1. Consider a plane EM wave in vacuum propagating in the direction $\hat{\mathbf{k}} = (1,1,1)/\sqrt{3}$. The wave has a planar polarization in the (x,y) plane.
   Write down the electric and the magnetic fields $\mathbf{E}(x,y,z,t)$ and $\mathbf{H}(x,y,z,t)$ of this wave, and make sure to spell out the directions of these fields.

2. Consider the stress tensor $\overleftrightarrow{T}$ of a plane EM wave. For simplicity, assume the wave propagates through the vacuum in the $+\hat{\mathbf{z}}$ direction. Show that regardless of the wave's polarization — planar, circular, elliptic, whatever, — the time-averaged stress tensor is uniform

$$\left\langle \overleftrightarrow{T} \right\rangle = \frac{I}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \left\langle T^{ij} \right\rangle = -\frac{I}{c} * \hat{k}^i \hat{k}^j \,. \tag{1}$$

   where I is the wave's intensity.

3. [15 points] Now consider the radiation pressure of an EM wave on a perfect head-on reflector. In class I have calculated $P = 2I/c$ using momentum conservation, but now let's do a more careful calculation of the actual EM forces on the reflector. Let the reflector fill up the $z > 0$ half-space, then

$$\text{for } z < 0: \quad \mathbf{E}(z,t) = \vec{\mathcal{E}}_i \exp(+ikz - i\omega t) + \vec{\mathcal{E}}_r \exp(-ikz - i\omega t) \tag{2}$$

   for some incident and reflected amplitudes $\vec{\mathcal{E}}_i$ and $\vec{\mathcal{E}}_r$, while

$$\text{for } z > 0: \quad \mathbf{E}(z,t) = 0. \tag{3}$$

   Note: a perfect reflector is a perfect conductor, so there should be some oscillating surface current density at its surface,

$$\mathbf{J}(z,t) = \mathbf{K}(t)\delta(z) = \vec{\mathcal{K}} e^{-i\omega t} \delta(z). \tag{4}$$

   (a) Spell out the the magnetic field on both sides of $z = 0$ and the boundary conditions for the $\mathbf{E}$ and $\mathbf{H}$ fields at $z = 0$.

(b) Solve these boundary conditions and find the surface current and the magnetic field in terms of the incident amplitude $\vec{\mathcal{E}}_i$.

(c) Find the time-average magnetic force on the surface current. Then show that this force corresponds to the radiation pressure $P = 2I/c$.

4. A light wave of intensity I_0 is emitted by an un-polarized source such as a lightbulb. But then the light passes through a sequence of polarizing filters, and your task is to find the intensity of light as it passes through each filter. Consider two sequences:

   (a) Four linearly polarized filters: the first filter's polarization axis is vertical, while the second, third and fourth filters' axes are rotated 45°, 90°, and 135° clockwise from the vertical.

   (b) The second sequence has four different filters: The first filter is circularly polarized left, the second filter is linearly polarized in a vertical direction, the third filter is circularly polarized right, and the fourth filter is linearly polarized in a horizontal direction.

5. Consider the sunlight reflected off the surface of a lake (which we assume to be perfectly smooth and horizontal). The incident sunlight is not polarized: the intensity of sunlight polarized in the incident plane is the same as for the sunlight polarized normal to the incident plane. But the two polarization have different reflectivities $R_\perp \neq R_\parallel$, so the reflected sunlight is partially polarized. It's *degree of polarization* is

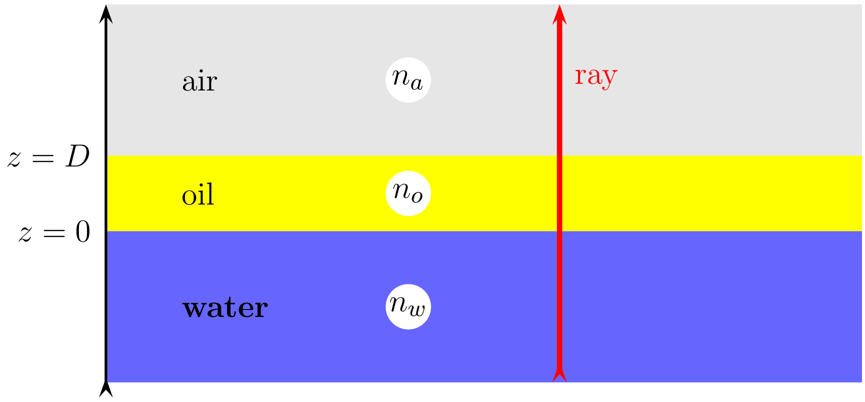
$$\Pi \stackrel{\text{def}}{=} \frac{I_\perp - I_\parallel}{I_\perp + I_\parallel} = \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel}, \tag{5}$$

   where the second equality assumes unpolarized incident sunlight.

   My notes on reflection and refraction have explicit formulae for the reflectivities $R_\perp$ and $R_\parallel$. Your task is to plot these reflectivities $R_\perp(\alpha)$ and $R_\parallel(\alpha)$ and the degree of polarization $\Pi(\alpha)$ as functions of the incidence angle α (*i.e.*, the angle between the Sun in the sky and the zenith). Assume $n_{\text{air}} \approx 1$ and $n_{\text{water}} \approx 1.33$.

   Use Mathematica or equivalent software to generate your plots, or just calculate them numerically using your favorite programming language. In any case, make sure to attach your code to the plots so I can see how you did them.

6. [20 points] Consider a light ray crossing head-on from one medium to another through a third medium, for example from water to air through an oil slick on the water's surface,



(6)

Approximate the light ray as a vertical plane wave, and remember that there is reflection at both air-oil and oil-water interfaces, thus

$$\begin{aligned} \text{for } z < 0: \quad \mathbf{E}(z,t) &= \vec{\mathcal{E}}_1 \exp(+ik_w z - i\omega t) \;+\; \vec{\mathcal{E}}_2 \exp(-ik_w z - \subset \omega t), \\ \text{for } 0 < z < D: \quad \mathbf{E}(z,t) &= \vec{\mathcal{E}}_3 \exp(+ik_o z - i\omega t) \;+\; \vec{\mathcal{E}}_4 \exp(-ik_o z - \subset \omega t), \\ \text{but for } z > D \quad \mathbf{E}(z,t) &= \vec{\mathcal{E}}_5 \exp(+ik_a z - i\omega t) \;+\; 0, \end{aligned} \tag{7}$$

where

$$k_w = (\omega/c) n_w \,, \quad k_o = (\omega/c) n_o \,, \quad k_a = (\omega/c) n_a \,, \tag{8}$$

$\vec{\mathcal{E}}_1$ is the incident amplitude, $\vec{\mathcal{E}}_2$ is the reflected amplitude, and $\vec{\mathcal{E}}_5$ is the transmitted amplitude.

(a) Write down the magnetic fields in all three media and the boundary conditions for both electric and magnetic fields at both interfaces $z = 0$ and $z = D$. Remember: all three media are non-magnetic, and there are no surface currents at the boundaries.

(b) Solve the boundary conditions for the reflection coefficient $r = \vec{\mathcal{E}}_2/\vec{\mathcal{E}}_1$ and the transmission coefficient $t = \vec{\mathcal{E}}_5/\vec{\mathcal{E}}_1$. Note that these coefficients depend not only on the 3 refraction indices n_a, n_o, and n_w, but also on the optical thickness $k_o \times D$ of the oil layer.

Hint: Use Mathematica or similar software. Trying to do this by hand would likely lead to mistakes and/or a lot of wasted time.

(c) Calculate the transmissivity T and the reflectivity R of this interface and make sure that $R + T = 1$.

7. [20 points] Finally, an example of a non-planar electromagnetic wave in the vacuum:

$$\mathbf{E}(r,\theta,\phi;t) \;=\; \mathrm{Re}\left(\frac{A}{r}\left(1+\frac{i}{kr}\right)\exp\bigl(ikr - i\omega t\bigr)\right) * \sin(\theta)\hat{\boldsymbol{\phi}} \tag{9}$$

for $k = \omega/c$. This wave is an example of a *divergent spherical wave*: its wave fronts are spheres whose radii grow with time as $r = ct + \text{const}$.

(a) Verify that the electric field (9) obeys the wave equation. Mathematical hint:

$$\begin{aligned} \text{for } \mathbf{F}(r,\theta,\phi) \;&=\; f(r) * \sin(\theta)\hat{\boldsymbol{\phi}} \\ \nabla^2\mathbf{F}(r,\theta,\phi) \;&=\; \left(f''(r) \,+\, \frac{2}{r}f'(r) \,-\, \frac{2}{r^2}f(r)\right) * \sin(\theta)\hat{\boldsymbol{\phi}}. \end{aligned} \tag{10}$$

(b) Show that the electric field (9) and the magnetic field

$$\mathbf{B}(r,\theta,\phi;t) \;=\; \mathrm{Re}\left(-\frac{A}{cr}\exp\bigl(ikr - i\omega t\bigr)\left[\sin\theta\,\hat{\boldsymbol{\theta}} \,+\, \frac{i}{rk}\left(1+\frac{i}{kr}\right)\bigl(\sin\theta\,\hat{\boldsymbol{\theta}} \,+\, 2\cos\theta\,\mathbf{n}\bigr)\right]\right) \tag{11}$$

obey all 4 Maxwell equations.

(c) Calculate the time-averaged pointing vector $\langle\mathbf{S}\rangle$ of this wave. Show that it points in the radial direction $\mathbf{n}$ and that its magnitude scales with the radius as $1/r^2$.

(d) Integrate $\langle\mathbf{S}\rangle\cdot\mathbf{d^2a}$ over a spherical surface of some radius R to find the net power carried by the spherical wave. Note: by energy conservation, this power should be the same for any radius R.