RELATIVISTIC ACTION for a CHARGED PARTICLE

In these notes I am going to derive the equation of motion for a relativistic charged particle in EM fields — and hence the EM forces on the particle — from the least action principle, and I am going to do it in a manifestly relativistic way. In particular, the action will be manifestly Lorentz invariant.

As a warm-up exercise, let me start with a free particle not subject to EM or other forces. In spacetime terms, the motion of the particle is described by its worldline, so the action is a *functional* of the worldline. This functional $S[\text{worldline}]$ should be invariant under all symmetries of the theory, so for a relativistic particle S should be invariant under the Lorentz symmetries. The simplest Lorentz invariant functional of a worldline — and the only such such functional which does not involve higher derivatives — is the net proper time along the worldline,

$$S \;=\; A\times\int\limits_{\text{worldline}} d\tau \tag{1}$$

for some constant coefficient A. In a moment, we shall see that getting the right energy and momentum of the particle calls for $A=-mc^2$.

Indeed, let's express the action (1) in the Lagrangian form

$$S \;=\; \int dt\, L(\mathbf{r},\mathbf{v}). \tag{2}$$

Since $d\tau=dt/\gamma$, it follows that the Lagrangian of a free relativistic particle has form

$$L(\mathbf{v}) \;=\; \frac{A}{\gamma(\mathbf{v})} \;=\; A\times\sqrt{1-(\mathbf{v}/c)^2}\,. \tag{3}$$

Consequently, the canonical momentum of the particle is

$$\mathbf{p}_{\text{can}} \;=\; \frac{\partial L}{\partial\mathbf{v}} \;=\; -\frac{A}{c^2}\,\frac{\mathbf{v}}{\sqrt{1-(\mathbf{v}/c)^2}} \;=\; -\frac{A}{c^2}\,\gamma\mathbf{v}. \tag{4}$$

Since the momentum should point in the same direction as the velocity, we need a negative A. Specifically, if we let $A=-mc^2$, then the canonical momentum becomes the relativistic momentum $\mathbf{p} = \gamma m \mathbf{v}$. Thus,

$$\text{the action} \quad \mathrm{S} \;=\; -mc^2 \int\limits_{\text{worldline}} d\tau, \tag{5}$$

$$\text{the Lagrangian} \quad \mathrm{L} \;=\; -mc^2\sqrt{1 - v^2/c^2}, \tag{6}$$

$$\text{the momentum} \quad \mathbf{p} \;=\; +m\gamma\mathbf{v} \;=\; \frac{m\mathbf{v}}{\sqrt{1-(v/c)^2}}\,. \tag{7}$$

Next, consider the relativistic energy and hence the Hamiltonian stemming from the Lagrangian (6). Given the canonical momentum (7), the energy function obtains as

$$E \;=\; \mathbf{v}\cdot\mathbf{p} \;-\; L \;=\; \gamma m\mathbf{v}^2 \;+\; \frac{mc^2}{\gamma} \;=\; \frac{mc^2}{\gamma}\left(\gamma^2\beta^2 \,+\, 1 \,=\, \gamma^2\right) \;=\; mc^2 \times \gamma(\mathbf{v}), \tag{8}$$

and this is precisely the relativistic energy we have obtained earlier in these notes. As to the Hamiltonian, we need to re-express this energy as a function of the canonical momentum rather than the velocity, as as we have seen before,

$$E^2(\mathbf{v}) \;=\; c^2\mathbf{p}^2(\mathbf{v}) \;+\; (mc^2)^2, \tag{9}$$

thus the Hamiltonian

$$H(\mathbf{p}) \;=\; +\sqrt{c^2\mathbf{p}^2 \,+\, (mc^2)^2}\,. \tag{10}$$

Charged relativistic particle in EM background

Now consider a charged particle interacting with some electromagnetic fields. In these notes, we are concerned with the particle's motion rather than the EM fields it produces, so let's treat the EM fields and potentials as a fixed background.

For a non-relativistic charged particle, the Lagrangian is

$$L(\mathbf{r},\mathbf{v}) \;=\; \frac{m\mathbf{v}^2}{2}\;-\;qV(\mathbf{r})\;+\;\frac{q}{c}\,\mathbf{A}(\mathbf{r})\cdot\mathbf{v}\qquad\langle\!\langle\,\text{in Gauss units}\,\rangle\!\rangle, \tag{11}$$

so we may write the net action as

$$S \;=\; S_{\text{free}}\;+\;S_{\text{EM}} \tag{12}$$

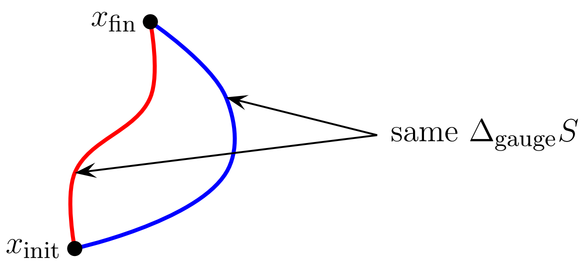
where S_{free} is the action of a free non-relativistic particle, while

$$\begin{aligned} S_{\text{EM}} \;&=\; -\frac{q}{c}\int dt\,\Big(cV(\mathbf{r})\;-\;\mathbf{A}(\mathbf{r})\cdot\mathbf{v}\Big)\;=\;-\frac{q}{c}\int\Big(cV(\mathbf{r})dt\;-\;\mathbf{A}(\mathbf{r})\cdot d\mathbf{r}(t)\Big)\\ &=\;-\frac{q}{c}\int\limits_{\text{worldline}} A_\mu(x(\tau))\,dx^\mu(\tau)\;=\;-\frac{q}{c}\int\limits_{\text{worldline}} d\tau\,A_\mu(x(\tau))\,\frac{dx^\mu}{d\tau}\,. \end{aligned} \tag{13}$$

is the action for interaction with the EM fields. The bottom line of eq. (13) is manifestly Lorentz invariant, so in a relativistic theory we should keep the interaction action exactly as in eq. (13) without any corrections. On the other hand, the free-particle action for a relativistic particle should be changed to (5), so the net action becomes

$$S \;=\; \int\limits_{\text{worldline}} d\tau\left(-mc^2\;-\;\frac{q}{c}\,A_\mu(x(\tau))\,\frac{dx^\mu}{d\tau}\right). \tag{14}$$

The action (14) is not quite gauge invariant, but its gauge variance depends only on the initial and the final positions of the particle but not on a particular worldline between them:



(15)

Indeed, under a gauge transform $\Lambda(x)$, the potential 4–vector $A^\mu(x)$ changes by

$$\Delta A_\mu(x) \;=\; -\partial_\mu\Lambda(x), \tag{16}$$

hence

$$\Delta A_\mu(x(\tau)) \times \frac{dx^\mu}{d\tau} \;=\; -\left.\frac{\partial\Lambda}{\partial x^\mu}\right|_x (\tau) \times \frac{dx^\mu}{d\tau} \;=\; -\frac{d}{d\tau}\Lambda(x(\tau)), \tag{17}$$

and therefore

$$\Delta S \;=\; +\frac{q}{c}\int d\tau\, \frac{d\Lambda(x(\tau))}{d\tau} \;=\; \frac{q}{c}\Big(\Lambda(x_{\rm fin}) \;-\; \Lambda(x_{\rm init})\Big), \tag{18}$$

regardless of the specific worldline between $x_{\rm init}$ and $x_{\rm fin}$. According to the least action principle, the classical worldline minimizes the action functional among all possible worldlines beginning at a given $x_{\rm init}$ and ending at the same $x_{\rm fin}$. In light of eq. (18), $\Delta_{\rm gauge}S$ is exactly the same for all such worldlines, so the worldline that minimizes S in one gauge would also minimize the $S + \Delta_{\rm gauge}S$ in any other gauge. *And that's how despite the gauge dependence of the action* (14)*, the equation of motion for the charged particle turns out to be gauge invariant.*

Now let's derive that equation of motion from the action (14). Instead of going through the Euler–Lagrange formalism which breaks Lorentz symmetry by treating the time t as a special variable, let's use the least action principle in a manifestly Lorentz-covariant way. That is, take an infinitesimal variation of the particle's worldline,

$$x^\mu(\tau) \;\to\; x^\mu(\tau) \;+\; \delta x^\mu(\tau), \tag{19}$$

calculate the first infinitesimal variation of the action (14), and demand that it vanishes for any $\delta x^\mu(\tau)$. Or rather, for any $\delta x^\mu(\tau)$ which vanishes at the beginning at the end of the worldline since the starting and the ending points should be fixed when minimizing the action; this will allow us to integrate by parts without worrying about the boundary terms.

Since the values of of the proper time τ at the beginning and at the end of a worldline may differ from a worldline to a worldline, let's re-parametrize the worldlines some generic parameter s rather than τ. In other world, instead of $x^\mu(\tau)$ we now have $x^\mu(s)$ where s monotonically increases along the worldline and has fixed values $s = 0$ at the beginning and

$s = 1$ at its end,

$$x^\mu(s=0) \;=\; x^\mu_{\rm init}\,, \qquad x^\mu(s=1) \;=\; x^\mu_{\rm fin}\,, \tag{20}$$

so the infinitesimal variations

$$x^\mu(s) \;\to\; x^\mu(s) \;+\; \delta x^\mu(s) \tag{21}$$

are constrained to

$$\delta x^\mu(s=0) \;=\; \delta x^\mu(s=1) \;=\; 0. \tag{22}$$

In terms of this re-parametrization, the action (14) becomes

$$S \;=\; \int\limits_0^1 ds \Big(-mc^2 \frac{d\tau}{ds} \;-\; \frac{q}{c}\, A_\mu(x(s))\, \frac{dx^\mu}{ds} \Big) \tag{23}$$

where

$$\left(c\frac{d\tau}{ds} \right)^2 \;=\; \frac{dx_\mu}{ds}\,\frac{dx^\mu}{ds}\,. \tag{24}$$

Under an infinitesimal variation (21),

$$\delta\left(c\frac{d\tau}{ds} \right)^2 \;=\; 2\frac{dx_\mu}{ds} \times \delta\left(\frac{dx^\mu}{ds} \right) \;=\; 2\left(\frac{dx_\mu}{d\tau}\,\frac{d\tau}{ds} \;=\; u_\mu\,\frac{d\tau}{ds} \right) \times \frac{d\delta x^\mu}{ds} \;=\; 2\frac{d\tau}{ds} \times u_\mu\,\frac{d\delta x^\mu}{ds}\,. \tag{25}$$

On the other hand,

$$\delta\left(c\frac{d\tau}{ds} \right)^2 \;=\; 2c^2\,\frac{d\tau}{ds} \times \delta\left(\frac{d\tau}{ds} \right), \tag{26}$$

hence

$$c^2\delta\left(\frac{d\tau}{ds} \right) \;=\; u_\mu\,\frac{d\delta x^\mu}{ds}\,. \tag{27}$$

At the same time,

$$\begin{aligned}
\delta\left(A_\nu(x(s))\,\frac{dx^\nu}{ds}\right) \;&=\; \delta\Big(A_\nu(x(s))\Big)\times\frac{dx^\nu}{ds}\;+\;A_\nu(x(s))\times\delta\left(\frac{dx^\nu}{ds}\right)\\
&=\;\Big(\partial_\mu A_\nu\times\delta x^\nu\Big)\times\frac{dx^\nu}{ds}\;+\;A_\nu\times\frac{d\delta x^\nu}{ds}\\
&=\;\left(\partial_\mu A_\nu\,\frac{dx^\nu}{ds}\right)\times\delta x^\mu\;+\;A_\mu\times\frac{d\delta x^\mu}{ds}\,.
\end{aligned}\tag{28}$$

Plugging these variations into eq. (23) for the action, we have

$$\begin{aligned}
\Delta S \;&=\; \int\limits_0^1 ds\left(-mu_\mu\times\frac{d\delta x^\mu}{ds}\;-\;\frac{q}{c}\left(\partial_\mu A_\nu\,\frac{dx^\nu}{ds}\right)\times\delta x^\mu\;-\;\frac{q}{c}\,A_\mu\times\frac{d\delta x^\mu}{ds}\right)\\
&\qquad\langle\!\langle\,\text{integrating the first and the third terms by parts}\,\rangle\!\rangle\\
&=\;\left[-mu_\mu\times\delta x^\mu\;-\;\frac{q}{c}A_\mu(x(s))\times\delta x^\mu\right]\Bigg|_{s=1}^{s=1}\\
&\qquad\langle\!\langle\,\text{the boundary term, which vanishes because } \delta x^\mu(s)=0 \text{ for } s=0 \text{ or } s=1\,\rangle\!\rangle\\
&\qquad+\int\limits_0^1 ds\bigg(+m\frac{du_\mu}{ds}\times\delta x^\mu\;-\;\frac{q}{c}\left(\partial_\mu A_\nu\,\frac{dx^\nu}{ds}\right)\times\delta x^\mu\\
&\qquad\qquad+\;\frac{q}{c}\left(\frac{dA_\mu(x(s))}{ds}\;=\;\partial_\nu A_\mu\times\frac{dx^\nu}{ds}\right)\times\delta x^\mu\bigg)\\
&=\;\int\limits_0^1 ds\,\delta x^\mu(s)\times\left(m\frac{du_\mu}{ds}\;-\;\frac{q}{c}\,\partial_\mu A_\nu\,\frac{dx^\nu}{ds}\;+\;\frac{q}{c}\,\partial_\nu A_\mu\,\frac{dx^\nu}{ds}\right).
\end{aligned}\tag{29}$$

This first variation of the action must vanish for any $\delta x^\mu(s)$, so in the integral on the bottom line of eq. (29), the expression inside $(\cdots)$ must vanish for all $0<s<1$. Thus, we should have

$$m\frac{du_\mu}{ds}\;-\;\frac{q}{c}\,\partial_\mu A_\nu\,\frac{dx^\nu}{ds}\;+\;\frac{q}{c}\,\partial_\nu A_\mu\,\frac{dx^\nu}{ds}\;=\;0\tag{30}$$

and hence

$$m\,\frac{du_\mu}{ds}\;=\;\frac{q}{c}\Big(\partial_\mu A_\nu-\partial_\nu A_\mu\;=\;F_{\mu\nu}\Big)\frac{dx^\nu}{ds}\tag{31}$$

where the $F_{\mu\nu}$ tensor is for $x=x(s)$.

Finally, let's go back to x^μ as functions of the proper time τ rather than some generic parameter s. Multiplying both sides of eq. (31) by $ds/d\tau$, we have

$$\frac{du_\mu}{ds} \times \frac{ds}{d\tau} = \frac{du^\mu}{d\tau}, \qquad \frac{dx^\nu}{ds} \times \frac{ds}{d\tau} = \frac{dx^\nu}{d\tau} = u^\nu, \tag{32}$$

and consequently

$$m\,\frac{du_\mu}{d\tau} = \frac{q}{c}\,F_{\mu\nu}(x(\tau)) \times u^\nu. \tag{33}$$

This is a manifestly Lorentz-covariant equation of motion for the relativistic charged particles. With raised index μ, it becomes

$$\frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = \frac{q}{c}\,F^{\mu\nu}u_\nu\,. \tag{34}$$

In terms of the Minkowski force 4–vector K^μ, eq. (34) means

$$K^\mu = \frac{q}{c}F^{\mu\nu}u_\nu \tag{35}$$

and hence ordinary force 3–vector

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}, \tag{36}$$

exactly as we saw in my previous set of notes. But now we know why the EM forces have this particular form.