DIPOLE RADIATION

Let's change the subject from the EM radiation by accelerating point charges to the radiation by time-dependent dipole moments, electric or magnetic. In these notes I allow for the general rather than harmonic time-dependence of $\mathbf{p}(t)$ or $\mathbf{m}(t)$. The harmonic time-dependence — such as happens in radio antennas — will be discussed in more detail in my next set of notes.

Let's start with a simple moment of an electric dipole moment: An actual dipole made from two point charges $+Q$ and $-Q$. The charge $-Q$ is stationary at some fixed position — which we take to be the coordinate origin $\mathbf{0}$, while the charge $+Q$ moves around along some path $\mathbf{w}(t)$, hence time-dependent electric dipole moment $\mathbf{p}(t) = Q\mathbf{w}(t)$. The moving charge generates both the generalized Coulomb electric field (which scales with the distance as $1/r^2$) and the radiation field (which scales with the distance as $1/r$), while the stationary charge generates only the Coulomb field. At long distances from the dipole, the net radiation field dominates over the Coulomb fields, thus

$$\begin{aligned} \mathbf{E}_{\text{rad}} &= \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} \\ \begin{matrix}\text{non-relativistic}\\ \text{limit}\end{matrix} &\to -\frac{Q\mu_0}{4\pi}\,\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{a})}{r} \\ &= -\frac{\mu_0}{4\pi}\,\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}})}{r}\,, \end{aligned} \tag{1}$$

where the last equality follows from

$$\mathbf{p} = Q\mathbf{w} \quad\Longrightarrow\quad \dot{\mathbf{p}} = Q\mathbf{v} \quad\Longrightarrow\quad \ddot{\mathbf{p}} = Q\mathbf{a}\,. \tag{2}$$

Likewise, the $1/r$ radiation term in the magnetic field amounts to

$$\mathbf{B}_{\text{rad}} = \frac{\mathbf{n}}{c}\times\mathbf{E}_{\text{rad}} = -\frac{\mu_0}{4\pi c}\,\frac{\mathbf{n}\times(\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}}))}{r} = +\frac{\mu_0}{4\pi c}\,\frac{\mathbf{n}\times\ddot{\mathbf{p}}}{r}\,. \tag{3}$$

Consequently, the leading (in the $r\to\infty$ limit) term in the Poynting vector is

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0}\mathbf{E}_{\text{rad}}\times\mathbf{B}_{\text{rad}} = -\frac{\mu_0}{16\pi^2 c}\,\frac{(\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}}))\times(\mathbf{n}\times\ddot{\mathbf{p}})}{r^2} = +\frac{\mu_0}{16\pi^2 c}\,(\mathbf{n}\ddot{\mathbf{p}})^2\,\frac{\mathbf{n}}{r^2}\,, \tag{4}$$

hence the EM power radiated per unit of solid angle is

$$\frac{dP}{d\Omega} \;=\; R^2\mathbf{n}\cdot\mathbf{S}_{\text{rad}} \;=\; \frac{\mu_0}{16\pi^2 c}\,(\mathbf{n}\times\ddot{\mathbf{p}})^2. \tag{5}$$

But what about electric dipole moments produced by multiple point charges — or by continuous charge distributions — rather than by just two charges? We want to focus on just the dipole moment and its effect without any contribution from the other multipole moments, so we take the ideal dipole limit: zero net charge, finite dipole moment **p**, and negligibly small quadrupole moment, octupole moment, *etc.*, *etc.* Mathematically, this corresponds to the singular charge density

$$\rho(\mathbf{r}) \;=\; -\mathbf{p}\cdot\nabla\delta^{(3)}(\mathbf{r}), \tag{6}$$

which in electrostatics produces the pure-dipole potential

$$\begin{aligned} 4\pi\epsilon_0 V(\mathbf{r}) \;&=\; \iiint\frac{1}{|\mathbf{r}-\mathbf{r}'|}(-\mathbf{p}\cdot\nabla')\delta^{(3)}(\mathbf{r}')\,d^3\mathrm{Vol}' \\ &=\; \iiint\left(+\mathbf{p}\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\delta^{(3)}(\mathbf{r}')\,d^3\mathrm{Vol}' \\ &\iiint\left(\frac{\mathbf{p}\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|^2}\right)\delta^{(3)}(\mathbf{r}')\,d^3\mathrm{Vol}' \\ &=\; \frac{\mathbf{p}\cdot\mathbf{n}}{|\mathbf{r}^2|}\,. \end{aligned} \tag{7}$$

In electrodynamics, the charge density

$$\rho(\mathbf{r},t) \;=\; -\mathbf{p}(t)\cdot\nabla\delta^{(3)}(\mathbf{r}) \tag{8}$$

for the time-dependent ideal dipole moment should be accompanied by the singular current density

$$\mathbf{J}(\mathbf{r},t) \;=\; \dot{\mathbf{p}}(t)\delta^{(3)}(\mathbf{r}) \tag{9}$$

to satisfy the continuity equation

$$-\frac{\partial\rho}{\partial t} \;=\; +\dot{\mathbf{p}}\cdot\nabla\delta^{(3)}(\mathbf{r}) \;=\; \nabla\cdot\mathbf{J}. \tag{10}$$

In your homework set#6, problem 4, I asked you to derive the electric field **E** and the magnetic field **B** for the the charge density (8) and the current density (9). Specifically, you should have verified that

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}-3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}}{r^3}+\frac{\dot{\mathbf{p}}-3(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}}{cr^2}+\frac{\ddot{\mathbf{p}}-(\mathbf{n}\cdot\ddot{\mathbf{p}})\mathbf{n}}{c^2r}\right]_{\text{ret}}, \qquad (11)\\
\mathbf{B}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi}\left[\frac{\mathbf{n}\times\dot{\mathbf{p}}}{r^2}+\frac{\mathbf{n}\times\ddot{\mathbf{p}}}{cr}\right]_{\text{ret}}. \qquad (12)
\end{aligned}$$

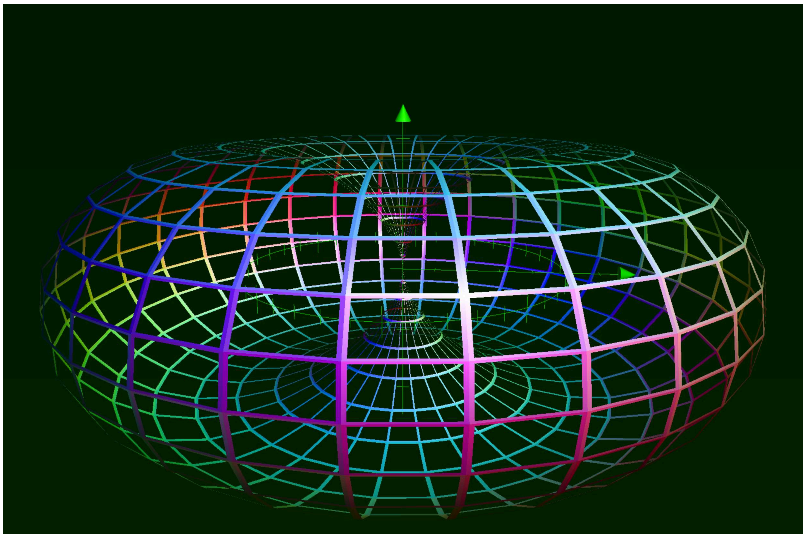
Note: these formulae are exact (for the ideal electric dipole) and include the static, the induction, and the radiation terms. But for our purposes of calculating the EM radiation which reaches all the way to infinity, we need only the radiation terms which scale with the distance as $1/r$, thus

$$\begin{aligned}
\mathbf{E}_{\text{rad}} &= \frac{\mu_0}{4\pi}\,\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}})}{r}, \qquad (13)\\
\mathbf{B}_{\text{rad}} &= -\frac{\mu_0}{4\pi c}\,\frac{\mathbf{n}\times\ddot{\mathbf{p}}}{r}, \qquad (14)
\end{aligned}$$

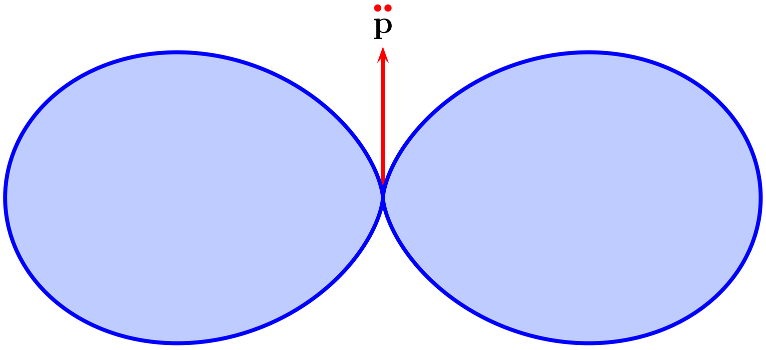
exactly as in the 2-point-charges model. Consequently, exactly as for that model, the EM power emitted into a solid angle Ω and reaching all the way to infinity is

$$\frac{dP}{d\Omega} = R^2\mathbf{n}\cdot\mathbf{S}_{\text{rad}} = \frac{\mu_0}{16\pi^2c}\,(\mathbf{n}\times\ddot{\mathbf{p}})^2. \qquad (5)$$

Note the angular distribution of this power: Taking the direction of the dipole moment's second time derivative $\ddot{\mathbf{p}}$ as our z axis, we have $(dP/d\Omega)\propto\sin^2\theta$. Thus, no power at all is emitted in the direction of the $\ddot{\mathbf{p}}$ or in the opposite direction, while maximal power density goes in the directions $\perp\ \ddot{\mathbf{p}}$. Here is the 3D *radiation power diagram* for the $\ddot{\mathbf{p}}$ pointing vertically up



and here is its vertical cross-section



Finally, the net power emitted by a time-dependent electric dipole moment is

$$P = \oiint \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0(\ddot{\mathbf{p}})^2}{16\pi^2 c} \oiint \sin^2\theta\, d\Omega = \frac{\mu_0(\ddot{\mathbf{p}})^2}{6\pi c}\,, \tag{15}$$

in perfect agreement with the Larmor formula

$$P = \frac{\mu_0 Q^2 \mathbf{a^2}}{6\pi c} \tag{16}$$

for the accelerated charge in the two-charge model. Indeed, the two two-charge model (with one charge being at rest at $\mathbf{r} = \mathbf{0}$), we have $Q\mathbf{w}(t) = \mathbf{p}(t)$, hence $Q\mathbf{a} = \ddot{\mathbf{p}}$, thus perfect agreement between eqs. (15) and (16).

Example: parallel-plate capacitor. For the sake of definiteness, let the capacitor has 2 parallel plates of area $A = 10.0\ \mathrm{cm}^2$ separated by $d = 1.00$ mm of vacuum. The capacitor is slowly charged to voltage $V_0 = 1000$ V, and then the plates are connected to each other through an $R = 1.00\ k\Omega$ resistor. As the capacitor is discharged, it's electric dipole moment changes with time in a non-linear fashion, $\ddot{\mathbf{p}} \neq 0$, so the capacitor emits some EM radiation. The question is: What fraction of the capacitor's initial energy $U_0 = \frac{1}{2}CV_0^2$ is emitted away as EM radiation?

Let's start with the capacitor's discharge through the resistor R. The current I through the resistor, the voltage V across the capacitor, and the charge Q stored in it are related as

$$I(t) = \frac{V(t)}{R} = \frac{Q(t)}{RC} \tag{17}$$

and also

$$\frac{dQ}{dt} = -I = -\frac{Q(t)}{RC}\,. \tag{18}$$

Solving this differential equation, we immediately see that the capacitor is discharged according to

$$Q(t) = Q_0 \times \exp(-t/RC). \tag{19}$$

For the capacitor at hand, the capacitance is

$$C = \epsilon_0 \frac{A}{d} = (8.85 \cdot 10^{-12}\ \mathrm{F/m}) \times \frac{10.0 \cdot 10^{-4}\ \mathrm{m}^2}{1.00 \cdot 10^{-3}\ \mathrm{m}} = 8.85 \cdot 10^{-12}\ \mathrm{F}, \tag{20}$$

so the time constant $\tau = RC$ of the discharge is $8.85 \cdot 10^{-9}$ s, rather fast.

Next, the radiation. The dipole moment of the charged capacitor is $p = Qd$, hence during the discharge

$$\ddot{p}(t) = d\ddot{Q}(t) = d(1/\tau)^2 Q(t) = \frac{d}{(RC)^2}\, Q_0 \exp(-t/RC), \tag{21}$$

so the EM power radiated by the capacitor is

$$P_{\mathrm{rad}} = \frac{\mu_0}{6\pi c}\,(\ddot{p})^2 = \frac{\mu_0 Q_0^2 d^2}{6\pi c (RC)^4}\, \exp(-2t/RC). \tag{22}$$

The net energy radiated by this capacitor obtains by integrating this power over time

$$U_{\text{rad}} \;=\; \int\limits_0^\infty P_{\text{rad}}\,dt \;=\; \frac{\mu_0 Q_0^2 d^2}{6\pi c(RC)^4}\int\limits_0^\infty \exp(-2t/RC)\,dt \;=\; \frac{\mu_0 Q_0^2 d^2}{6\pi c(RC)^4}\times\frac{RC}{2} \;=\; \frac{\mu_0 Q_0^2 d^2}{12\pi c(RC)^3}\,. \tag{23}$$

OOH, the energy initially stored in the capacitor is

$$U_0 \;=\; \frac{CV_0^2}{2} \;=\; \frac{Q_0^2}{2C}\,, \tag{24}$$

hence

$$\frac{U_{\text{rad}}}{U_0} \;=\; \frac{\mu_0 Q_0^2 d^2}{12\pi c(RC)^3}\bigg/\frac{Q_0^2}{2C} \;=\; \frac{\mu_0 C d^2}{6\pi c(RC)^3}\,. \tag{25}$$

Plugging eq. (20) for the capacitance C into this formula, we get

$$\frac{U_{\text{rad}}}{U_0} \;=\; \frac{\mu_0 d^2}{6\pi c R^3}\times\frac{1}{C^2} \;=\; \frac{\mu_0 d^2}{6\pi c R^3}\times\frac{d^2}{\epsilon_0^2 A^2} \tag{26}$$

where

$$\frac{\mu_0}{c\epsilon_0^2} \;=\; \frac{1}{c^3\epsilon_0^3} \;=\; Z_0^3, \tag{27}$$

hence

$$\frac{U_{\text{rad}}}{U_0} \;=\; \frac{1}{6\pi}\left(\frac{Z_0}{R}\right)^3\times\frac{d^4}{A^2} \tag{28}$$

For the capacitor at hand

$$\frac{d^4}{A^2} \;=\; \frac{(1.00\cdot 10^{-3}\ \text{m})^4}{(1.00\cdot 10^{-3}\ \text{m}^2)^2} \;=\; 1.00\cdot 10^{-6} \tag{29}$$

while

$$\left(\frac{Z_0}{R}\right)^3 \;=\; \left(\frac{377\ \Omega}{\underline{100}0\ \Omega}\right)^3 \;=\; 0.377^3 \;=\; 0.0536, \tag{30}$$

hence altogether

$$\frac{U_{\text{rad}}}{U_0} \;=\; \frac{53.6\cdot 10^{-3}}{6\pi}\times 1.00\cdot 10^{-6} \;=\; 2.84\cdot 10^{-9}. \tag{31}$$

We see that only a very small fraction of the capacitor's initial energy got radiated as EM energy. But since the capacitor was never designed as an antenna, this should not be too surprising.

Magnetic Dipole Radiation

We do not have a useful toy model of a magnetic dipole, so let's start with the singular electric current density of the ideal magnetic dipole,

$$\mathbf{J}(\mathbf{r},t)\;=\;\mathbf{m}(t)\times\nabla\delta^{(3)}(\mathbf{r}). \tag{32}$$

The ideal magnetic dipole also has $\rho\equiv 0$, which is compatible with the continuity equation since

$$\nabla\cdot\mathbf{J}\;=\;\mathbf{m}\cdot\big(\nabla\times\nabla\delta^{(3)}(\mathbf{r})\big)\;=\;0. \tag{33}$$

The **E** and **B** fields of this ideal magnetic dipole follow from the current (32) via the Jefimenko equations, but it's easier to re-derive them from scratch starting with the retired vector potential

$$\begin{aligned}\mathbf{A}(\mathbf{r},t)\;&=\;\frac{\mu_0}{4\pi}\iiint\frac{\mathbf{m}(t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}\times\nabla'\delta^{(3)}(\mathbf{r}')\\ &\qquad\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle\\ &=\;\frac{\mu_0}{4\pi}\iiint\left(\nabla'\times\frac{\mathbf{m}(t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}\right)\delta^{(3)}(\mathbf{r}')\\ &=\;\frac{\mu_0}{4\pi}\left[\nabla'\times\frac{\mathbf{m}(t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}\right]_{@\mathbf{r}'=\mathbf{0}}.\end{aligned} \tag{34}$$

On the bottom line here

$$\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\;=\;-\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\;\xrightarrow[\mathbf{r}'=0]{}\;+\frac{\mathbf{n}}{r^2} \tag{35}$$

while the magnetic moment $\mathbf{m}(t_{\rm ret})$ depends on $\mathbf{r}'$ because the retarded time depends on the $\mathbf{r}'$, thus

$$\begin{aligned}\nabla'\times\mathbf{m}(t_{\rm ret})\;&=\;(\nabla' t_{\rm ret})\times\dot{\mathbf{m}}(t_{\rm ret}),\\ \nabla' t_{\rm ret}\;&=\;-\frac{1}{c}\nabla'|\mathbf{r}'-\mathbf{r}|\;=\;-\frac{1}{c}\,\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'}\;\longrightarrow\;+\frac{\mathbf{n}}{c}\,,\\ \text{hence}\quad\nabla'\times\mathbf{m}(t_{\rm ret})\;&=\;+\frac{\mathbf{n}}{c}\times\dot{\mathbf{m}}(t_{\rm ret}),\end{aligned} \tag{36}$$

and therefore

$$\nabla' \times \frac{\mathbf{m}(t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{r}'|} \;=\; \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{m}(t_{\mathrm{ret}}) \;+\; \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla \mathbf{m}(t_{\mathrm{ret}}) \;\xrightarrow[\mathbf{r}'=0]{}\; \frac{\mathbf{n}}{r^2} \times \mathbf{m}(t_{\mathrm{ret}}) \;+\; \frac{\mathbf{n}}{rc} \times \dot{\mathbf{m}}(t_{\mathrm{ret}}). \tag{37}$$

Plugging this formula into eq. (34), we get

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\left(\frac{\mathbf{n}}{r^2} \times \mathbf{m}(t_{\mathrm{ret}}) \;+\; \frac{\mathbf{n}}{rc} \times \dot{\mathbf{m}}(t_{\mathrm{ret}})\right). \tag{38}$$

Note: in this formula, the retarded time is for $\mathbf{r}' = 0$, the fixed location of the ideal magnetic dipole, thus

$$t_{\mathrm{ret}} \;=\; t \;-\; \frac{r}{c}. \tag{39}$$

Next, the electric field $\mathbf{A} = -\frac{\partial \mathbf{A}}{\partial t}$. The time derivative here is taken at fixed $\mathbf{r}$, so the retarded time runs at the same rate as the observer time t. Consequently,

$$\frac{\partial}{\partial t}\mathbf{m}(t_{\mathrm{ret}}) \;=\; \dot{\mathbf{m}}(t_{\mathrm{ret}}) \tag{40}$$

and likewise

$$\frac{\partial}{\partial t}\dot{\mathbf{m}}(t_{\mathrm{ret}}) \;=\; \ddot{\mathbf{m}}(t_{\mathrm{ret}}), \tag{41}$$

hence

$$\mathbf{E}(\mathbf{r},t) \;=\; -\frac{\mu_0}{4\pi}\left(\frac{\mathbf{n}}{r^2} \times \dot{\mathbf{m}}(t_{\mathrm{ret}}) \;+\; \frac{\mathbf{n}}{rc} \times \ddot{\mathbf{m}}(t_{\mathrm{ret}})\right). \tag{42}$$

Finally, the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, where the curl is taken at fixed observer time t but variable retarded time t_{ret}. Consequently,

$$\nabla^i m^j(t_{\mathrm{ret}}) \;=\; (\nabla^i t_{\mathrm{ret}})\dot{m}^j(t_{\mathrm{ret}}) \;=\; -\frac{n^i}{c} * \dot{m}^j(t_{\mathrm{ret}}) \tag{43}$$

and likewise

$$\nabla^i \dot{m}^j(t_{\mathrm{ret}}) \;=\; (\nabla^i t_{\mathrm{ret}})\ddot{m}^j(t_{\mathrm{ret}}) \;=\; -\frac{n^i}{c} * \ddot{m}^j(t_{\mathrm{ret}}). \tag{44}$$

In the context of

$$\mathbf{B}(\mathbf{r},t) \;=\; \nabla\times\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\left[\nabla\times\left(\frac{\mathbf{n}}{r^2}\times\mathbf{m}(t_{\text{ret}})\right) + \nabla\times\left(\frac{\mathbf{n}}{cr}\times\dot{\mathbf{m}}(t_{\text{ret}})\right)\right] \tag{45}$$

this means

$$\begin{aligned}
\left[\nabla\times\left(\frac{\mathbf{n}}{r^2}\times\mathbf{m}(t_{\text{ret}})\right)\right]^i \;&=\; \epsilon^{ijk}\nabla^k\left(\epsilon^{k\ell p}\,\frac{n^\ell}{r^2}\,m^p(t_{\text{ret}})\right)\\
&=\; \epsilon^{ijk}\epsilon^{k\ell p}\left[\left(\nabla^j\frac{n^\ell}{r^2}\right)\,m^p(t_{\text{ret}}) \;+\; \frac{n^\ell}{r^2}\Big(\nabla^j m^p(t_{\text{ret}})\Big)\right]\\
&=\; \left(\delta^{i\ell}\delta^{jp}-\delta^{ip}\delta^{j\ell}\right)\left[\frac{\delta^{j\ell}-3n^jn^\ell}{r^3}*m^p(t_{\text{ret}}) \;+\; \frac{n^\ell}{r^2}*\frac{-n^j}{c}\dot{m}^p(t_{\text{ret}})\right]\\
&\qquad\langle\!\langle\,\text{after some algebra}\,\rangle\!\rangle\\
&=\; \frac{\delta^{ip}-3n^in^p}{r^3}\,m^p(t_{\text{ret}}) \;+\; \frac{\delta^{ip}-n^in^p}{r^2c}\,\dot{m}^p(t_{\text{ret}}),
\end{aligned} \tag{46}$$

and likewise

$$\begin{aligned}
\left[\nabla\times\left(\frac{\mathbf{n}}{rc}\times\dot{\mathbf{m}}(t_{\text{ret}})\right)\right]^i \;&=\; \epsilon^{ijk}\nabla^k\left(\epsilon^{k\ell p}\,\frac{n^\ell}{rc}\,\dot{m}^p(t_{\text{ret}})\right)\\
&=\; \epsilon^{ijk}\epsilon^{k\ell p}\left[\left(\nabla^j\frac{n^\ell}{rc}\right)\,\dot{m}^p(t_{\text{ret}}) \;+\; \frac{n^\ell}{rc}\Big(\nabla^j\dot{m}^p(t_{\text{ret}})\Big)\right]\\
&=\; \left(\delta^{i\ell}\delta^{jp}-\delta^{ip}\delta^{j\ell}\right)\left[\frac{\delta^{j\ell}-2^jn^\ell}{r^2c}*\dot{m}^p(t_{\text{ret}}) \;+\; \frac{n^\ell}{rc}*\frac{-n^j}{c}\ddot{m}^p(t_{\text{ret}})\right]\\
&\qquad\langle\!\langle\,\text{after some algebra}\,\rangle\!\rangle\\
&=\; \frac{-2n^in^p}{r^2c}\,\dot{m}^p(t_{\text{ret}}) \;+\; \frac{\delta^{ip}-n^in^p}{rc^2}\,\ddot{m}^p(t_{\text{ret}}).
\end{aligned} \tag{47}$$

Altogether, we get

$$B^i(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\left[\frac{\delta^{ip}-3n^in^p}{r^3}\,m^p(t_{\text{ret}}) + \frac{\delta^{ip}-3n^in^p}{r^2c}\,\dot{m}^p(t_{\text{ret}}) + \frac{\delta^{ip}-n^in^p}{rc^2}\,\ddot{m}^p(t_{\text{ret}})\right], \tag{48}$$

or in vector notations

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}-3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})}{r^3} + \frac{\dot{\mathbf{m}}-3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{m}})}{r^2c} + \frac{\ddot{\mathbf{m}}-\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{m}})}{rc^2}\right]_{\text{ret}}. \tag{49}$$

Note electric-magnetic symmetry between the ideal dipoles: for an electric dipole $\mathbf{p}(t)$

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}-3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}}{r^3}+\frac{\dot{\mathbf{p}}-3(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}}{cr^2}+\frac{\ddot{\mathbf{p}}-(\mathbf{n}\cdot\ddot{\mathbf{p}})\mathbf{n}}{c^2r}\right]_{\mathrm{ret}}, && (11)\\
\mathbf{B}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi}\left[\frac{\mathbf{n}\times\dot{\mathbf{p}}}{r^2}+\frac{\mathbf{n}\times\ddot{\mathbf{p}}}{cr}\right]_{\mathrm{ret}}, && (12)
\end{aligned}$$

while for a magnetic dipole $\mathbf{m}(t)$

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}-3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})}{r^3}+\frac{\dot{\mathbf{m}}-3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{m}})}{r^2c}+\frac{\ddot{\mathbf{m}}-\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{m}})}{rc^2}\right]_{\mathrm{ret}}, && (49)\\
\mathbf{E}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi}\left[\frac{\mathbf{n}\times\dot{\mathbf{m}}}{r^2}+\frac{\mathbf{n}\times\ddot{\mathbf{m}}}{rc}\right]_{\mathrm{ret}}, && (42)
\end{aligned}$$

and these formulae are clearly related by

$$\mathbf{B}_{\mathrm{el}} = +\frac{1}{c}\mathbf{E}_{\mathrm{mag}}, \qquad \mathbf{E}_{\mathrm{mag}} = -c\mathbf{B}_{\mathrm{mag}}, \qquad \mathbf{m}_{\mathrm{mag}} = c\mathbf{p}_{\mathrm{el}}. \tag{50}$$

That is, the exact EM fields of a magnetic dipole $\mathbf{m}(t)$ obtains by swapping the electric and the magnetic fields (modulo overall signs and factors of c) of an electric dipole $\mathbf{p}(t)=\mathbf{m}(t)/c$. In particular, the power radiated by the time-dependent magnetic dipole is exactly the same as radiated by the electric dipole $\mathbf{p}(t)=\mathbf{m}(t)/c$.

Indeed, in the long-distance (from the dipole) limit, the magnetic dipole's $\mathbf{E}$ and $\mathbf{B}$ fields are dominated by the radiation terms,

$$\begin{aligned}
\mathbf{E}_{\mathrm{rad}} &= -\frac{\mu_0}{4\pi c}\,\frac{\mathbf{n}\times\ddot{\mathbf{m}}}{r},\\
\mathbf{B}_{\mathrm{rad}} &= +\frac{\mu_0}{4\pi c^2}\,\frac{\ddot{\mathbf{m}}-\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{m}})}{r} = -\frac{\mu_0}{4\pi c^2}\,\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{m}})}{r},
\end{aligned}\tag{51}$$

hence the leading terms in the Poynting vector

$$\mathbf{S}_{\mathrm{rad}} = \frac{1}{\mu_0}\mathbf{E}_{\mathrm{rad}}\times\mathbf{B}_{\mathrm{rad}} = \frac{\mu_0}{16\pi^2c^3}\,\frac{(\mathbf{n}\times\ddot{\mathbf{m}})\times(\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{m}}))}{r^2} = \frac{\mu_0\big(\mathbf{n}\times\ddot{\mathbf{m}}\big)^2}{16\pi^2c^3}\,\frac{\mathbf{n}}{r^2}, \tag{52}$$

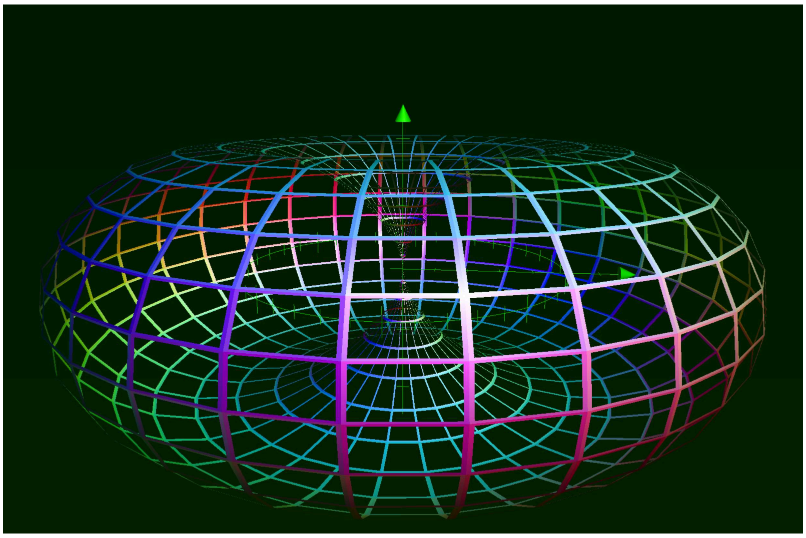
and therefore the EM power radiated per unit of solid angle

$$\frac{dP}{d\Omega}\ =\ r^2\mathbf{n}\cdot\mathbf{S}_{\text{rad}}\ =\ \frac{\mu_0\bigl(\mathbf{n}\times\ddot{\mathbf{m}}\bigr)^2}{16\pi^2c^3}\,. \tag{53}$$

For a magnetic dipole moment $\mathbf{m}(t)$ that has a fixed direction (or rather, a fixed axis, *i.e.* fixed direction modulo overall sign) the angular distribution of the radiated EM power is

$$\frac{dP}{d\Omega}\ =\ \frac{\mu_0\bigl(\ddot{\mathbf{m}}\bigr)^2}{16\pi^2c^3}\times\sin^2\theta \tag{54}$$

where θ is the angle between the dipole moment and the direction towards the observer. This is the same angular distribution as the radiation by an electric dipole, or radiation by a non-relativistic accelerating charged particle: no radiation is emitted along the dipole's axis, while the strongest emission is in the directions ⊥ to the axis.



Finally, the net EM power emitted by the magnetic dipole is

$$P\ =\oiint\frac{dP}{d\Omega}\,d\Omega\ =\ \frac{\mu_0\bigl(\ddot{\mathbf{m}}\bigr)^2}{16\pi^2c^3}\oiint\sin^2\theta\,d\Omega\ =\ \frac{\mu_0\bigl(\ddot{\mathbf{m}}\bigr)^2}{16\pi^2c^3}\times\frac{8\pi}{3}\ =\ \frac{\mu_0\bigl(\ddot{\mathbf{m}}\bigr)^2}{6\pi c^3}\,. \tag{55}$$

Example: Coil Antenna

As an example of EM radiation by a time-dependent magnetic dipole, consider a coil antenna: A coil of $N = 10$ turns wrapped around a wooden frame of area $A = 1.00\ \mathrm{m}^2$. The coil is powered by a harmonic current $I(t) = I_0\cos(\omega t)$ of amplitude $I_0 = 14.1$ A (*i.e.*, $I_{\rm rms} = 10.0$ A) and frequency $(\omega/2\pi) = 5$ MHz. Let's find how much power is radiated by this antenna.

The magnetic moment of this antenna is $m(t) = NAI(t)$, hence

$$\ddot{m} \;=\; -\omega^2 NAI_0 \times \cos(\omega t), \tag{56}$$

and therefore

$$P \;=\; \frac{\mu_0}{6\pi c^3} \times \omega^4 N^2 A^2 I_0^2 \times \cos^2(\omega t), \tag{57}$$

which after averaging over the current's period of oscillation becomes

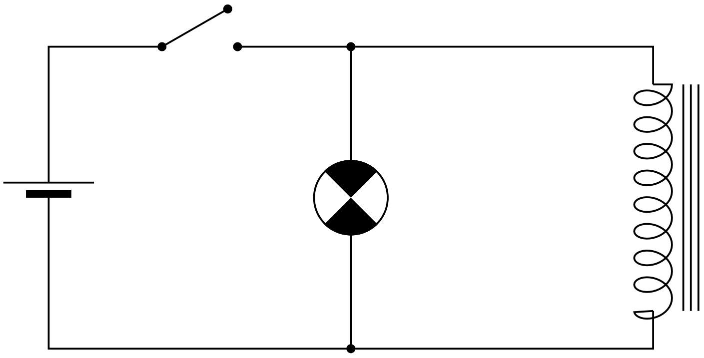
$$\langle P\rangle \;=\; \frac{\mu_0}{6\pi c^3} \times \omega^4 N^2 A^2 \times \left(\frac{I_0^2}{2} \;=\; I_{\rm rms}^2\right). \tag{58}$$

Numerically, for the data at hand, $\langle P\rangle \approx 24$ W.

BTW, the direction of this antenna's dipole moment is ⊥ to the plane of the coil. Consequently, most radiation is emitted within the plane of the coil or close to it, but almost not radiation is emitted ⊥ to that plane. The earliest radio direction finders were used on this fact, plus the *antenna reciprocity theorem*, which says that the receiving antenna sensitivity to incoming signals has exactly the same direction dependence as emission by the same antenna on the same frequency. Thus, the coil antenna would have strongest sensitivity to radio signals coming from the sides (close to the plane of the coil) but almost no sensitivity to signals coming directly from the front or from the back of the antenna. Having a sailor rotate the antenna while he listens to the radio communications from an airplane and noticing the orientation of the antenna when the signals seems to disappear let the crew find out where that plane was (relative to the ship). Here is the WWI era picture.

Example: Inductor Discharge

Remember a freshmen demo of inductor energy: A light bulb is connected in parallel with a honking big inductor to a low-voltage DC power supply.



The inductor coil has much smaller Ohmic resistance than the light bulb, so while the switch is open there is much stronger current through the inductor than through the bulb, so the bulb is barely lit and stays rather dim. But when the switch is suddenly thrown open, the inductor current — which cannot instantly stop — has to go through the bulb. This makes the bulb suddenly flush bright, and then dim out as the energy stored in the inductor drains through the bulb.

Treating the bulb as a resistor with fixed resistance R and the inductor as a perfect inductor, we have (after the switch has been opened):

$$V_{\text{inductor}} = -L \times \frac{dI}{dt} = V_{\text{bulb}} = R \times I, \tag{59}$$

hence

$$\frac{dI}{dt} = -\frac{R}{L} \times I \tag{60}$$

and therefore

$$I(t) = I_0 \exp\bigl(-\tfrac{R}{L}t\bigr). \tag{61}$$

For example, suppose the inductor coil has length $\ell = 20$ cm, $N = 400$ turns, radius

$a = 1.5$ cm, and a soft ferrite core with $\mu = 2000$. Consequently, it's self-inductance is

$$L \;=\; \frac{N\Phi}{I} \;=\; N\mu\mu_0(N/\ell)(\pi a^2) \;=\; 1.42\ \text{H}, \tag{62}$$

so assuming the bulb's resistance $R = 100\ \Omega$, we find the decay rate of the inductor current (61) to be

$$\tau \;=\; \frac{L}{R} \;=\; 0.014\ \text{s}, \tag{63}$$

slow enough to see the bulb's flush but too fast to perceive its duration.

Now let's calculate what fraction of the energy initially stored in the inductor

$$U_0 \;=\; \tfrac{1}{2}LI_0^2 \tag{64}$$

get emitted as EM waves due to rapidly changing magnetic dipole moment of the coil. Without the ferrite core, a current I through the coil gives rise to the net magnetic dipole moment

$$m \;=\; N \times \pi a^2 \times I \tag{65}$$

but the core amplifies this magnetic moment by the factor of μ, thus

$$m(t) \;=\; \mu N(\pi a^2) \times I(t). \tag{66}$$

For the exponentially decaying current (61), we have

$$\ddot{m} \;=\; \mu N(\pi a^2) \times \ddot{I} \;=\; \mu N(\pi a^2) \times \frac{I}{\tau^2} \tag{67}$$

where $\tau = L/R$ is the time constant of the current's decay. Consequently, the EM power is emitted at the rate

$$P = \frac{\mu_0(\ddot{m})^2}{6\pi c^3} = \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^4} \times I^2(t) = \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^4} \times I_0^2 \exp(-2t/\tau). \tag{68}$$

Integrating this power over time, we find the net EM energy emitted by the coil to be

$$\begin{aligned} U_{\text{rad}} &= \int\limits_0^\infty P(t)\,dt = \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^4} \int\limits_0^\infty I_0^2 \exp(-2t/\tau)\,dt \\ &= \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^4} \times \frac{I_0^2\tau}{2} = \frac{I_0^2}{2} \times \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^3}, \end{aligned} \tag{69}$$

or as a fraction of the initial energy stored in the coil,

$$\begin{aligned} \frac{U_{\text{rad}}}{U_0} &= \frac{I_0^2}{2} \times \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^3} \Big/ \frac{I_0^2 L}{2} \\ &= \frac{\mu_0\mu^2 N^2(\pi a^2)^2}{6\pi c^3\tau^3 L} = \frac{\mu_0\mu^2 N^2(\pi a^2)^2 R^3}{6\pi c^3 L^4} \\ &= \frac{\mu_0\mu^2 N^2(\pi a^2)^2 R^3}{6\pi c^3} \times \frac{1}{[N(N/\ell)(\pi a^2)\mu\mu_0]^4} \\ &= \frac{1}{6\pi}\left(\frac{R}{\mu_0 c}\right)^3 \times \left(\frac{\ell^2}{\mu N^3(\pi a^2)}\right)^2. \end{aligned} \tag{70}$$

For the inductor at hand

$$\frac{\ell^2}{\mu N^3(\pi a^2)} = 4.42 \cdot 10^{-8} \tag{71}$$

while

$$\frac{R}{\mu_0 c} = \frac{R}{Z_0} = \frac{100\ \Omega}{377\ \Omega} = 0.265, \tag{72}$$

hence

$$\frac{U_{\text{rad}}}{U_0} = \frac{1}{6\pi} \times (4.42 \cdot 10^{-8})^2 \times (0.265)^3 = 1.94 \cdot 10^{-18}. \tag{73}$$

Similar to the capacitor from an earlier example, the coil radiates only a tiny fraction of its initial energy. But like the capacitor, the coil was never designed to work as an antenna.