tag{47}$$

But the internal energy change is only $\Delta U = 2\hbar\omega$, so the kinetic energy must also change by

$$\Delta K \;=\; (\gamma - 1) \times 2\hbar\omega. \tag{48}$$

On the other hand, we know that in its original frame the nucleus does not recoil, so in any other frame its velocity should also stay constant, whatever it was. And the only way to change the kinetic energy of a nucleus without changing its velocity is by changing its mass,

$$m_2 \;=\; m_1 \;-\; \Delta m, \tag{49}$$

such that

$$(\gamma - 1)c^2 \times \Delta m \;=\; \Delta K \;=\; (\gamma - 1) \times 2\hbar\omega, \tag{50}$$

thus

$$\Delta m \;=\; \frac{2\hbar\omega}{c^2} \tag{51}$$

Note velocity independence of this formula!

In terms of the internal energy U of the nucleus — *i.e.*, the energy it has when it's not moving —

$$\Delta m \;=\; \frac{\Delta U}{c^2}\,, \tag{52}$$

or in terms of the net energy $E = K + U$,

$$\Delta E \;=\; c^2 \times \Delta\mathcal{M}(v). \tag{53}$$

The same formula applies to other inelastic processes, even when there is a recoil, and this have lead Einstein to his famous formula

$$E = \mathcal{M}(v) \times c^2 = \gamma(v) mc^2 = \frac{mc^2}{\sqrt{1-(v/c)^2}}. \tag{54}$$

In particular, a particle at rest has a tremendous hidden energy $E_0 = mc^2$. In any inelastic process, this energy increases or decreases, and this leads to an increase or decrease of the net mass of the system.

In general, any change of net energy changes the net mass of the system, even in such mundane non-relativistic processes as pool-ball collisions or chemical reactions, although the resulting Δm is too small to measure. But in nuclear reactions Δm is typically of the order $10^{-3} \times m$, and that can be easily measured by a mass spectrometer. In fact, once can calculate the energy released or consumed in some nuclear reaction by simply looking up the masses of initial and final nuclei and calculating the Δm. For example, in the deuterium-tritium fusion reaction

$$D_1^2 + T_1^3 = \mathrm{He}_2^4 + n_0^1, \tag{55}$$

the masses are

$$\begin{aligned} m(D) &= 2.014\,102 \text{ u}, \\ m(T) &= 3.016\,049 \text{ u}, \\ m(\mathrm{He}^4) &= 4.002\,602 \text{ u}, \\ m(n) &= 1.008,\, 665 \text{ u}, \end{aligned} \tag{56}$$

so in the fusion reaction the net mass is reduced by

$$\Delta m = 18.884 \cdot 10^{-3} \text{ u}, \tag{57}$$

where u is the atomic mass unit,

$$\mathrm{u} = 1.660\,539 \cdot 20^{-27} \text{ kg} = 931.494 \text{ MeV}/c^2. \tag{58}$$

Consequently, the fusion reaction should release energy

$$\Delta E = (18.884 \cdot 10^{-3} \text{ u}) \times c^2 = 17.59 \text{ MeV}.$$

and the experimentally measured fusion energy indeed agrees with this value.

In a more extreme example, an electron and a positron can annihilate each other, so their entire hidden energy $2 \times m_e c^2$ is converted to the energy of the photons produced in the annihilation. On the other hand, when highly energetic elementary particles collide, their kinetic energies can be converted to the mass of some heavy new particles. For example, at the Large Hadron Collider at CERN, the protons are accelerated till their γ factor reaches about 7000; in other words, their kinetic energy is 7000 larger than the rest energy $m_p c^2$. In GeV units, each proton has energy about 6500 GeV, and when two protons collide, a notable fraction of the net 13,000 GeV energy is converted to the masses of many particles particles created in the collision. Some of these particles can be much heavier than the original proton, for example the Higgs particle has mass $M_H = 126\ \mathrm{GeV}/c^2$.

Energy–momentum 4–vector.

The relativistic energy and momentum of a free particle form a Lorentz 4–vector

$$p^\mu \;=\; \left(\frac{E}{c},\mathbf{p}\right) \;=\; mu^\mu \tag{59}$$

where u^μ is the 4–velocity of the particle. Indeed, in components

$$p^i \;=\; \gamma v^i m \;=\; u^i m \quad\text{and}\quad p^0 \;=\; \frac{E}{c} \;=\; \gamma cm \;=\; u^0 m. \tag{60}$$

Consequently, the net energy and momentum of any multi-particle system also form a 4–vector

$$P^\mu_{\text{net}} \;=\; \left(\frac{E_{\text{net}}}{c},\mathbf{P}_{\text{net}}\right) \;=\; \sum_a^{\text{particles}} m_a u_a^\mu \tag{61}$$

which transforms under Lorentz symmetries as any other 4–vector,

$$\begin{aligned} E'^{\,\text{net}} &= \gamma(E^{\text{net}} - vP^{\text{net}}_\parallel),\\ P'^{\,\text{net}}_\parallel &= \gamma\left(P^{\text{net}}_\parallel - \frac{v}{c^2}\,E^{\text{net}}\right),\\ \mathbf{P}'^{\,\text{net}}_\perp &= \mathbf{P}^{\text{net}}_\perp\,. \end{aligned} \tag{62}$$

Therefore, *if the energy and the momentum are conserved in one frame of reference, then they are also conserved in any other reference frame!*

Now consider the Lorentz square of the energy-momentum 4–vector p^μ. For a single particle we have

$$p^\mu p_\mu \;=\; \frac{E^2}{c^2} \;-\; \mathbf{p}^2 \;=\; (\gamma mc)^2 \;-\; (\gamma m\mathbf{v})^2 \;=\; m^2\gamma^2(c^2-\mathbf{v}^2) \;=\; m^2c^2, \tag{63}$$

or in terms of the particle's 4–velocity u^μ,

$$p^\mu \;=\; mu^\mu \quad\Longrightarrow\quad p^\mu p_\mu \;=\; m^2\times u^\mu u_\mu \;=\; m^2c^2. \tag{64}$$

Either way, the energy and the momentum of a particle are related as

$$E^2 \;=\; c^2\mathbf{p}^2 \;+\; m^2c^4. \tag{65}$$

In the 4–momentum space, this formula defines a hyperbolic hypersurface called the *mass shell*.

In the non-relativistic limit, the mass shell condition becomes

$$E \;=\; mc^2 \;+\; \frac{p^2}{2m} \;-\; \frac{p^4}{8m^3c^2} \;+\; \cdots, \tag{66}$$

while in the ultra-relativistic limit $\gamma \gg 1$ and hence $|\mathbf{p}| \gg mc$ we get

$$E \;=\; c|\mathbf{p}| \;+\; \frac{m^2c^3}{2|\mathbf{p}|} \;+\; \cdots. \tag{67}$$

In particular, for the massless particles like the photons

$$E \;=\; c|\mathbf{p}|, \tag{68}$$

in perfect agreement with $\omega = c|\mathbf{k}|$ for the EM waves in vacuum and the quantum formulae $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$.

In general, massless particles (*i.e.*, particles having zero *rest mass*) must move with the speed of light — otherwise, they would have zero energy and zero momentum — while particles with non-zero rest masses must move slower than light. Among the presently known particles, only the photon is exactly massless. The neutrinos do have masses, although they are very small — less then eV/c^2 – and since we cannot detect neutrinos with energies much less than an MeV, all the neutrinos we have ever detected were ultra-relativistic with $\gamma > 10^6$, and their speeds were experimentally indistinguishable from the speed of light. The only reason we know about the neutrino masses is because of the quantum oscillations between the 3 neutrino species!

To see how the neutrino oscillations work, note that in quantum mechanics, eq. (67) for the energy of an ultra-relativistic particle becomes the Hamiltonian operator

$$\hat{H} \approx c|\mathbf{p}| + \frac{c^3}{2|\mathbf{p}|}\hat{M}^2 \qquad (69)$$

where $\hat{M}^2$ is the 3×3 matrix in the Hilbert space of the neutrino species. This matrix is non-diagonal in the basis in which we make and detect neutrinos, and this causes neutrinos to oscillate from one species to another. Specifically, if the neutrino is created at time $t = 0$ in a state vector $|\psi_0\rangle$, then after it has traveled through long distance L in time $t = L/c$, its state vector becomes

$$|\psi\rangle = e^{i\,\text{some overall phase}} \times \exp\left(i\,\frac{c^3\hbar L}{2E}\,\hat{M}^2\right)|\psi_0\rangle\,. \qquad (70)$$

The matrix exponential of a non-diagonal matrix is itself non-diagonal, and the off-diagonal matrix elements of this exponential are probability amplitudes for changing the neutrino's species. For more information, please see the Wikipedia article on neutrino oscillations.

Relativistic Collisions

Having dealt with a single relativistic particle, let's consider a collision of two relativistic particles. The collision can be elastic or inelastic, and (in the inelastic case) may have any number of particles in the final state. But come hell or high water, the net energy-momentum of the colliding particles must be conserved, thus

$$P^\mu_{\rm net} \;=\; p^\mu_1 \,+\, p^\mu_2 \;=\; \sum_a p'^\mu_a\,. \tag{71}$$

A convenient Lorentz-invariant parameter of the collision is the Lorentz-square of the net energy-momentum, or rather

$$s \;=\; c^2 P_\mu P^\mu. \tag{72}$$

This s parameter has the same value in all inertial frames, but its physical meaning becomes particularly clear in the center-of-inertia frame — often called the center-of-mass frame — where the net 3–momentum happens to vanish, $\mathbf{P}_{\rm net} = 0$. In this frame

$$s \;=\; E^2_{\rm net} \;-\; c^2\mathbf{P}^2_{\rm net} \;=\; E^2_{\rm net}\,, \tag{73}$$

so $\sqrt{s}$ is the collision energy in the center of mass frame,

$$E_{\rm cm} \;=\; \sqrt{s}. \tag{74}$$

In any other frame, the net energy is larger,

$$E^2_{\rm net} \;=\; s^2 \,+\, c^2\mathbf{P}^2_{\rm net} \;=\; E^2_{\rm cm} \,+\, c^2\mathbf{P}^2_{\rm net} \;>\; E^2_{\rm cm}\,, \tag{75}$$

but the extra energy is completely tied up the the conserved net momentum, so it does nothing but make the system's center of mass move at constant velocity

$$\mathbf{v} \;=\; \frac{c^2\mathbf{P}_{\rm net}}{E_{\rm net}}\,. \tag{76}$$

The only energy available for making new particles, — or anything else besides the center of mass motion — is the $E_{\rm cm} = \sqrt{s}$.

Let's calculate the center-of-mass energy in the lab frame of a fixed-target experiment: In this frame one initial particle has high speed while and hence high energy E_1 and high momentum $\mathbf{p}_1$, while the second initial particle is at rest, thus $E_2 = M_2c^2$ and $\mathbf{p}_2 = 0$. Consequently,

$$\begin{aligned} s \;&=\; c^2(p_1+p_2)^2 \;=\; (E_1+M_2c^2)^2 \;-\; c^2\mathbf{p}_1^2 \\ &=\; E_1^2 \;+\; 2E_1M_2c^2 \;+\; M_2^2c^4 \;-\; c^2\mathbf{p}_1^2 \;=\; M_1^2c^4 \;+\; 2E_1M_2c^2 \;+\; M_2^2c^4 \\ &=\; (M_1+M_2)^2c^4 \;+\; 2(E_1-M_1c^2)\times M_2c^2. \end{aligned} \tag{77}$$

In the non-relativistic limit, this formula leads to

$$E_{\rm cm} \;=\; \sqrt{s} \;\approx\; (M_1+M_2)c^2 \;+\; (E_1-M_1c^2)\times\frac{M_2}{M_1+M_2}\,, \tag{78}$$

or in terms of the kinetic energies $K = E - Mc^2$,

$$K_{\rm cm} \;=\; K_1\times\frac{M_2}{M_1+M_2}\,. \tag{79}$$

On the other hand, in the ultra-relativistic limit of $E_1 \gg M_1c^2$, eq. (77) becomes

$$E_{\rm cm} \;\approx\; \sqrt{2E_1\times M_2c^2} \;\ll\; E_1\,. \tag{80}$$

In particle physics, there are two types of collision experiments: the fixed-target experiments, in which a beam of accelerated particles collides with a solid or liquid target, and the collider experiments in which two particle beams moving in opposite directions are focused on some point where they collide with each other. In the fixed target experiments, all the accelerated particles collide with some particle in the target, which makes for a much higher collision rate than in a collider where most accelerated particles miss each other. On the other hand, in a collider the entire energy of the two accelerated particles is available in the CM frame for discovering new physics, while in a fixed-target experiment most of the energy is wasted on the center of mass motion, and only a small fraction $E_{\rm cm} \ll E_1$ goes towards the interesting physics.

For example, the oldest proton-proton collider at CERN — the PS, which started working back in 1959 — had two 28 GeV proton beams colliding head on, so the CM-frame energy available for discovering new physics was 56 GeV. To reach the same energy at a fixed-target experiment — a proton beam hitting a tank of liquid hydrogen — we need a proton beam of energy

$$E_1 \;=\; \frac{s=E_{\rm cm}^2}{2M_pc^2} \;=\; 1670\ \text{GeV}, \tag{81}$$

and no accelerator had reached this energy level until 2010, half a century after the PS. Ironically, the first accelerator producing proton beams with energy higher than (81) was the LHC at CERN which is also a collider. Today, the CM-frame collision energy at LHS is about 13,000 GeV; to reach this energy at a fixed target experiment, we would need a proton beam of energy

$$E_1 \;\approx\; 90\cdot 10^6\ \text{GeV}, \tag{82}$$

and we would be lucky to reach this energy level in another half-a-century.

Examples of Relativistic Kinematics

Consider a collision of two particles or a decay of one unstable particle into two or more particles. There are several equations relating energies and momenta of all initial-state and final state particles involved in such a collision or a decay. First, the the net energy-momentum must be conserved; in the 4–vector form,

$$\sum_i^{\text{initial}} p_i^\mu \;=\; \sum_f^{\text{final}} p_f'^\mu. \tag{83}$$

Second, for every initial-state or final-state particle, it's energy and momentum must be related by eq. (65), or in covariant form

$$\forall i:\; p_i^2 \;=\; m_i^2c^2 \quad\text{and}\quad \forall f:\; p_f'^2 \;=\; m_f^2c^2. \tag{84}$$

In many situations, these relations allow us to find the final particles' energies, or at least establish the relations between their energies and directions of motion.

EXAMPLE#1: **decay of a charged pion into a muon and a neutrino**,

$$\pi^+ \;\to\; \mu^+ \;+\; \nu_\mu\,. \tag{85}$$

By energy-momentum conservation

$$p_\pi^\alpha \;=\; p_\mu^\alpha \;+\; p_\nu^\alpha \tag{86}$$

(where I use α for the Lorentz vector index since μ, ν, and π are used up as the species labels), hence

$$\begin{aligned} p_\nu^2 \;&=\; (p_\pi - p_\mu)^2 \;=\; p_\pi^2 \;+\; p_\mu^2 \;-\; 2p_\pi\cdot p_\mu\,,\\ p_\mu^2 \;&=\; (p_\pi - p_\nu)^2 \;=\; p_\pi^2 \;+\; p_\nu^2 \;-\; 2p_\pi\cdot p_\nu\,. \end{aligned} \tag{87}$$

At the same time, there is a *mass shell* condition for the 4–momentum of each particle,

$$p_\pi^2 \;=\; M_\pi^2 c^2, \quad p_\mu^2 \;=\; M_\mu^2 c^2, \quad p_\nu^2 \;=\; M_\nu^2 c^2 \;\approx\; 0. \tag{88}$$

Plugging these conditions into eqs. (87), we immediately get

$$2p_\pi\cdot p_\mu \;=\; M_\pi^2 c^2 \;+\; M_\mu^2 c^2 \quad\text{while}\quad 2p_\pi\cdot p_\nu \;=\; M_\pi^2 c^2 \;-\; M_\mu^2 c^2. \tag{89}$$

Now, in the rest frame of the initial pion

$$p_\pi\cdot p_\mu \;=\; (M_\pi c)\times(E_\mu/c) \;-\; \mathbf{p}_\pi\cdot\mathbf{p}_\mu \;=\; M_\pi\times E_\mu \;-\; 0, \tag{90}$$

and likewise

$$p_\pi\cdot p_\nu \;=\; M_\pi\times E_\nu, \tag{91}$$

so eqs. (89) give us the energies of the muon and the neutrino as

$$E_\mu \;=\; \frac{M_\pi^2 + M_\mu^2}{2M_\pi}\times c^2, \quad E_\nu \;=\; \frac{M_\pi^2 - M_\mu^2}{2M_\pi}\times c^2.$$

Numerically, $M_\pi c^2 = 139$ MeV, $M_\mu c^2 = 105$ MeV, hence after the decay $E_\mu = 109$ MeV and $E_\nu = 30$ MeV.

Example#2: **Compton scattering**.

This is the elastic scattering of a photon and and electron,

$$\gamma \,+\, e^- \,\longrightarrow\, \gamma \,+\, e^-, \tag{92}$$

usually analyzed in the lab frame where the initial electron is at rest. In this frame, some of the initial photon's energy is transferred to the electron, so the final photon has a lower energy than the initial photon. By the Planck formula $E = \hbar\omega$, this means that the scattered photon has lower frequency — and hence longer wavelength — than the initial photon. Arthur Compton experimentally discovered this effect in X-ray scattering in 1922, and in 1923 he explained it in terms of X-ray quanta being relativistic particles. Today, the Compton effect is a simple (or at least simple enough for this class) consequence of special relativity and basic quantum theory, but back in 1923 understanding photons was the bleeding edge of science, so in 1927 Arthur Compton got a Nobel Prize for his discovery.

Let's work out the relativistic kinematics of the Compton scattering (92).

- The initial photon has frequency ω and direction $\mathbf{n}$, hence wave vector $\mathbf{k} = (\omega/c)\mathbf{n}$, and therefore energy–momentum

$$p_\gamma^\mu \;=\; \frac{\hbar\omega}{c}\bigl(1,\mathbf{n}\bigr). \tag{93}$$

- Likewise, the scattered photon has direction $\mathbf{n}'$, frequency ω', hence wave vector $\mathbf{k}' = (\omega'/c)\mathbf{n}'$ and therefore energy–momentum

$$p_\gamma'^\mu \;=\; \frac{\hbar\omega'}{c}\bigl(1,\mathbf{n}'\bigr). \tag{94}$$

- The initial electron is at rest in the lab frame, thus

$$p_e^\mu \;=\; m_e c(1,\mathbf{0}). \tag{95}$$

- Finally, we don't have any direct knowledge of the scattered electron, but we know that the net energy–momentum must be conserved,

$$p_e'^\mu \;+\; p_\gamma'^\mu \;=\; p_e^\mu \;+\; p_\gamma^\mu\,, \tag{96}$$

hence

$$p_e'^\mu \;=\; p_e^\mu \;+\; p_\gamma^\mu \;-\; p_\gamma'^\mu\,. \tag{97}$$

Now, let's take the Lorentz squares of the two sides of eq. (97):

$$\begin{aligned}(p_e')^2 \;&=\; (p_e)^2 \;+\; (p_\gamma)^2 \;+\; (p_\gamma')^2\\ &\qquad +\;2(p_e\cdot p_\gamma)\;-\;2(p_e\cdot p_\gamma')\;-\;2(p_\gamma\cdot p_\gamma').\end{aligned} \tag{98}$$

On the top line of this formula, we have

$$(p_e')^2 \;=\; (p_e)^2 \;=\; m_e^2c^2 \tag{99}$$

because that's true for any electron in the Universe, and likewise

$$(p_\gamma')^2 \;=\; (p_\gamma)^2 \;=\; 0 \tag{100}$$

because that's true for any photon. Consequently, the whole top line of eq. (98) cancels out between the LHS and the RHS or vanishes altogether, so the whole equation reduces to vanishing of its second line,

$$2(p_e\cdot p_\gamma)\;-\;2(p_e\cdot p_\gamma')\;-\;2(p_\gamma\cdot p_\gamma')\;=\;0. \tag{101}$$

In this formula, in the lab frame where the initial electron is at rest,

$$(p_e\cdot p_\gamma)\;=\;m_ec(1,\mathbf{0})\cdot\frac{\hbar\omega}{c}(1,\mathbf{n})\;=\;m_e\hbar\omega\bigl(1\cdot1\,-\,\mathbf{0}\cdot\mathbf{n}\,=\,1\bigr)\;=\;m_e\hbar\omega, \tag{102}$$

and likewise

$$(p_e\cdot p_\gamma')\;=\;m_e\hbar\omega'. \tag{103}$$

Finally,

$$(p_\gamma\cdot p_\gamma')\;=\;\frac{\hbar\omega}{c}(1,\mathbf{n})\cdot\frac{\hbar\omega'}{c}(1,\mathbf{n}')\;=\;\frac{\hbar^2\omega\omega'}{c^2}\Bigl((1\,\mathbf{n})\cdot(1,\mathbf{n}')\;=\;1-\mathbf{n}\cdot\mathbf{n}'\Bigr)\;=\;\frac{\hbar^2\omega\omega'}{c^2}\,(1-\cos\theta), \tag{104}$$

where θ is the angle between the scattered photon's direction $\mathbf{n}'$ and the initial photon direction $\mathbf{n}$, $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$. Plugging all these formulae into eq. (101), we arrive at

$$2m_e\hbar\omega \;-\; 2m_2\hbar\omega' \;=\; 2\frac{\hbar^2\omega\omega'}{c^2}\,(1-\cos\theta). \tag{105}$$

Finally let's divide both sides of this formula by $2m_e\hbar\omega\omega'$, thus

$$\frac{1}{\omega'} \;-\; \frac{1}{\omega} \;=\; \frac{\hbar}{m_e c^2}\times(1-\cos\theta). \tag{106}$$

Or in terms the wavelengths $\lambda = 2\pi c/\omega$ and $\lambda' = 2\pi c/\omega'$,

$$\lambda' \;-\; \lambda \;=\; \frac{2\pi\hbar}{m_e c}\times(1-\cos\theta). \tag{107}$$

, which is how Arthur Compton himself wrote his famous formula.

Numerically,

$$\frac{2\pi\hbar}{m_e c} \;\approx\; 2.43\ \text{pm}, \tag{108}$$

which is why it's easy to see that $\lambda' > \lambda$ for the scattering of hard X rays, and the effect becomes overwhelming for the hard gamma rays. On the other hand, the Compton effect is harder to observe for the soft X rays, much harder for the UV rays, and almost impossible at longer wavelengths.

Finally, note the $\hbar$ factor in the numerator of the RHS of eq. (107). Thanks to this factor, the Compton effect disappears in the classical limit $\hbar\to 0$. And indeed, in the purely classical electromagnetism there is no Compton effect, and the scattered EM wave has the same wavelength as the incident EM wave. It takes quantum mechanics — packaging a discrete amount of EM energy $\hbar\omega$ and EM momentum $\hbar\mathbf{k}$ into a single photon — to generate the Compton effect.

Relativistic Dynamics

Newton Laws

Let's start with the three Laws of Newton in the relativistic context. The **First Law** is built into the principle of relativity, so in the relativistic mechanics it holds without any changes from the days of Newton. Likewise, the **Second Law** remain valid in its momentum form: *In any inertial frame*

$$\frac{d\mathbf{p}_{\text{body}}}{dt} = \mathbf{F}^{\text{net}}_{\text{on body}} = \sum \begin{pmatrix} \text{all forces acting} \\ \text{on the body} \end{pmatrix}, \tag{109}$$

provided the $\mathbf{p}$ *here is the relativistic momentum* $\underline{\mathbf{p} = \gamma m\mathbf{v}}$.

But the **Third Law** becomes tricky. The net relativistic momentum is always conserved, so when two particles collide at the same point in space, their momenta change by equal and opposite amounts, $\Delta\mathbf{p}_1 = -\Delta\mathbf{p}_2$, just as they would in the non-relativistic mechanics. But when two particles (or extended bodies) interact at non-zero distance from each other, we generally do not have $\mathbf{F}_{12}(t) = -\mathbf{F}_{21}(t)$. One reason for this is the simultaneity of the two forces acting at different places depends on the frame of reference, so in the would-be Third Law formula

$$\mathbf{F}_{12}(t_1) \;=?=\; -\mathbf{F}_{21}(t_2) \tag{110}$$

we would not be able to identify $t_1 = t_2 = t$ in any invariant way.

Also, a force between two bodies separated in space should be mediated but some kind of a field, electric, magnetic, gravitational, whatever. For the EM fields, we saw that they can carry energy and momentum of their own and exchange them with the charged bodies. So instead of the naive Third Law, we end up with

$$\mathbf{F}_{12}(t) + \mathbf{F}_{21} + \frac{d\mathbf{P}^{\text{net}}_{\text{EM}}}{dt} = 0. \tag{111}$$

Note: every term here depends on a particular frame of reference, but the net sum vanishes in any inertial frame.

Formulae similar to (111) exist for the bodies interacting with each other by means of gravitational fields, or some other kind of non-EM field, but dealing with such forces is way beyond the scope of this class.

Motion Under a Constant Force

Consider a particle of rest mass m subject to a constant force $\mathbf{F}$. For simplicity, suppose the particle has zero velocity at time $t_0 = 0$, so the Second Law

$$\frac{d\mathbf{p}}{dt} \;=\; \mathbf{F} \tag{109}$$

leads to

$$\mathbf{p}(t) \;=\; t\mathbf{F}. \tag{112}$$

In terms of the particle's velocity $\mathbf{v}$,

$$\mathbf{p} \;=\; \gamma m\mathbf{v} \quad\Longrightarrow\quad \mathbf{v} \;=\; \frac{\mathbf{p}}{\gamma m} \;=\; \frac{c^2\mathbf{p}}{E} \tag{113}$$

where $E = \gamma mc^2$ is the relativistic energy of the particle. In terms of the 3-momentum,

$$E \;=\; c\sqrt{m^2c^2 + \mathbf{p}^2} \quad\Longrightarrow\quad \mathbf{v} \;=\; \frac{c\mathbf{p}}{\sqrt{m^2c^2 + \mathbf{p}^2}}\,. \tag{114}$$

In particular, for motion under the constant force $\mathbf{F}$ and hence momentum (112),

$$\mathbf{v}(t) \;=\; \frac{ct\mathbf{F}}{\sqrt{m^2c^2 + F^2t^2}}\,. \tag{115}$$

Now let's integrate eq. (115) to find the particle's displacement $\mathbf{r}(t)$ as a function of time t. Let's use the coordinates where the motion is along the x axis, starting with $x(0) = 0$.

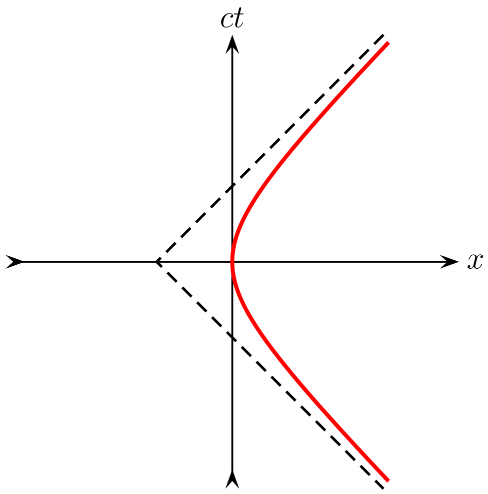
Then

$$
\begin{aligned}
x(t) &= \int\limits_0^t \frac{cFt'\,dt'}{\sqrt{m^2c^2+F^2t'^2}} \\
&= \frac{c}{F}\int\limits_{t'=0}^{t'=t} d\Big(\sqrt{m^2c^2+F^2t'^2}\Big) \\
&= \frac{c}{F}\Big(\sqrt{m^2c^2+F^2t^2}\;-\;mc\Big) \\
&= \sqrt{(ct)^2+(mc^2/F)^2}\;-\;(mc^2/F).
\end{aligned}
\tag{116}
$$

This is an example of a hyperbolic motion,

$$
x(t) \;=\; \sqrt{(ct)^2+b^2}\;+\;\text{const} \qquad \text{for } b \;=\; \frac{mc^2}{F}\,. \tag{117}
$$

so called because the world-line of the particle is a hyperbola,



(118)

For a more complicated example, suppose the particle has non-zero initial velocity $\mathbf{v}_0$ in a direction $\perp$ to the force. Specifically, let the force point in the x direction while $\mathbf{v}_0$ is in the y direction. In this case,

$$
\begin{aligned}
p_x(t) &= Ft, \\
p_y(t) &= \text{const} \;=\; m\gamma_0 v_0\,,
\end{aligned}
$$

Consequently, the energy grows with time as

$$E(t) \;=\; c\sqrt{m^2c^2 + p_x^2(t) + p_y^2} \;=\; c\sqrt{m^2c^2 + p_y^2 + F^2t^2}, \tag{119}$$

and therefore

$$\mathbf{v}(t) \;=\; \frac{c^2\mathbf{p}(t)}{E(t)} \tag{120}$$

becomes in components

$$\begin{aligned} v_x(t) \;&=\; \frac{cFt}{\sqrt{(m^2c^2 + p_y^2) + F^2t^2}}\,, \\ v_y(t) \;&=\; \frac{cp_y}{\sqrt{(m^2c^2 + p_y^2) + F^2t^2}}\,. \end{aligned} \tag{121}$$

Note: while the force F accelerates the particle in x direction, the y component of the velocity becomes smaller due to larger inertia γm while p_y stays constant.

Finally, integrating the velocity components (121) over dt we arrive at the displacement from the original position,

$$\begin{aligned} \Delta x \;&=\; \int_0^t v_x(t')dt' \;=\; \frac{c}{F}\Bigl(\sqrt{m^2c^2 + p_y^2 + F^2t^2} \;-\; \sqrt{m^2c^2 + p_y^2}\Bigr), \\ \Delta y \;&=\; \int_0^t v_y(t')dt' = \;\frac{cp_y}{F}\;\operatorname{ar\,sinh}\frac{Ft}{\sqrt{m^2c^2 + p_y^2}}\,. \end{aligned} \tag{122}$$

Next, consider the work-energy theorem

$$\Delta E \;=\; \text{Work} \;=\; \int \mathbf{F}\cdot d\mathbf{r} \tag{123}$$

for the relativistic energy (119). Infinitesimally,

$$\begin{aligned}
d\text{Work} \;&=\; \mathbf{F}\cdot d\mathbf{r} \;=\; \frac{d\mathbf{p}}{dt}\cdot\mathbf{v}dt \;=\; d\mathbf{p}\cdot\mathbf{v} \;=\; d\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)\cdot\mathbf{v} \\
&=\; \left(\frac{md\mathbf{v}}{\sqrt{1-v^2/c^2}} \;+\; \frac{m\mathbf{v}(\mathbf{v}\cdot d\mathbf{v})}{c^2[1-v^2/c^2]^{3/2}}\right)\cdot\mathbf{v} \\
&=\; \frac{(\mathbf{v}\cdot d\mathbf{v})}{c^2[1-v^2/c^2]^{3/2}}\Big(mc^2(1-v^2/c^2) \;+\; m\mathbf{v}^2 \;=\; mc^2\Big) \\
&=\; d\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right) \;=\; dE_{\text{rel}}\,,
\end{aligned} \tag{124}$$

so indeed we should get

$$\Delta E_{\text{rel}} \;=\; \int\mathbf{F}\cdot d\mathbf{r}. \tag{125}$$

In particular, for the motion under the constant force **F** starting from motion $\perp$ to the force,

$$\begin{aligned}
E(t) \;-\; E(0) \;&=\; \mathbf{F}\cdot\Delta\mathbf{r} \;=\; F*\Delta x \\
&=\; F*\frac{c}{F}\Big(\sqrt{m^2c^2+F^2t^2} \;-\; \sqrt{m^2c^2+p_y^2}\Big) \\
&=\; c\Big(\sqrt{m^2c^2+F^2t^2} \;-\; \sqrt{m^2c^2+p_y^2}\Big).
\end{aligned} \tag{126}$$

Moreover, the initial energy $E(0)$ in terms of p_y is

$$E(0) \;=\; c\sqrt{m^2c^2+p_y^2}, \tag{127}$$

so at later times

$$E(t) \;=\; c\sqrt{m^2c^2+p_y^2+F^2t^2}, \tag{128}$$

in perfect agreement with eq. (119) for the relativistic energy.

Lorentz Transforming the Force

By the first law of Newton, the vanishing of the net force on some body holds true in all inertial frames. But for a non-vanishing force, its values in different inertial frames are related by a rather messy formula; it's somewhat similar to the relativistic velocity addition formula (for the non-parallel velocities), but even uglier. Indeed, similar to $\mathbf{v}$, the force $\mathbf{F}$ is a derivative of the space components $\mathbf{p}$ of the 4–vector p^μ with respect to the ordinary time t, — which is a time component of a 4–vector x^μ. But for the velocity $\mathbf{v}$ the $d\mathbf{r}$ and the dt are respective space/time components of the same 4–vector, while for the force $\mathbf{F}$ the $d\mathbf{p}$ is a space component of one 4–vector while dt is the time component of a different 4–vector. In the velocity case, the key to its Lorentz transforms is the proper velocity $u^\mu = (dx^\mu/d\tau)$. Likewise, the fey to the Lorentz transforms of the force 3–vector is the *proper force* AKA the *Minkowski force*

$$K^\mu \stackrel{\text{def}}{=} \frac{dp^\mu}{d\tau} \tag{129}$$

where τ is the proper time of the body in question. By construction, K^μ is a derivative of a 4–vector WRT a 4–scalar, so it's a proper 4–vector which transforms under any Lorentz transform as

$$K'^\mu = L^\mu_{\ \nu} K^\nu. \tag{130}$$

In particular, under a Lorentz boost in the x direction,

$$\begin{aligned} K'^0 &= \gamma_{\text{rel}} K^0 - \gamma_{\text{rel}}\beta_{\text{rel}} K^x, \\ K'^x &= \gamma_{\text{rel}} K^x - \gamma_{\text{rel}}\beta_{\text{rel}} K^0, \\ K'^y &= K^y, \\ K'^z &= K^z. \end{aligned} \tag{131}$$

In components,

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \frac{d\mathbf{p}}{dt}\frac{dt}{d\tau} = \mathbf{F}\gamma, \tag{132}$$

while

$$K^0 = \frac{1}{c}\frac{dE}{d\tau} = \frac{1}{c}\left(\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}\right)\frac{dt}{d\tau} = (\mathbf{F}\cdot\boldsymbol{\beta})\gamma. \tag{133}$$

Consequently,

$$(K\cdot u) \;=\; K^\mu u_\mu \;=\; K^0 u^0 \;-\; \mathbf{K}\cdot\mathbf{u} \;=\; \gamma(\mathbf{F}\cdot\boldsymbol{\beta})*\gamma c \;-\; \gamma\mathbf{F}\cdot\gamma\mathbf{v} \;=\; \gamma^2\Big(\mathbf{F}\cdot\mathbf{v} - \mathbf{F}\cdot\mathbf{v}\Big) \;=\; 0. \tag{134}$$

The same result obtains in a manifestly covariant form from

$$K^\mu u_\mu \;=\; \frac{d(p^\mu = mu^\mu)}{d\tau} u_\mu \;=\; \frac{d}{d\tau}\left(\tfrac{1}{2}mu^\mu u_\mu\right) \;=\; 0 \tag{135}$$

because

$$\tfrac{1}{2}mu^\mu u_\mu \;=\; \tfrac{1}{2}mc^2 \;=\; \text{const.} \tag{136}$$

Now let's re-express the Lorentz boost (131) of the Minkowski force K^μ in terms of the ordinary force components $F^{x,y,z}$:

$$\begin{aligned}
\gamma' F'^x \;&=\; \gamma_{\text{rel}}\gamma F^x \;-\; \gamma_{\text{rel}}\beta_{\text{rel}}\gamma(\boldsymbol{\beta}\cdot\mathbf{F}),\\
\gamma' F'^y \;&=\; \gamma F^y,\\
\gamma' F'^z \;&=\; \gamma F^z,
\end{aligned} \tag{137}$$

where γ' obtains from the Lorentz boost of the particle's proper velocity:

$$c\gamma' \;=\; u'^0 \;=\; \gamma_{\text{rel}} u^0 \;-\; \gamma_{\text{rel}}\beta_{\text{rel}} u^x \;=\; c\gamma_{\text{rel}}\gamma\big(1 \,-\, \beta_{\text{rel}}\beta^x\big). \tag{138}$$

Therefore

$$\begin{aligned}
F'^x \;&=\; \frac{F^x \,-\, \beta_{\text{rel}}(\boldsymbol{\beta}\cdot\mathbf{F})}{(1 \,-\, \beta_{\text{rel}}\beta^x)}\,,\\
F'^y \;&=\; \frac{F^y}{\gamma_{\text{rel}}(1 \,-\, \beta_{\text{rel}}\beta^x)}\,,\\
F'^z \;&=\; \frac{F^z}{\gamma_{\text{rel}}(1 \,-\, \beta_{\text{rel}}\beta^z)}\,.
\end{aligned} \tag{139}$$

Note that similar to the 3–velocity transformation rule, the component of the 3–force **F** in the direction of the Lorentz boost transforms differently than the components in the directions $\perp$ to the boost.

Electromagnetic Forces

A non-relativistic charged particle moving through electric and magnetic fields feels the force

$$\mathbf{F} \;=\; Q\mathbf{E} \;+\; Q\mathbf{v}\times\mathbf{B} \tag{140}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$ should be evaluated at the particle location $\mathbf{r}(t)$. For a relativistic charged particle, the net EM force obtains from exactly the same eq. (140) without any relativistic correction. In my next set of notes I shall write this equation in a manifestly relativistic form

$$K^\mu \;=\; \frac{Q}{c}F^{\mu\nu}u_\nu \tag{141}$$

where u_ν is the particle's proper velocity and $F^{\mu\nu} = -F^{\nu\mu}$ is the antisymmetric Lorentz tensor comprising both the electric and the magnetic fields. But for the moment, let's accept the force equation (140) for a relativistic particle as a basic experimental fact and focus on the charged particle's motion in the given EM fields.

For simplicity, let's focus on the motion in the *uniform and constant* electromagnetic fields. There are three possibilities here:

1. $\mathbf{E} = \text{const} \neq 0$ while $\mathbf{B} = 0$. In this case, the charged particle moves under the constant force $\mathbf{F} = Q\mathbf{E}$, and we have already studied this motion in some detail.

2. $\mathbf{B} = \text{const} \neq 0$ while $\mathbf{E} = 0$. I shall address this case in a moment.

3. Both $\mathbf{E} = \text{const} \neq 0$ and $\mathbf{B} = \text{const} \neq 0$. Let me postpone addressing this case until we have learned how the EM fields mix with each other under Lorentz transforms, *cf.* my next set of notes.

Cyclotron Motion in Magnetic Field.

When a non-relativistic charged particle moves through a (uniform and constant) magnetic field $\mathbf{B}$, it moves in a circle (if the initial velocity is $\perp \mathbf{B}$) or in a helix at constant speed v. Indeed, the magnetic field performs no mechanical work on the particle, so its kinetic energy — and hence its speed — remains constant; instead, the magnetic force changes the velocity's direction. Let the magnetic field $\mathbf{B}$ point in the z direction, then the components $v^{x,y,z}$ of the particle's velocity obey differential equations

$$\begin{aligned} m\frac{dv^x}{dt} &= Q(\mathbf{v}\times\mathbf{B})^x &= +QBv^y, \\ m\frac{dv^y}{dt} &= Q(\mathbf{v}\times\mathbf{B})^y &= -QBv^x, \\ m\frac{dv^z}{dt} &= Q(\mathbf{v}\times\mathbf{B})^z &= 0. \end{aligned} \tag{142}$$

Solving these equations, we get

$$v^z = \text{const} \tag{143}$$

while

$$\begin{aligned} v^x(t) &= v^\perp \sin(\omega t + \phi_0), \\ v^y(t) &= v^\perp \cos(\omega t + \phi_0), \end{aligned} \tag{144}$$

where

$$\omega = \frac{QB}{m} \tag{145}$$

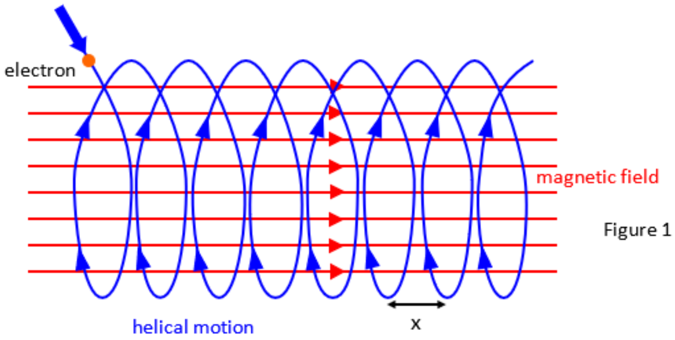
while $v^\perp$ and ϕ_0 depend on the initial conditions. For $v^z = 0$, eqs. (144) describe *uniform motion in a circle of radius*

$$R = \frac{v^\perp}{\omega} = \frac{mv^\perp}{QB}. \tag{146}$$

The direction of this motion is clockwise for $QB > 0$ and counterclockwise of $QB < 0$.

Note that while the radius (146) of the circular motion depends on the particle's speed, the frequency (145) depends only on the magnetic field and on the particles charge-to-mass ratio. This fact used by the *cyclotron* accelerators of protons and nuclei, so the circular motion in a uniform and constant magnetic field is often called the *cyclotron motion.*

For $v^z \neq 0$, the motion is a superposition of circular motion in the xy plane and uniform linear motion in z direction, so the particle moves in a helix:



In particular, electrons or ions trapped in the Earth's magnetic field move along such helixes.

A relativistic charged particle in a magnetic field follows a similar circular or helical path, but in equations (145) and (146) for its angular velocity or the circle's radius we should replace the mass m with the relativistic inertial mass γm. Indeed, for a force normal to the particle's velocity — such as the magnetic Lorentz force — the Second Law of Newton becomes

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{v})}{dt} = \gamma m\mathbf{a} \tag{147}$$

(where m is the rest mass of the particle). For a circular or helical motion, the centripetal acceleration (in the lab frame) is

$$a_c = \omega \times v^{\perp} \tag{148}$$

without any relativistic corrections, so the centripetal force is

$$F_c = \gamma m\omega v^{\perp}. \tag{149}$$

OOH, the centripetal force is the Lorentz force

$$F_c = F_L = Qv^{\perp}B, \tag{150}$$

thus

$$QBv^{\perp} = \gamma m\omega v^{\perp} \implies \omega = \frac{QB}{\gamma m}. \tag{151}$$

Thus, for a relativistic particle the angular velocity of its circular motion depends not only on the B field and the particle Q/m ratio but also on the particle's relativistic energy E/mc^2. In a cyclotron, this effect causes de-synchronization between the particle's motion and the electric field which accelerates it, so the cyclotrons cannot accelerate protons or nuclei to relativistic speeds. In practice, the highest speed of a particle accelerated by a cyclotron is about 20% of c.

As to the radius of the particle's circle or helix,

$$R = \frac{v^\perp}{\omega} = \frac{\gamma m v^\perp}{QB} = \frac{p^\perp}{QB}. \tag{152}$$

Note that this radius depends only in the relativistic momentum of the particle (or rather, the component $p^\perp$ of this momentum in the directions $\perp$ **B**) but not on its rest mass. Consequently, any charged particle born in a high-energy collision with the same Q and the same $p^\perp$ would move in a circle (or helix) of the same radius in a given magnetic field. For example, in a detector tracking the particle's motion in $B = 1$ Tesla, any particle of charge $+e$, — be it a positron, a proton, a muon, a pion, *etc.*, — and having $p^\perp = 300$ MeV/c would move in a circle or helix of radius $R = 1$ meter. Consequently, by following the particle's track in this detector we can measure its relativistic momentum.

Eq. (152) is particularly important to the synchrotrons — accelerators where the particles follow a fixed track in a vacuum pipe surrounded by magnets, whose B field is adjusted to the particle's speed so as to keep turning along the track's curvature: In light of eq. (152), the synchrotron needs

$$B = \frac{p_{\text{particle}} = \gamma m v}{QR}. \tag{153}$$

The phase of the accelerating electric field is also adjusted to synchronize with the particle's motion, hence the name *synchro*tron.

The maximal momentum to which a synchrotron is capable of accelerating particles follows from the synchrotron's radius of curvature R and the maximal strength B_{max} of its magnets,

$$p_{\text{max}} = QB_{\text{max}}R. \tag{154}$$

For example, the LHS has length of 27 km, but some of that length is taken by the straight sections, so the circular arcs where the magnets makes the protons turn have curvature radius $R \approx 2800$ m. The maximal magnetic field produced by the superconducting magnets is $B \approx 8.4$ Tesla, hence the maximal momenta or a proton circulating around LHC is

$$p_{\max}[\text{in GeV/c}] \;=\; \frac{c}{10^9\ \text{V}} \times B_{\max} \times R \;=\; 7060, \tag{155}$$

which corresponds to maximal proton energy

$$E_{\max} \;=\; c\sqrt{p_{\max}^2 + m_p^2 c^2} \;\approx\; c p_{\max} \;=\; 7060\ GeV. \tag{156}$$

And indeed, the current stage of LHS collides 2 proton beams of energy 7 TeV each.

To make a more powerful accelerator producing particles — be they protons or anything else — of higher relativistic momentum, one needs a bigger curvature radius and/or more powerful magnets. For example, CERN is planning to build the next synchrotron that would be 90 km long and hopes to develop 18 Tesla superconducting magnets to manage 50 TeV protons.

PS: Eq. (155) gives the maximal relativistic momenta of a particle going around a synchrotron regardless of whether that particle is a proton, and antiproton, a muon, or anything else, as long as its charge is $\pm e$ and it lives long enough to be accelerated. But for the electron and positron synchrotrons there is a much tighter limit due to synchrotron radiation. As we saw in my notes on radiation by moving charges, eq. (206), for a light particle moving at a very high γ, for each turn around a synchrotron it loses a notable fraction of its energy,

$$\frac{\Delta E}{E} \;\approx\; \frac{Q^2}{3\epsilon_0 mc^2} \times \frac{\beta^3\gamma^3}{R} \;=\; \frac{e^2}{3\epsilon_0 c} \times \frac{p^3}{(mc)^4 R}\,. \tag{157}$$

For a given relativistic momentum, this loss is proportional to m^{-4}, and that's why its much larger for the electrons or positrons than for heavier particles. Consequently, for a future electron-positron synchrotron we would need $R \propto p_{\max}^3$ in order to to keep the energy loss per turn down to some reasonable limit like 2%. For example, a $125+125$ GeV Higgs factory would need to have radius at least 3 times larger than LEP, for the total length of 80 km or longer.