ELECTROMAGNETIC WAVES

Wave Equation and Plane Waves

In one space dimension, the wave equation has general form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Psi(x,t) = 0, \tag{1}$$

where $\Psi(x,t)$ is some kind of quantity that happens to obey this equation: a displacement of a stretched string, a current in a cable, whatever... The most general solution of the 1D wave equation is a superposition of a pulse — of any profile — traveling to the right at speed v and another pulse traveling to the left at the same speed, thus

$$\Psi(x,t) = f_1(x-vt) + f_2(x+vt) \tag{2}$$

for any two functions f_1 and f_2 of a single argument. Of particular importance are the harmonic (AKA monochromatic) waves

$$\Psi(x,t) = \mathrm{Re}\Big(A_1\times\exp(ikx-i\omega t)\Big) + \mathrm{Re}\Big(A_2\times\exp(-ikx-i\omega t)\Big), \tag{3}$$

where $\omega = 2\pi f$ is the (angular) frequency and

$$k = \frac{2\pi}{\text{wavelength}} = \frac{\omega}{v} \tag{4}$$

is the *wave number.*

In three space dimensions, the wave equation becomes

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Psi(x,y,z;t) = 0, \tag{5}$$

or in components

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = 0. \tag{6}$$

Unlike the 1D wave equation (1), the 3D wave equation does not have a simple general solution like (2); instead, we have a wide variety of solutions, much wider than in 1D.

However, the particularly important solutions — the *harmonic plane waves* — have rather simple form

$$\Psi(\mathbf{r},t) \;=\; \mathrm{Re}\Big(A\,\exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t)\Big) \tag{7}$$

where $\mathbf{k}$ is the *wave vector* of magnitude $|\mathbf{k}| = \omega/v$ and any direction we like. Such waves are called *plane* waves because the *wave fronts* — the surfaces of constant phase

$$\text{phase} \;=\; \mathbf{k}\cdot\mathbf{r} \,-\, \omega t \,+\, \arg(A) \;=\; \text{const} \tag{8}$$

— are planes $\perp$ to the wave vector $\mathbf{k}$. Also, these wave fronts happen to move in the direction of $\mathbf{k}$ at the wave speed v, as illustrated at this page.

It is easy to verify that the harmonic plane wave (7) obey the 3D wave equation (5) as long as

$$|\mathbf{k}| \;=\; \frac{\omega}{v} \quad\Longleftrightarrow\quad \mathbf{k}^2 \;=\; \frac{\omega^2}{v^2}\,. \tag{9}$$

Indeed, the space and time derivatives act on the

$$\exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;=\; \exp(ik_x x + ik_y y + ik_z z - i\omega t), \tag{10}$$

by multiplying it by $+i\times$ component of $\mathbf{k}$ or $-i\omega$,

$$\nabla^j \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;=\; ik^j \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t), \tag{11}$$

$$\frac{\partial}{\partial t}\exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;=\; -i\omega * \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t), \tag{12}$$

hence

$$\nabla^2 \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;=\; -\mathbf{k}^2 * \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t), \tag{13}$$

$$\frac{\partial^2}{\partial t^2}\exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;=\; -\omega^2 * \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t), \tag{14}$$

and therefore

$$\begin{aligned}\left(\nabla^2 \,-\, \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t) \;&=\; \left(-\mathbf{k}^2 \,+\, \frac{\omega^2}{v^2}\right) * \exp(i\mathbf{k}\cdot\mathbf{r} \,-\, i\omega t)\\ &=\; 0 \quad \text{for } \mathbf{k}^2 \;=\; \frac{\omega^2}{v^2}\,.\end{aligned} \tag{15}$$

Electromagnetic Wave Equation

For simplicity, let's focus on the electromagnetic fields in the vacuum or in a linear, isotropic, and uniform medium where

$$\mathbf{D}(\mathbf{r},t) \;=\; \epsilon\epsilon_0\mathbf{E}(\mathbf{r},t), \qquad \mathbf{B}(\mathbf{r},t) \;=\; \mu\mu_0\mathbf{H}(\mathbf{r},t), \tag{16}$$

for constant ϵ and μ. In the absence of any free charges or conduction currents, the Maxwell equations for the EM fields become

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 0, && \text{(M1)}\\
\nabla\cdot\mathbf{B} &= 0, && \text{(M2)}\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, && \text{(M3)}\\
\nabla\times\mathbf{B} &= +\frac{1}{v^2}\frac{\partial\mathbf{E}}{\partial t}, && \text{(M4)}\\
\text{where } \frac{1}{v^2} &\stackrel{\text{def}}{=} \mu\mu_0\epsilon\epsilon_0\,; && (17)
\end{aligned}$$

this specific coefficient in eq. (M4) comes from

$$\nabla\times\mathbf{B} \;=\; \mu\mu_0\nabla\times\mathbf{H} \;=\; \mu\mu_0\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \;=\; \mathbf{0}+\epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \;=\; \mu\mu_0\epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\,. \tag{18}$$

Equations (M3) and (M4) are coupled first-order differential equations for the electric and magnetic fields. We can decouple these equations by applying an extra curl:

$$\nabla\times(\nabla\times\mathbf{E}) \;=\; \nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \;=\; -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \;=\; -\frac{\partial}{\partial t}\left(\frac{1}{v^2}\frac{\partial\mathbf{E}}{\partial t}\right) \;=\; -\frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{19}$$

$$\nabla\times(\nabla\times\mathbf{B}) \;=\; \nabla\times\left(\frac{1}{v^2}\frac{\partial\mathbf{E}}{\partial t}\right) \;=\; \frac{1}{v^2}\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) \;=\; \frac{1}{v^2}\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \;=\; -\frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2}\,. \tag{20}$$

At the same time, a double curl of a vector field $\mathbf{V}$ is related to its Laplacian according to

$$\nabla\times(\nabla\times\mathbf{V}) \;=\; \nabla(\nabla\cdot\mathbf{V}) \;-\; \nabla^2\mathbf{V}. \tag{21}$$

In particular, for the electric and magnetic fields which have zero divergences according to eqs. (M1–2), we have

$$\nabla\times(\nabla\times\mathbf{E}) \;=\; -\nabla^2\mathbf{E}, \qquad \nabla\times(\nabla\times\mathbf{B}) \;=\; -\nabla^2\mathbf{B}. \tag{22}$$

Consequently, the decoupled second-order equations (19) and (20) become

$$\begin{aligned} \nabla^2\mathbf{E}(x,y,z,t) &= +\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\,\mathbf{E}(x,y,z,t), \\ \nabla^2\mathbf{B}(x,y,z,t) &= +\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\,\mathbf{B}(x,y,z,t). \end{aligned} \tag{23}$$

In other words, every component $E_x, E_y, E_z, B_x, B_y, B_z$ of the electric or magnetic field obeys the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\text{component}(x,y,z,t) \;=\; 0 \tag{24}$$

with the wave speed

$$v \;=\; \frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}}\,. \tag{25}$$

In particular, in the vacuum this speed of EM waves is

$$v_{\text{vac}} \;=\; \frac{1}{\sqrt{\epsilon_0\mu_0}} \;=\; \text{light speed } c \;=\; 299\,792\,458 \text{ m/s}. \tag{26}$$

Back in Maxwell's time, the measurements of speed of light in the vacuum were a few percent off, about $3.15\cdot 10^8$ m/s. The Coulomb constant $1/(4\pi\epsilon_0)$ was also a few percent off, so plugging it into eq. (26), Maxwell got $v \approx 3.11\cdot 10^8$ m/s. The sheer coincidence between the experimental speed of light in the vacuum and the theoretical speed of the EM waves (also in the vacuum) immediately suggested to Maxwell that light is an electromagnetic wave. Later, with better measurements, both speeds were corrected by a few percent, but the Maxwell's conclusion stands: the light is an electromagnetic wave.

In a non-vacuum transparent medium, the light is also an EM wave but moving at a lesser speed

$$v \;=\; \frac{c}{\sqrt{\mu\epsilon}} \;=\; \frac{c}{n} \tag{27}$$

where $n = \sqrt{\mu\epsilon}$ is the *index of refraction* of the medium at hand. The refraction here refers to the bending of light rays as they cross from one transparent medium into another; the degree of such bending depends on the refraction indices n_1 and n_2 of the two media, hence the name 'refraction index'.

Although Maxwell's identification of light as a kind of EM wave was rather persuasive (as there were a lot more similarities than just the speed of the wave), the ultimate proof of Maxwell's theory required a different kind of an EM wave that could be generated by an oscillating electric current and detected by some equipment specifically sensitive to the oscillating electric field. In 1886, Heinrich Rudolf Hertz did experimentally generate and detect such EM waves; they were called the *Hertzian waves* for a couple of decades but eventually got renamed the *radio waves*.

Since then, may other kinds of EM waves were discovered, or rather identified as EM waves: the microwaves, the infrared and the ultraviolet light-like waves, the X-rays, and the gamma-rays. All of these rays are electromagnetic waves, but of very different frequencies. This Wikipedia article gives a basic overview of the electromagnetic spectrum.

Plane Electromagnetic Waves

In a harmonic plane electromagnetic wave, both the electric and the magnetic fields have the same frequency ω and the same wave vector $\mathbf{k}$, thus

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\Big(\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)\Big), \\ \mathbf{B}(\mathbf{r},t) &= \mathrm{Re}\Big(\vec{\mathcal{B}}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)\Big), \end{aligned} \tag{28}$$

for some complex amplitude vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$. To obey the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\mathbf{E}(\mathbf{r},t)\\ \mathbf{B}(\mathbf{r},t)\end{pmatrix} = 0 \tag{29}$$

for the EM waves in vacuum, we need

$$\mathbf{k}^2 - \frac{\omega^2}{v^2} = 0 \quad\Longrightarrow\quad \mathbf{k} = \frac{\omega}{v}\hat{\mathbf{k}} \tag{30}$$

for some unit vector $\hat{\mathbf{k}}$.

However, there is more to the Maxwell equations for the EM fields than just the wave equation (29), and this leads to several constraints on the 6 component amplitudes $\mathcal{E}_{x,y,z}$ and $\mathcal{B}_{x,y,z}$. Let's start with the Gauss Law constraints

$$\nabla\cdot\mathbf{E} \;=\; 0 \quad\text{and}\quad \nabla\cdot\mathbf{B} \;=\; 0. \tag{31}$$

When acting on $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, the ∇ acts by multiplying by $i\mathbf{k}$,

$$\nabla\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t) \;=\; i\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{32}$$

hence

$$\nabla\cdot\mathbf{E} \;=\; \nabla\cdot\Big(\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big) \;=\; i(\mathbf{k}\cdot\vec{\mathcal{E}})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t) \tag{33}$$

and likewise

$$\nabla\cdot\mathbf{B} \;=\; \nabla\cdot\Big(\vec{\mathcal{B}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big) \;=\; i(\mathbf{k}\cdot\vec{\mathcal{B}})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t). \tag{34}$$

Thus, to satisfy the Gauss Law constraints (31), the amplitude vectors must obey

$$\mathbf{k}\cdot\vec{\mathcal{E}} \;=\; 0 \quad\text{and}\quad \mathbf{k}\cdot\vec{\mathcal{B}} \;=\; 0. \tag{35}$$

In other words, both the electric and the magnetic amplitude vectors must be $\perp$ to the wave direction $\hat{\mathbf{k}}$. Consequently, *at every place* $\mathbf{r}$ *and every time* t*, the* $\mathbf{E}$ *and* $\mathbf{B}$ *fields of an EM wave are transverse to the wave's direction*; that's why we say that *the electromagnetic waves are transverse waves*.

Next, the time-dependent equations

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}\,, \qquad \nabla\times\mathbf{B} \;=\; +\frac{1}{v^2}\frac{\partial\mathbf{E}}{\partial t}\,. \tag{36}$$

Again, when acting on $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, the ∇ acts by multiplying by $i\mathbf{k}$ while $\partial/\partial t$ acts by multiplying by $-i\omega$. Thus,

$$\nabla\times\mathbf{E} \;=\; i(\mathbf{k}\times\vec{\mathcal{E}})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{37}$$

$$\nabla\times\mathbf{B} = i(\mathbf{k}\times\vec{\mathcal{B}})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{38}$$
$$\frac{\partial}{\partial t}\mathbf{E} = -i\omega\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{39}$$
$$\frac{\partial}{\partial t}\mathbf{B} = -i\omega\vec{\mathcal{B}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{40}$$

so eqs. (36) become

$$i\mathbf{k}\times\vec{\mathcal{E}} = +i\omega\vec{\mathcal{B}}, \qquad i\mathbf{k}\times\vec{\mathcal{B}} = -\frac{i\omega}{v^2}\,\vec{\mathcal{E}}. \tag{41}$$

These two linear equations become consistent with each other for $\mathbf{k}=(\omega/v)\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ is a unit vector in the direction of the wave's propagation. Indeed, plugging this $\mathbf{k}$ into eqs. (41) and dividing both sides of each equation bu $i\omega$, we get

$$\hat{\mathbf{k}}\times\vec{\mathcal{E}} = v\vec{\mathcal{B}}, \qquad \hat{\mathbf{k}}\times\vec{\mathcal{B}} = -\frac{1}{v}\vec{\mathcal{E}}, \tag{42}$$

and these two equations are equivalent for $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ that are $\perp\hat{\mathbf{k}}$. Indeed:

$$\begin{aligned} &\hat{\mathbf{k}}\times\vec{\mathcal{B}} = -\frac{1}{v}\vec{\mathcal{E}} \implies \vec{\mathcal{E}} = -v\hat{\mathbf{k}}\times\vec{\mathcal{B}} \implies \\ \implies\ &\hat{\mathbf{k}}\times\vec{\mathcal{E}} = -v\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\vec{\mathcal{B}}) = -v(\hat{\mathbf{k}}\cdot\vec{\mathcal{B}})\hat{\mathbf{k}} + c\vec{\mathcal{B}} = 0 + v\vec{\mathcal{B}} \quad \langle\!\langle\text{because } \vec{\mathcal{B}}\perp\hat{\mathbf{k}}\rangle\!\rangle, \end{aligned} \tag{43}$$

and likewise

$$\begin{aligned} &\hat{\mathbf{k}}\times\vec{\mathcal{E}} = v\vec{\mathcal{B}} \implies \vec{\mathcal{B}} = \frac{1}{v}\hat{\mathbf{k}}\times\vec{\mathcal{E}} \implies \\ \implies\ &\hat{\mathbf{k}}\times\vec{\mathcal{B}} = \frac{1}{v}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\vec{\mathcal{E}}) = \frac{1}{v}(\hat{\mathbf{k}}\cdot\vec{\mathcal{E}})\hat{\mathbf{k}} - \frac{1}{v}\vec{\mathcal{E}} = 0 - \frac{1}{v}\vec{\mathcal{E}} \quad \langle\!\langle\text{because } \vec{\mathcal{E}}\perp\hat{\mathbf{k}}\rangle\!\rangle. \end{aligned} \tag{44}$$

Thus, according to either equation, the magnitudes of the electric and the magnetic amplitudes are related as

$$|\vec{\mathcal{B}}| = \frac{1}{v}\,|\vec{\mathcal{E}}|\,, \tag{45}$$

while their directions in the plane $\perp\hat{\mathbf{k}}$ differ by 90°.

Moreover, eqs. (42) relating the electric and the magnetic amplitudes have no phase factors other than the $\pm$ signs. Consequently,

$$\begin{aligned} v\mathbf{B}(\mathbf{r},t) &= \mathrm{Re}\Big(v\vec{\mathcal{B}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big) \;=\; \mathrm{Re}\Big((\hat{\mathbf{k}}\times\vec{\mathcal{E}})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big) \\ &= \hat{\mathbf{k}}\times\mathrm{Re}\Big(\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big) \;=\; \hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t), \end{aligned} \tag{46}$$

which means that *at any particular place* $\mathbf{r}$ *and at any particular time* t*, the electric and the magnetic fields of a plane harmonic wave are related as*

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{1}{v}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t). \tag{47}$$

Both fields are transverse to the wave direction $\hat{\mathbf{k}}$ and also $\perp$ to each other! Specifically:

- If you look at the wave such that it comes into your eye (the Optics convention), then the magnetic field points 90° to the left from the electric field.
- But if you look at the wave from the direction of its source so that $\hat{\mathbf{k}}$ points away from you (the particle physics convention), then the magnetic field points 90° to the right of the electric field.

⋆ this web page has a 3D illustration of these directions.

In term of the magnetic strength field $\mathbf{H}$ rather than the induction field $\mathbf{B}$, eq. (47) becomes

$$\mathbf{H}(\mathbf{r},t) \;=\; \frac{1}{Z}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) \tag{48}$$

where

$$Z \;=\; \mu\mu_0 v \;=\; \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} \tag{49}$$

is the *wave impedance* of the medium at hand. In particular, the vacuum has *wave impedance of the free space*

$$Z_0 \;=\; \mu_0 c \;=\; \sqrt{\frac{\mu_0}{\epsilon_0}} \;=\; \frac{1}{\epsilon_0 c} \;\approx\; 377\ \Omega. \tag{50}$$

For a non-vacuum medium, the wave impedance is

$$Z = \sqrt{\frac{\mu}{\epsilon}} \times Z_0 = \frac{\mu}{n} \times Z_0 \tag{51}$$

(where $n = \sqrt{\mu\epsilon}$ is the refraction index), and since most transparent media are non-magnetic, we may approximate $\mu \approx 1$, $n \approx \sqrt{\epsilon}$, and

$$Z \approx \frac{Z_0}{n}. \tag{52}$$

We shall use this approximation in by next set of notes about refraction and reflection of EM waves.

Energy and Momentum of Plane EM Waves

The energy density of the EM fields in a linear medium is

$$u = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} + \tfrac{1}{2}\mathbf{H}\cdot\mathbf{B} = \frac{\epsilon\epsilon_0}{2}\,\mathbf{E}^2 + \frac{\mu\mu_0}{2}\,\mathbf{H}^2, \tag{53}$$

and in a plane wave where the electric and the magnetic fields are transverse ($\mathbf{E} \perp \hat{\mathbf{k}}$ and $\mathbf{H} \perp \hat{\mathbf{k}}$) and related to each other as

$$Z\mathbf{H}(\mathbf{r},t) = \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t), \tag{54}$$

the electric and the magnetic terms in eq. (53) are equal to each other. Indeed, for $\mathbf{E} \perp \hat{\mathbf{k}}$

$$\mathbf{H}^2 = \frac{1}{Z^2}\,(\hat{\mathbf{k}} \times \mathbf{E})^2 = \frac{\epsilon\epsilon_0}{\mu\mu_0}\,\mathbf{E}^2, \tag{55}$$

hence

$$u_{\rm mag} = \frac{\mu\mu_0}{2}\,\mathbf{H}^2 = \frac{\epsilon\epsilon_0}{2}\,\mathbf{E}^2 = u_{\rm el}\,, \tag{56}$$

and therefore

$$u_{\rm net} = \cancel{\tfrac{1}{2}}\,\epsilon\epsilon_0\mathbf{E}^2. \tag{57}$$

For a harmonic plane wave

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\Big(\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)\Big), \tag{58}$$

it's convenient to re-express the wave's energy density (57) in terms of the electric amplitude $\vec{\mathcal{E}}$. Earlier in class — *cf.* my notes — we saw that for a harmonic AC current and voltage

$$I(t) = \mathrm{Re}\left(I_0 e^{-i\omega t}\right), \qquad V(t) = \mathrm{Re}\left(V_0 e^{-i\omega t}\right), \tag{59}$$

the *time-averaged* electric power is

$$\langle P\rangle = \langle IV\rangle = \tfrac{1}{2}\,\mathrm{Re}(I_0^* V_0) = \tfrac{1}{2}\,\mathrm{Re}(I_0 V_0^*). \tag{60}$$

Likewise, in a harmonic plane wave (58) we have

$$\text{time-averaged } \left\langle \mathbf{E}^2\right\rangle = \tfrac{1}{2}\,\mathrm{Re}\big(\vec{\mathcal{E}}^* \cdot \vec{\mathcal{E}}\big) = \tfrac{1}{2}\left|\vec{\mathcal{E}}\right|^2, \tag{61}$$

hence time-averaged energy density (57) of the wave is

$$\langle u\rangle = \frac{\epsilon\epsilon_0}{2}\left|\vec{\mathcal{E}}\right|^2. \tag{62}$$

Next, the energy *flow* density of the plane wave, which obtains from the Poynting vector

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{63}$$

For the harmonically oscillating electric and magnetic fields, the time-average of this Poynting vector is related to the electric and magnetic amplitudes as

$$\langle \mathbf{S}\rangle = \tfrac{1}{2}\,\mathrm{Re}\left(\vec{\mathcal{E}}^* \times \vec{\mathcal{H}}\right), \tag{64}$$

where in light of eq. (54) the magnetic amplitude $\vec{\mathcal{H}}$ follows from the electric amplitude $\vec{\mathcal{E}}$ as

$$\vec{\mathcal{H}} = \frac{1}{Z}\,\hat{\mathbf{k}}\times\vec{\mathcal{E}}. \tag{65}$$

Consequently,

$$\vec{\mathcal{E}}^* \times \vec{\mathcal{H}} = \frac{1}{Z}\,\vec{\mathcal{E}}^* \times (\hat{\mathbf{k}}\times\vec{\mathcal{E}}) = \frac{1}{Z}\Big(\hat{\mathbf{k}}\big(\vec{\mathcal{E}}^*\cdot\vec{\mathcal{E}}\big) - \vec{\mathcal{E}}\big(\vec{\mathcal{E}}^*\cdot\hat{\mathbf{k}}\big)\Big) = \frac{1}{Z}\Big(\hat{\mathbf{k}}\left|\vec{\mathcal{E}}\right|^2 - 0\Big), \tag{66}$$

where the last equality follows from $(\vec{\mathcal{E}}^*\cdot\hat{\mathbf{k}}) = (\vec{\mathcal{E}}\cdot\hat{\mathbf{k}})^* = 0$. Consequently, the time-averaged energy flow density of the plane EM wave is

$$\langle \mathbf{S} \rangle \;=\; \frac{|\vec{\mathcal{E}}|^2}{2Z}\,\hat{\mathbf{k}}. \tag{67}$$

Taking the ratio of this energy flow density to the energy density (62), we get

$$\frac{|\langle \mathbf{S} \rangle|}{\langle u \rangle} \;=\; \frac{|\vec{\mathcal{E}}|^2/2Z}{(\epsilon\epsilon_0/2)|\vec{\mathcal{E}}|^2} \;=\; \frac{1}{Z\epsilon\epsilon_0} \;=\; \frac{1}{\sqrt{\epsilon\epsilon_0\,\mu\mu_0}} \;=\; v_{\mathrm{wave}}\,, \tag{68}$$

and therefore

$$\langle \mathbf{S} \rangle \;=\; \langle u \rangle\;\mathbf{v}_{\mathrm{wave}}\,. \tag{69}$$

In other words, the energy of the plane EM wave moves in space with exactly the same velocity vector $\mathbf{v}_{\mathrm{wave}} = v\hat{\mathbf{k}}$ as the phase fronts, *i.e.* the planes of constant phase, $\mathbf{k}\cdot\mathbf{r} - \omega t =$ const.

A point of terminology: the *intensity* of an EM wave is the (time-averaged) power it transmits per unit of cross-sectional area. In terms of the (time-averaged) Poynting vector $\langle \mathbf{S} \rangle$,

$$I \;=\; \langle \mathbf{S} \rangle \cdot \hat{\mathbf{k}} \;=\; |\langle \mathbf{S} \rangle|\,, \tag{70}$$

hence

$$I \;=\; \frac{|\vec{\mathcal{E}}|^2}{2Z} \quad\text{while}\quad \langle u \rangle \;=\; \frac{I}{v} \;=\; \frac{nI}{c}\,. \tag{71}$$

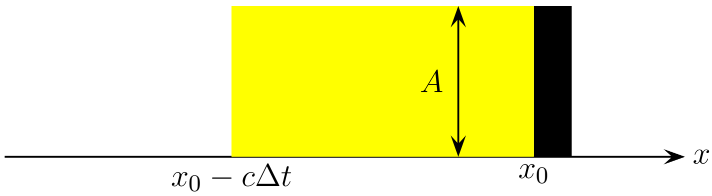
Finally, consider the momentum density of the EM wave,

$$\mathbf{g} \;=\; \mathbf{D}\times\mathbf{B} \;=\; \epsilon\epsilon_0\mu\mu_0\,\mathbf{S} \;=\; \frac{1}{v^2}\,\mathbf{S} \;=\; \frac{n^2}{c^2}\,\mathbf{S}\,. \tag{72}$$

After time-averaging over the wave's period, this momentum density becomes

$$\langle \mathbf{g} \rangle \;=\; \frac{n^2}{c^2}\,\langle \mathbf{S} \rangle \;=\; \frac{n^2}{c^2}\,I\hat{\mathbf{k}}. \tag{73}$$

Like any EM momentum, the wave's momentum can be transferred to a mechanical momentum of some body which absorbs or reflects the EM wave, thus exerting a *radiation pressure* on that body. For an example, consider a plane EM wave in the vacuum traveling in $+\hat{\mathbf{x}}$ and hitting a perfect absorber at some location x_0. Look at the fields in the volume between $x_1 = x_0 - c\Delta t$ and $x_2 = x_0$ in the x direction and of the same cross-section A as the absorber in the (y, z) directions.



(74)

During time Δt, the fields initially located in this volume would travel to the absorber and disappear, while the net momentum of these fields

$$\mathbf{p}_{\text{net}} \;=\; \text{volume} * \mathbf{g} \;=\; Ac\Delta t * \frac{I\hat{\mathbf{x}}}{c^2} \tag{75}$$

would be transferred to the absorber. This momentum transfer amounts to the radiation force

$$\mathbf{F} \;=\; \frac{\mathbf{p}_{\text{net}}}{\Delta t} \;=\; A * \frac{I\hat{\mathbf{x}}}{c}\,, \tag{76}$$

and since this force is proportional to the cross–sectional area A, there is the *radiation pressure*

$$P \;=\; \frac{F_x}{A} \;=\; \frac{I}{c}\,. \tag{77}$$

Note: this is a light pressure on a perfect absorber. The pressure on the perfect reflector is twice that, $P = 2(I/c)$, but only if the light hits the reflector head-on and reflected back where it came from. Indeed, in this set up, during the time Δt the radiation in the yellow volume $A \times c\Delta t$ on the diagram (74) woupld be reflected back rather than absorbed, so its momentum would reverse its direction. Consequently

$$\Delta\mathbf{p}_{\text{rad}} \;=\; -2 * \text{volume} * \mathbf{g} \;=\; -2Ac\Delta t * \frac{I\hat{\mathbf{x}}}{c^2}, \tag{78}$$

hence

$$\Delta\mathbf{p}_{\text{reflector}} \;=\; -\Delta\mathbf{p}_{\text{rad}} \;=\; +2A\Delta t\frac{I}{c}\,\hat{\mathbf{x}}, \tag{79}$$

so the net radiation force on the reflector is

$$\mathbf{F} = \frac{\delta\mathbf{p}}{\Delta t} = 2A\frac{I}{c}\,\hat{\mathbf{x}} \tag{80}$$

and the pressure is

$$P = \frac{F_x}{A} = 2\frac{I}{c}\,. \tag{81}$$

Historically, the light pressure was predicted by Maxwell in 1862 and experimentally discovered by Pyotr Lebedev in 1900. This was a hard experiment for the times, and could only be done in a very good vacuum to avoid gas pressure disturbances on the absorber being heated by the light it absorbs. Several physicists tried to measure the light pressure before Lebedev, and they all had problem with gas pressure effects in a poor vacuum that were much bigger than the radiation pressure they were tried to measure.

Polarizations of EM Waves

Both electric and magnetic fields of a plane EM wave are linearly related to the electric amplitude vector $\vec{\mathcal{E}}$, which is a complex vector in the 2D plane $\perp$ to the wave direction $\hat{\mathbf{k}}$. Hence, all superpositions of waves with the same frequency ω and wave vector $\mathbf{k}$ follow from the superpositions

$$\vec{\mathcal{E}}_{\text{net}} = \alpha_1\vec{\mathcal{E}}_1 + \alpha_2\vec{\mathcal{E}}_2 \tag{82}$$

of such 2D complex vectors. In this section, we shall see how it works, and how to decompose a general amplitude vector $\vec{\mathcal{E}}$ into two independent wave *polarizations*. But to simplify our notations, let's focus on the waves traveling in the positive z direction, $\hat{\mathbf{k}} = (0,0,+1)$. Consequently, the amplitude vectors of all such waves have form

$$\vec{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, 0) \tag{83}$$

with 2 independent complex components $\mathcal{E}_x$ and $\mathcal{E}_y$. Depending on the relative phases — and also relative magnitudes — of these two components, an EM wave can be linearly polarized, circularly polarized, or elliptically polarized.

Linear polarizations

Linear polarizations (AKA planar polarizations) of the EM waves obtain when the complex amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ have the same phase (up to a sign). In general, the linearly polarized waves have

$$\mathcal{E}_x = \mathcal{E}_0 \times \cos\phi, \qquad \mathcal{E}_y = \mathcal{E}_0 \times \sin\phi \tag{84}$$

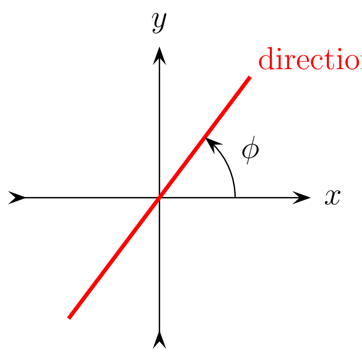
for some real angle ϕ, so when the electric field oscillates in time and space,

$$\mathbf{E}(z,t) = \mathrm{Re}\big(\vec{\mathcal{E}}\, e^{ikz-i\omega t}\big), \tag{85}$$

we get

$$\begin{aligned}\mathbf{E}(z,t) &= (\cos\phi, \sin\phi, 0) * \mathrm{Re}\big(\mathcal{E}_0\, e^{ikz-i\omega t}\big)\\ &= |\mathcal{E}_0| * (\cos\phi, \sin\phi, 0) * \cos(kz-\omega t+\delta)\\ &\quad \text{where } \delta = \arg(\mathcal{E}_0).\end{aligned} \tag{86}$$

Thus, in a linearly polarized wave, the electric field always points in the same direction $(\cos\phi, \sin\phi, 0)$ (modulo the overall sign), namely along the line in the (x,y) plane making angle ϕ with the x axis,



(87)

That's why such polarizations are called *linear*. The same polarizations are called *planar* after the shape of the 3D plot of the electric wave $\mathbf{E}(z)$ (for any fixed time t): Such a plot is restricted to a single 2D plane, spanning the z axis and the red line on the above diagram, for example this plot.

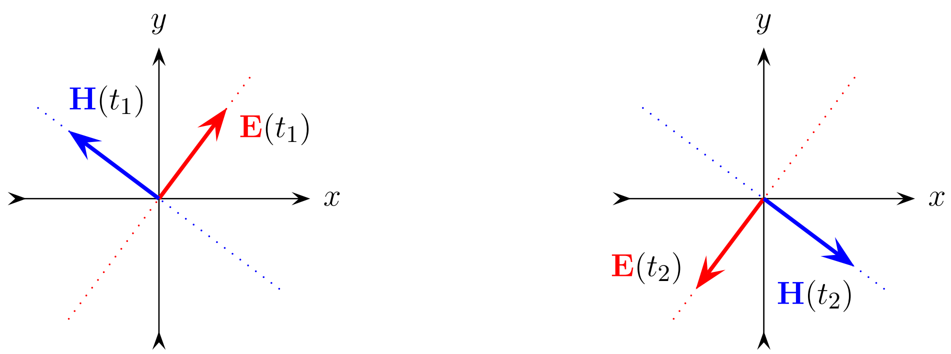
As to the magnetic field of a linearly polarized EM wave,

$$\vec{\mathcal{H}} = \frac{1}{Z}\hat{\mathbf{k}}\times\vec{\mathcal{E}} = \frac{\mathcal{E}_0}{Z}(0,0,1)\times(\cos\phi,\sin\phi,0) = \frac{\mathcal{E}_0}{Z}(-\sin\phi,+\cos\phi,0), \tag{88}$$

hence

$$\mathbf{H}(z,t) = \frac{|\mathcal{E}_0|}{Z}*(\sin\phi,+\cos\phi,0)*\cos(kz-\omega t+\delta). \tag{89}$$

In other words, the magnetic fields oscillates with the same phase as the electric field, but its direction is rotated 90° counterclockwise (in the (x,y) plane) from the electric field's direction. Here are snapshots of the electric and the magnetic fields at two instances of time:



(90)

Note: on this diagram, the $+z$ direction of the wave is towards your face, that's why the magnetic field points 90° to the left the electric field. If you were looking at the field from the opposite direction of the wave's source, the magnetic field would point 90° to the *right* of the electric field. This web page has an animated 3D diagram that clarifies the relative directions of the two fields.

Circular polarizations

In a circularly polarized wave, the complex amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ have similar magnitudes but their phases differ by 90°, $\mathcal{E}_y = \pm i\mathcal{E}_x$ and hence

$$\vec{\mathcal{E}} = \frac{\mathcal{E}_0}{\sqrt{2}}(1,\pm i,0). \tag{91}$$

Consequently, the x and the y components of the electric field $\mathbf{E}(z,t)$ oscillate with phases differing by 90°:

$$E_x(z,t) = \frac{|\mathcal{E}_0|}{\sqrt{2}} \times \cos(kz - \omega t + \delta), \tag{92}$$

$$\begin{aligned} E_y(z,t) &= \frac{|\mathcal{E}_0|}{\sqrt{2}} \times \cos\big(kz - \omega t + \delta \pm \tfrac{\pi}{2}\big) \\ &= \mp\frac{|\mathcal{E}_0|}{\sqrt{2}} \times \sin(kz - \omega t + \delta). \end{aligned} \tag{93}$$

Thus, the electric field $\mathbf{E}$ keeps constant magnitude $|\mathbf{E}| = |\mathcal{E}_0|/\sqrt{2}$, but its direction moves in a circle in the (x, y) plane,

$$\text{direction}(\mathbf{E}) = \pm(\omega t - kz - \delta). \tag{94}$$

Here is a 3D animated illustration from wikipedia.

The two circular polarizations — one with $\vec{\mathcal{E}}_+ \propto (1, +i, 0)$ and the other with $\vec{\mathcal{E}}_- \propto (1, -i, 0)$ — correspond to the two opposite direction of the $\mathbf{E}$ field's rotation. But which direction of rotation we call 'right' and which we call 'left' depends on a convention:

- In both conventions, we look at the electric field vector $\mathbf{E}(t)$ as a function of time at a fixed location $\mathbf{r}$.
- In the Optics convention, we look at the incoming wave — the unit wave vector $\hat{\mathbf{k}}$ points into your eye.
- But in the Particle Physics convention, we look at the outgoing wave, with the $\hat{\mathbf{k}}$ vector pointing away from you.

⋆ Consequently, the same physical direction of rotation that appears clockwise (right) in one convention would appear counterclockwise (left) in the other convention, and vice verse.

In particular, for the wave traveling in the $+z$ direction, looking at the (x, y) plane drawn on a horizontal piece of paper from above corresponds to the Optics convention: the wave travels up, towards your eyes. In this convention, the positive direction of angles in the (x, y) plane is counterclockwise (left), while eq. (94) tells us that the direction of $\mathbf{E}$ moves in the positive direction for $\vec{\mathcal{E}}_+ \propto (1,+i,0)$ and in the negative direction for $\vec{\mathcal{E}}_- \propto (1,-i,0)$. Consequently,

$$\begin{aligned}&\text{In the Optical convention:}\\ &\vec{\mathcal{E}}_+ \ \propto\ (1,+i,0) \quad \text{is the left circular polarization,}\\ &\vec{\mathcal{E}}_- \ \propto\ (1,-i,0) \quad \text{is the right circular polarization.}\end{aligned} \tag{95}$$

And of course,

$$\begin{aligned}&\text{In the Particle Physics convention, it's the other way around:}\\ &\vec{\mathcal{E}}_+ \ \propto\ (1,+i,0) \quad \text{is the right circular polarization,}\\ &\vec{\mathcal{E}}_- \ \propto\ (1,-i,0) \quad \text{is the left circular polarization.}\end{aligned} \tag{96}$$

The reason for this particular convention in the particle physics is that a circularly polarized plane EM wave corresponds to a beam of photons of definite helicity

$$\lambda \stackrel{\text{def}}{=} \hat{\mathbf{k}}\cdot\mathbf{Spin} \quad (\text{in units of } \hbar). \tag{97}$$

For a photon, the two allowed values of its helicity are $+1$ and -1 (but not 0), and its convenient to call a photon with $\lambda = +1$ as polarized right while a photon with $\lambda = -1$ as polarized left.

Altogether, we have the following correspondence table for the circular polarizations:

λ	Particle	Optics	Equation for $\vec{\mathcal{E}}$
$+1$	right	left	$i\hat{\mathbf{k}}\times\vec{\mathcal{E}}_+ = +\vec{\mathcal{E}}_+$
-1	left	right	$i\hat{\mathbf{k}}\times\vec{\mathcal{E}}_- = -\vec{\mathcal{E}}_-$

The last column here helps to write down the electric amplitudes $\vec{\mathcal{E}}$ for the 2 circular polarizations for a general direction $\hat{\mathbf{k}}$ of the plane wave: The two $\vec{\mathcal{E}}$'s are eigenvectors of a Hermitian linear operator $\vec{\mathcal{E}} \to i\hat{\mathbf{k}}\times\vec{\mathcal{E}}$ for its two non-zero eigenvalues $\lambda = \pm 1$. In particular, for $\hat{\mathbf{k}} = (0, 0, 1)$, the eigenvector equation becomes

$$i(0, 0, 1) \times (\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z) \;=\; (-i\mathcal{E}_y, +i\mathcal{E}_x, 0) \;=\; \pm(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z) \tag{98}$$

whose eigenvectors are indeed

$$\vec{\mathcal{E}}_\pm \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\,(1, \pm i, 0). \tag{99}$$

Elliptic polarizations

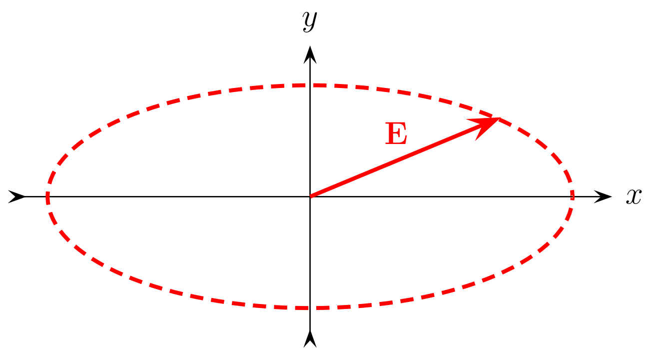
For generic $\mathcal{E}_x$ and $\mathcal{E}_y$ amplitudes of a plane wave — two complex numbers of different magnitudes and different phases, — the $\mathbf{E}(t)$ vector moves along an ellipse in the (x, y) plane, so such polarizations are called *elliptic*. For example, consider

$$\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}_0}{\sqrt{2 - r^2}}\Big(1, \pm i\sqrt{1 - r^2}, 0\Big) \tag{100}$$

and hence

$$\begin{aligned} E_x(z, t) \;&=\; \frac{|\mathcal{E}_0|}{\sqrt{2 - r^2}} \times \cos(\omega t - kz - \delta), \\ E_y(z, t) \;&=\; \pm\sqrt{1 - r^2} \times \frac{|\mathcal{E}_0|}{\sqrt{2 - r^2}} \times \sin(\omega t - kz - \delta). \end{aligned} \tag{101}$$

As a function of time (at a fixed z), the electric field vector with these components indeed follows an ellipse of eccentricity r in the (x, y) plane:



(102)

For this particular ellipse, its major axis is along the x axis while the minor axis is along the

y axis, but one may easily generalize this example to any other axis direction by taking

$$\vec{\mathcal{E}} = \frac{|\mathcal{E}_0|}{\sqrt{2-r^2}} * \Big((\cos\phi, \sin\phi, 0) \pm i\sqrt{1-r^2} * (-\sin\phi, +\cos\phi, 0) \Big). \tag{103}$$

Indeed, any complex 2D vector $\vec{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, 0)$ can be written in the form (103) for some overall complex amplitude $\mathcal{E}_0$, a real angle ϕ, and a real eccentricity r between 0 and 1. For $r = 1$ we get a linear polarization in the direction ϕ, for $r = 0$ we get a circular polarization, and for any other $0 < r < 1$ we get an elliptic polarization.

Polarization bases

Consider superpositions of two (or several) EM waves of the same frequency ω traveling in the same direction $\hat{\mathbf{k}}$ (and hence having the same wave vector $\mathbf{k} = (\omega/v)\hat{\mathbf{k}}$). Since EM fields of a plane wave depend linearly on the electric amplitude vector $\vec{\mathcal{E}}$, superposition of all fields follow from superposition of these amplitude vectors:

$$\begin{aligned} \text{IF} \quad \vec{\mathcal{E}}_{\text{net}} &= \alpha_1 \vec{\mathcal{E}}_1 + \alpha_2 \vec{\mathcal{E}}_2 \\ \text{THEN} \quad \mathbf{E}_{\text{net}}(\mathbf{r},t) &= \alpha_1 \mathbf{E}_1(\mathbf{r},t) + \alpha_2 \mathbf{E}_2(\mathbf{r},t) \\ \text{AND} \quad \mathbf{H}_{\text{net}}(\mathbf{r},t) &= \alpha_1 \mathbf{H}_1(\mathbf{r},t) + \alpha_2 \mathbf{H}_2(\mathbf{r},t). \end{aligned} \tag{104}$$

So let's take a closer look at the linear space of the amplitude vectors $\vec{\mathcal{E}}$.

For a given direction $\hat{\mathbf{k}}$ of the plane waves, their electric amplitudes $\vec{\mathcal{E}}$ are complex two-dimensional vectors in the plane $\perp \hat{\mathbf{k}}$. Consequently, there are two independent polarizations $\mathbf{e}_1$ and $\mathbf{e}_2$, and all the amplitudes are linear combination of these polarizations,

$$\textbf{any} \quad \vec{\mathcal{E}} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \quad \text{for some complex } \alpha_1 \text{ and } \alpha_2\,. \tag{105}$$

For example, for a wave traveling in the $+z$ direction, we may use

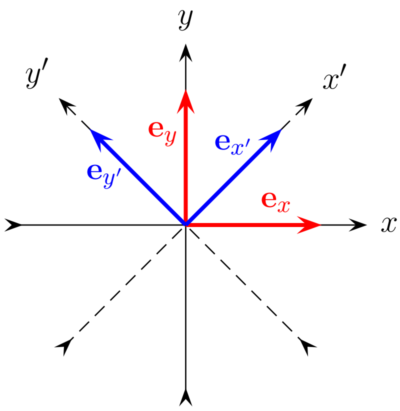
$$\mathbf{e}_x = (1,0,0),\ \mathbf{e}_y = (0,1,0) \quad \vec{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, 0) = \mathcal{E}_x \mathbf{e}_x + \mathcal{E}_y \mathbf{e}_y\,. \tag{106}$$

However, there are infinitely many other bases of a complex 2D vector space. Indeed, take any 2 complex unit vectors that are $\perp \hat{\mathbf{k}}$ and $\perp$ to each other, and they would form a basis:

$$\text{IF} \quad \mathbf{e}_1^* \cdot \mathbf{e}_1 = \mathbf{e}_2^* \cdot \mathbf{e}_2 = 1 \quad \text{AND} \quad \hat{\mathbf{k}} \cdot \mathbf{e}_1 = \hat{\mathbf{k}} \cdot \mathbf{e}_2 = \mathbf{e}_1^* \cdot \mathbf{e}_2 = 0$$

$$\text{THEN for any } \vec{\mathcal{E}} \perp \hat{\mathbf{k}}: \quad \vec{\mathcal{E}} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$$
$$\text{where } \alpha_1 = \mathbf{e}_1^* \cdot \vec{\mathcal{E}} \text{ and } \alpha_2 = \mathbf{e}_2^* \cdot \vec{\mathcal{E}}. \tag{107}$$

In particular, any pair of *real* unit vectors $\mathbf{e}_1 \perp \mathbf{e}_2$ (in the plane $\perp \hat{\mathbf{k}}$) forms a *basis of linear polarizations.* Indeed, a plane wave with amplitude $\alpha_1 \mathbf{e}_1$ for a real vector $\mathbf{e}_1$ is linearly polarized, and so is the wave with amplitude $\alpha_2 \mathbf{e}_2$, but their superposition may have any polarization we like, linear, circular, or elliptic, depending on the complex coefficients α_1 and α_2. Earlier in these notes, we have seen how this works in the $\mathbf{e}_x, \mathbf{e}_y$ basis (for wave moving in the $z+$ direction), but it would work in exactly the same way for any other pair of linear polarizations $\perp$ to each other. For example, for the same wave direction we may use a basis of $(\mathbf{e}_{x'}, \mathbf{e}_{y'})$ for some coordinate axes (x', y') rotated through some angle relative to (x, y):



$$\text{any } \vec{\mathcal{E}} = \mathcal{E}_x \mathbf{e}_x + \mathcal{E}_y \mathbf{e}_y$$
$$= \mathcal{E}_{x'} \mathbf{e}_{x'} + \mathcal{E}_{y'} \mathbf{e}_{y'} . \tag{108}$$

The 2 circular polarizations

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}(1, +i, 0), \quad \mathbf{e}_- = \frac{1}{\sqrt{2}}(1, -i, 0), \tag{109}$$

also form a basis of all polarizations. For example, any linear polarization is a superposition of the two circular polarizations as

$$\vec{\mathcal{E}} = \mathcal{E}_0(\cos\phi, \sin\phi, 0) = \frac{\mathcal{E}_0}{\sqrt{2}} e^{-i\phi} \mathbf{e}_+ + \frac{\mathcal{E}_0}{\sqrt{2}} e^{+i\phi} \mathbf{e}_- , \tag{110}$$

— note coefficients of equal magnitudes but different phases, — while for any elliptic polarization one generally has

$$\vec{\mathcal{E}} = (\mathbf{e}_+^* \cdot \vec{\mathcal{E}})\mathbf{e}_+ + (\mathbf{e}_-^* \cdot \vec{\mathcal{E}})\mathbf{e}_- \tag{111}$$

where the two coefficient generally have both different magnitudes and different phases.

In principle, one may also use a pair of elliptic polarizations as a basis, but this is rarely done. In practice, one chooses a basis according to the available filters selecting a particular polarization of light, while the light of the other polarization is either absorbed or redirected somewhere else. Most such filters select either a linear polarization along some axis, or a specific circular polarization, right or left.

A perfect polarizer transmit 100% of the slected polarization's energy, thile the other polarization is absorbed or redirected in its entirety, so nothing passes through. So suppose an EM wave polarized along a complex unit vector $\mathbf{e}_0$, — *i.e.*, having amplitude $\vec{\mathcal{E}} = \mathcal{E}_0\mathbf{e}_0$, — goes through a filter preferring the polarization $\mathbf{e}_1$. In this case, the EM wave emerging from the filter has amplitude

$$\vec{\mathcal{E}}' \;=\; \mathbf{e}_1\bigl(\mathbf{e}_1^*\vec{\mathcal{E}}\bigr)\,: \tag{112}$$

the direction of its polarization is $\mathbf{e}' = \mathbf{e}_1$, precisely as specified by the filter, while its magnitude is

$$\mathcal{E}' \;=\; (\mathbf{e}_1^*\cdot\mathbf{e}_0)\mathcal{E}_0\,. \tag{113}$$

Consequently, the intesitity ration of the fitered wave to the initial wave is

$$\frac{I}{I_0} \;=\; \frac{|\mathcal{E}'|^2}{|\mathcal{E}_0|^2} \;=\; \left|\mathbf{e}_1^*\cdot\mathbf{e}_0\right|^2. \tag{114}$$

In particular, when the filter selects a planar polarization and the initial wave is also planarly polarized — hence both unit vectors $\mathbf{e}_1$ and $\mathbf{e}_0$ are real, —

$$\frac{I}{I_0} \;=\; \cos^2\bigl(\text{angle between } \mathbf{e}_1 \text{ and } \mathbf{e}_0\bigr). \tag{115}$$

This is the *Malus Law*, discovered by Étienne–Louis Malus back in 1808.

Another useful rule is that when the light from a completely unpolarized source — like an incandecent light bulb or the Sun — goes through any polarizing filter, the light after the filter is 100% polarized but has only 50% of its initial intensity. But when that light goes through a second filter, its intensity then follows from the Malus Law (115) or its generalization (114).