PHY–352 L. Problem set #5 Due February 17, 2026 (Tuesday).

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1. [12 points] Consider a vacuum-filled rectangular 5.00 cm $\times$ 2.00 cm waveguide.

   (a) List all TE and TM modes which can propagate through this waveguide at frequency $f = 10.0$ GHz.

   (b) Find the range of frequencies at which precisely one mode can propagate through the waveguide.

   (c) What is the group velocity of this mode at $f = 5$ GHz?

   (d) The wave guide is 10 m long. What is the maximal rate of 5 GHz pulses that can travel through this waveguide without merging due to dispersion?

2. Consider the $\mathrm{TE}_{1,0}$ mode of a rectangular $a \times b$ vacuum-filled waveguide. The electric and the magnetic fields of this mode are

$$\begin{aligned}
H_z(x,y,z,t) &= H_0 \times \cos\frac{\pi x}{a} \times \exp(ikz - i\omega t),\\
H_x(x,y,z,t) &= -\frac{ika}{\pi} H_0 \times \sin\frac{\pi x}{a} \times \exp(ikz - i\omega t),\\
E_y(x,y,z,t) &= +\frac{i\omega\mu_0 a}{\pi} H_0 \times \sin\frac{\pi x}{a} \times \exp(ikz - i\omega t),\\
H_y \equiv E_x &\equiv E_z \equiv 0.
\end{aligned} \tag{1}$$

   (a) Calculate the time-averaged Poynting vector and the time-averaged energy density of this wave. Then integrate both over the waveguide's cross-section to get the EM wave power and the energy per unit of length.

   (b) Show that the velocity of the EM energy in the waveguide is equal to the group velocity of the $\mathrm{TE}_{1,0}$ mode.

   ⋆ Actually, for any TE or TM mode of a waveguide, the energy velocity is equal to the group velocity. But proving *that* is beyond the scope of this homework.

3. Consider a coaxial cable used as a waveguide: It has an inner wire of (outer) radius a, the outer wire of (inner) radius b, and the space between the wires is filled with a dielectric of relative permittivity ϵ. The TEM mode of a wave down this waveguide has fields

$$\mathbf{E}(s,\phi,z,t) \;=\; A\exp(ikz-i\omega t)\,\frac{\hat{\mathbf{s}}}{s}\,, \qquad \mathbf{B}(s,\phi,z,t) \;=\; \frac{A\sqrt{\epsilon}}{c}\exp(ikz-i\omega t)\,\frac{\hat{\boldsymbol{\phi}}}{s}\,, \tag{2}$$

for $k=\sqrt{\epsilon}\omega/c$ and some constant A.

(a) Verify that the EM fields (2) obey all 4 Maxwell equations.

(b) Find the voltage between the wires and the currents in each wire as functions of z and t. Compare their amplitudes and find the circuit impedance Z of the coaxial cable.

Note: the wave impedance of the coaxial waveguide and its circuit impedance as a cable are different quantities. In this question, we ask about the circuit impedance $Z=\hat{V}/\hat{I}$ of the cable.

(c) The inner wire of the cable has radius $a=1.00$ mm, and the polyethylene dielectric separating the wires has $\epsilon=2.25$. What should be the inner radius b of the outer wire so that the cable would have standard impedance $Z=75\ \Omega$?

4. Suppose

$$V(x,t)\equiv\;0, \qquad \mathbf{A}(x,t) \;=\; -\frac{N}{4c}\,f(ct-|x|)\hat{\mathbf{y}} \tag{3}$$

for some constant N and

$$f(ct-|x|) \;=\; \begin{cases} (ct-|x|)^4 & \text{for } ct>|x|,\\ 0 & \text{for } ct<|x|. \end{cases} \tag{4}$$

(a) Find the electric and the magnetic fields, $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$.

(b) Verify that these fields obey all 4 Maxwell equations for appropriate charge and current densities ρ and $\mathbf{J}$, and write down these $\rho(x,t)$ and $\mathbf{J}(x,t)$.

Hint: $|x|$ has a singular second derivative, so watch out for singular terms in the derivatives of the $\mathbf{E}$ and $\mathbf{B}$.

5. Now let

$$V(\mathbf{r},t) \;\equiv\; 0, \qquad \mathbf{A}(\mathbf{r},t) \;=\; -\frac{Qt}{4\pi\epsilon_0}\,\frac{\mathbf{n}}{r^2}\,. \tag{5}$$

   (a) Show that for these potentials, the magnetic field $\mathbf{B}$ is entirely absent while the electric field $\mathbf{E}$ is the *static* field of the point charge Q at the origin.

   (b) Find the gauge transform $\Lambda(\mathbf{r},t)$ that would bring the potentials (5) to $V' = V_{\text{Coulomb}}(r)$ and $\mathbf{A}' \equiv 0$.

6. [8 points] This time, let

$$V(x,y,z,t) \;\equiv\; 0, \qquad \mathbf{A}(x,y,z,t) \;=\; A_0\cos(kx-\omega t)\hat{\mathbf{y}} \tag{6}$$

   for some constant parameters A_0, k, and ω.

   Show that the EM fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ for these potentials describe a plane wave, and write down the conditions on the parameters A_0, k, and ω needed for this wave to obey Maxwell equations for the vacuum.

7. [8 points] Which of the potentials (3), (5), (6) (if any) is in the Coulomb gauge and which (if any) is in the Landau gauge? Note: these are non mutually exclusive, some potentials may obey both gauge conditions.

8. [15 points] Consider a time-dependent point charge $Q(t)$ fed by a radially symmetric current,

$$\rho(\mathbf{r},t) \;=\; Q(t)\delta^{(3)}(\mathbf{r}), \qquad \mathbf{J}(\mathbf{r},t) \;=\; -\frac{\dot{Q}(t)}{4\pi r^2}\,\mathbf{n} \quad \langle\!\langle\, \text{where } \dot{Q} = dQ/dt \,\rangle\!\rangle. \tag{7}$$

   (a) Verify that these charge and current density obey the continuity equation.

   (b) Find the scalar and the vector potentials in the Coulomb gauge by solving equations

$$\nabla^2 V \;=\; -\frac{1}{\epsilon_0}\rho, \tag{8}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} \;-\; \nabla^2\mathbf{A} \;=\; \mu_0\mathbf{J} \;-\; \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right). \tag{9}$$

   (c) Find the electric and the magnetic fields and check that they obey the Maxwell equations.

9. A thin but infinitely long wire is stretched along the z axis. At $t = 0$ there is a sudden but very short burst of current in the wire,

$$I(t) \;=\; N\delta(t) \qquad \text{for some constant } N. \tag{10}$$

(a) Find the potentials $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ is the Landau gauge.

(b) Find the electric and the magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$.