1. Two concentric spherical shells carry uniformly distributed electric charges, $+Q$ on the inner shell of radius $a$, and $-Q$ on the outer shell of radius $b > a$. The whole system is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

   (a) Find the angular momentum of the EM fields of this system (relative to the spheres' center).

   (b) Now let's slowly turn off the magnetic field. Find the net torque of the induced electric field on the two spherical shells. Also, find the net angular momentum the two shells would acquire once the magnetic field is turned off.

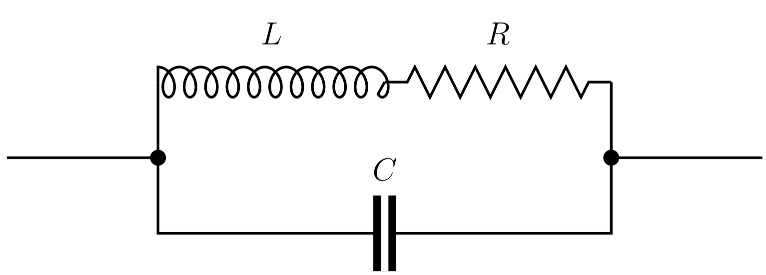
2. Consider a solid iron ball or radius $R$ that carries electric charge $Q$ and uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. Initially, the ball is not rotating.

   (a) Find the net angular momentum (relative to the ball's center) of the EM fields generated by the ball.

   Let's slowly heat up the ball. As the temperature raises, the iron's magnetization decreases, until it completely disappears at the Curie point. For simplicity, assume uniform temperature and hence uniform magnetization of the ball at all times.

   (b) Find the torque on the ball by the electric field induced by decreasing magnetic field. Show that the net angular momentum imparted on the ball in this process is precisely the EM angular momentum you found in part (a).

3. [20 points] Consider a circuit made from a capacitor $C$ and an inductor $L$. The coils of a real inductor have some Ohmic resistance $R$, so we formally represent this circuit as LRC,



(a) Let's start with a free LRC circuit, not connected to anything else. Show that the voltage across this circuit obeys the *dumped oscillator equation*

$$\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right) V(t) \;=\; 0 \tag{1}$$

where

$$\omega_0 \;=\; \frac{1}{\sqrt{LC}} \quad \text{and} \quad \gamma \;=\; \frac{R}{L}\,. \tag{2}$$

For $\gamma < 2\omega_0$, the general solution to this differential equation is

$$V(t) \;=\; V_0 \times \cos(\omega' t + \phi_0) \times e^{-\gamma t/2} \quad \text{for} \quad \omega' \;=\; \sqrt{\omega_0^2 - (\gamma/2)^2}. \tag{3}$$

(b) Show that the current through the circuit depends on time as

$$I(t) \;=\; -\omega_0 C V_0 \times \sin(\omega' t + \phi_0 + \delta) e^{-\gamma t/2} \quad \text{for} \quad \delta \;=\; \arcsin\frac{\gamma}{2\omega_0}\,. \tag{4}$$

Then calculate the energy stored in the circuit and show that it decreases with time as

$$\begin{aligned} U(t) \;&=\; U_0 \times e^{-\gamma t} \times \left[1 \;-\; \frac{\gamma}{2\omega_0}\sin(2\omega' t + \text{const})\right] \\ &\approx\; U_0 \times e^{-\gamma t} \quad \text{for} \quad \gamma \ll \omega_0\,. \end{aligned} \tag{5}$$

Hint:

$$\cos^2(\alpha) \;+\; \sin^2(\alpha + \delta) \;=\; 1 \;-\; \sin\delta \times \sin(2\alpha + \delta). \tag{6}$$

The ratio

$$Q \;=\; \frac{\omega_0}{\gamma} \;=\; \frac{\text{angular frequency}}{\text{energy loss rate}} \tag{7}$$

is called the *quality* of the oscillator such as LRC circuit. The same ratio also governs the strength and the width of the resonance when the LRC circuit is connected to an outside source of harmonic current $I(t) = \hat{I} e^{-i\omega t}$.

(c) Calculate the impedance of the LRC circuit and show that it's frequency dependence has a resonant form

$$Z(\omega) \;=\; \frac{\mathit{smooth\_function}(\omega)}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{8}$$

where the *smooth_function* in the numerator does not have any peaks or troughs for $\omega$ near $\omega_0$.

(d) Calculate the width of the resonance peak (at the level $|Z|^2 = \frac{1}{2}|Z|^2_{\rm peak}$) and show that for $Q \gg 1$ the width is

$$\Delta\omega \;\approx\; \gamma \;=\; \frac{\omega_0}{Q}\,. \tag{9}$$

4. Now a largish reading assignment: §9.1 of Griffith's textbook about waves in 1 space dimension. All this material should be familiar to you from the lower-division Physics classes, but you need to refresh your knowledge of the wave basics before I explain the *electromagnetic* waves in class.

   This reading assignment is not graded. Instead, the following problems will test your understanding of the material.

5. (a) Which of the following would-be waveforms $f(z,t)$ obeys the wave equation and which does not:

$$f_1(z,t) \;=\; A\exp\left(-\frac{(z-vt)^2}{2b^2}\right), \tag{10}$$

$$f_2(z,t) \;=\; Ae^{-z^2/2b^2}\cos(\omega t), \tag{11}$$

$$f_3(z,t) \;=\; \frac{Ab^2}{(z+vt)^2+b^2}\,, \tag{12}$$

$$f_4(z,t) \;=\; A\cos^3(kz)\cos(\omega t) \quad \text{for } k=\frac{\omega}{v}\,, \tag{13}$$

$$f_5(z,t) \;=\; A\cos^5(k(z+vt)). \tag{14}$$

   (b) Show that the *standing wave*

$$f_s(z,t) \;=\; 2A\sin(kz)\cos(\omega t) \tag{15}$$

   does obey the wave equation, then write it as a linear superposition of a wave running to the right and a wave running to the left.

6. [15 points] Two long strings are tied to each other at $z = 0$. The two strings have the same tension $T_1 = T_2 = T$ but different mass densities: the left string (at $z < 0$) is 4 times lighter than the right string (at $z > 0$), $(m/\ell)_1 = \frac{1}{4}(m/\ell)_2$.
An incident harmonic wave $A_i \exp(ik_1 z - i\omega t)$ comes from the left side; at $z = 0$, this wave is partially transmitted to the right side and partially reflected back.
   (a) Write down the wave equations for the two sides and the boundary conditions at $z = 0$.
   (b) Solve the boundary conditions for the amplitudes $A_r$ and $A_t$ of the reflected and the transmitted waves in terms of the incident amplitude $A_i$.
   (c) Calculate the power carried by the incident wave.
   (d) Calculate the powers of the reflected and the transmitted waves. What fraction of the incident power is transmitted to the right side and what fraction is reflected back to the left?

7. A wave on a string is transverse, but in 3D there are two directions transverse to the string, say $x$ and $y$ for the string stretched in $z$ direction. Thus, a general polarized harmonic wave running in $+z$ direction has form

$$x(z,t) \;=\; \mathrm{Re}\big(Ae^{ikz-i\omega t}\big), \quad y(z,t) \;=\; \mathrm{Re}\big(Be^{ikz-i\omega t}\big), \tag{16}$$

for a complex 2D amplitude vector $(A, B)$.
   (a) Suppose the complex amplitudes $A$ and $B$ have the same phase, hence

$$A \;=\; C \times \cos\theta, \quad B \;=\; C \times \sin\theta \tag{17}$$

   for the same complex $C$ and a real angle $\theta$. Such polarizations are called *planar* because the vibrating string always stays in the same plane, namely the plane spanning the $z$ axis and the unit vector $\mathbf{n} = (\cos\theta, \sin\theta, 0)$.
   Prove this.

(b) Now suppose the $A$ and $B$ amplitudes have similar magnitudes but phases different by $\pm 90°$,

$$B = +iA \quad \text{or} \quad B = -iA. \tag{18}$$

Such polarization are called *circular* because every point $z$ of the string moves in a circle parallel to the $(x, y)$ plane.

Prove this.

(c) A circular polarization is called *right* if the circular motion is clockwise and *left* if the motion is counterclockwise. In Optics convention, you look at this motion from the direction of the wave's target, *i.e.* the wave moves towards your eye. For the wave moving in the $+z$ direction, this means looking at the $(x, y)$ plane from the above.

Which of the two circular polarizations (18) is right and which is left?

(d) How would you shake the string in order to produce a circularly polarized wave?