1. Let's start with a few basic questions about attenuating EM waves and skin effect.

   (a) At 20° C temperature, pure silicon has conductivity $\sigma \approx 4.3 \cdot 10^{-4}$ ℧/m and dielectric constant $\epsilon \approx 11.7$. If you plant an electric charge in the middle of a silicon crystal, how long would it take for that charge to drift to the crystal's boundary?

   (b) Gold is a good electric conductor — $\sigma = 4.1 \cdot 10^{7}$ ℧/m — and it does not oxidize when exposed to air, but its quite expensive. The high-frequency electronics equipment often uses wires made of a cheap metal but covered with a thin layer of gold.
   If you were designing an apparatus working with signals of 10 GHz frequency, how thick a gold coating would you need for your wires?

2. The water in lake Austin has electric conductivity about $\sigma \approx 0.05$ ℧/m and dielectric constant $\epsilon \approx 80$.

   (a) Check that this water acts as a good conductor with $\sigma \gg \omega\epsilon\epsilon_0$ — at the frequency $\omega_1 = 2\pi \times 1$ MZh, but as a poor conductor with $\sigma \ll \omega\epsilon\epsilon_0$ at the higher frequency $\omega_2 = 2\pi \times 100$ MHz.

   (b) Calculate the attenuation rates of the radio waves in the water of Lake Austin at the 1 MHz and the 100 MHz frequencies.

3. Consider an attenuating EM wave in a good conductor. Show that the magnetic field lags behind the electric field by the 45° phase and has higher time-averaged energy than the electric field. Specifically,

$$\frac{\langle u\rangle_{\text{mag}}}{\langle u\rangle_{\text{el}}} = \frac{|\epsilon_{\text{eff}}|}{\text{Re}(\epsilon_{\text{eff}})} \xrightarrow[\text{for a good conductor}]{} \frac{\sigma}{\omega\epsilon\epsilon_0} \gg 1. \tag{1}$$

4. At the near-infrared and visible-light frequencies, the metals are less conductive than at the low frequencies, and consequently they also have lower-than-expected reflectivities to the IR and visible EM waves. For example, at the orange-color frequency 500 THz (vacuum wavelength $\lambda = 600$ nm), polished copper has head-on reflectivity $R = 85\%$.
   Infer the conductivity σ_{HF} of copper at this high frequency from its reflectivity, and then compare it to the copper's DC conductivity $\sigma_{\text{DC}} = 5.96 \cdot 10^7$ ℧/m.

5. Consider the gravity waves on the water's surface. In shallow water, the waves are not dispersive: they travel with a fixed speed $v = \sqrt{gh}$ (where h is the water's depth and g is the gravitational field). But in deep water, the waves cannot feel the bottom, so they behave as if the depth were only $\lambda/2\pi$. In other words, the waves in deep water have *dispersion relation*

$$\omega(k) = \sqrt{gk}. \tag{2}$$

   (a) Show that for such waves the phase velocity is *twice* the group velocity.

   (b) Calculate the dispersion of the waves in deep water and hence the maximal signal rate you can sent using such waves to a destination at distance L.

6. In a related problem, consider wave functions in quantum mechanics. A free particle traveling in x direction has wave function

$$\Psi(x,t) = A \exp\left(\frac{i}{\hbar}(px - Et)\right) \tag{3}$$

   where p is the particle's momentum and $E = p^2/2m$ is its kinetic energy.

   (a) Find the phase velocity and the group velocity of this wave. In particular, show that this time its the group velocity is twice the phase velocity.

   (b) Which velocity – if any – is the physical velocity of the moving particle?

7. Consider the dielectric constant $\epsilon(\omega)$ — and hence the refraction index $n(\omega)$ — of a gas. At frequencies ω that are not too close to any of the resonance frequencies ω_i of the gas's molecules, the $\epsilon(\omega)$ is approximately real and obtains as

$$\epsilon(\omega) \;=\; n^2(\omega) \;=\; 1 \;+\; \frac{Ne^2}{\epsilon_0 m_e} \sum_i \frac{f_i}{\omega_i^2 - \omega^2} \tag{4}$$

where N is the gas's density, and each $f_i > 0$ is the relative strength of the resonance at frequency ω_i.

Use this formula to show that

$$\frac{dn}{d\omega} > 0 \quad \langle\!\langle \text{ normal dispersion } \rangle\!\rangle \tag{5}$$

$$\text{and} \quad n^2 \;+\; \frac{\omega}{2}\frac{dn^2}{d\omega} \;>\; 1, \tag{6}$$

hence for any frequency at which we may use eq. (4), the phase velocity and the group velocity of the EM waves in the gas obey

$$v_{\text{group}}(\omega) \;<\; v_{\text{phase}}(\omega) \quad \text{and} \quad v_{\text{group}}(\omega) \times v_{\text{phase}}(\omega) \;<\; c^2 \tag{7}$$

and therefore

$$v_{\text{group}}(\omega) \;<\; c. \tag{8}$$

8. [15 points] Consider a toy model of a hydrogen atom: The electron forms a uniformly charged ball of radius a, with the proton at its center. The electron may move around as a rigid ball so its center vibrates around the proton. This makes for a single-resonance model of the atom.

   (a) Find the resonant frequency ω_0 of this model.

   (b) Get the numeric value of this frequency for $a = 0.5$ Å. In what part of the electromagnetic spectrum does this frequency lie?

At low frequencies, eq. (4) for the refraction index of a gas becomes the Cauchy formula

$$n(\lambda) \;\approx\; 1 \;+\; A\left(1 + \frac{B}{\lambda^2}\right) \tag{9}$$

where $\lambda = 2\pi c/\omega$ is the vacuum wavelength for the frequency ω, while A and B are parameters of the gas; A is called the *coefficient of refraction* and B the *coefficient of dispersion*. The Cauchy formula is derived at the very end of the textbook §9.4.3.

(c) Calculate the coefficients A and B for the gas made of the single-resonance model atoms. Assume standard conditions (temperature $0°$ C and pressure 1 atm) to get the density of the gas. Compare your results to the measured values for the hydrogen gas at standard conditions: $A_H = 1.36 \cdot 10^{-4}$, $B_H = 7.7 \cdot 10^{-15}\ \mathrm{m}^2$.