PHY–352 L.   Problem set #6   Due February 26, 2026 (Thursday).

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0. First, read the solutions to the homework set#5 that you did not do. Pay particular attention to the problems 4 through 9 on the subject of EM potentials, gauge transforms, and causal solutions of the wave equation.

1. In class we have learned that in the Landau gauge, the potentials of continuous charges and currents obtain as

$$V(\mathbf{r},t) \;=\; \frac{1}{4\pi\epsilon_0}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\rho(\mathbf{r}',t_{\mathrm{ret}}), \qquad \mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}). \tag{1}$$

Your task in this problem is to verify that these potentials indeed obey the Landau gauge condition.

(a) First, show that

$$\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;+\; \nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;=\; \frac{(\nabla\cdot\mathbf{J})@(\text{fixed}\,t_{\mathrm{ret}})}{\mathcal{R}}\,. \tag{2}$$

(b) Now use eq. (2) and the continuity equation to verify the Landau gauge condition for the potentials (1).

2. A circular ring of radius a has a non-uniform electric charge density

$$\frac{dQ}{d\ell} \;=\; \lambda_0\,\left|\sin(\phi/2)\right|. \tag{3}$$

The ring — with the electric charges stuck to it — rotates around its axis with a constant angular frequency ω.

Find the exact scalar and vector potentials at the center of the ring.

3. Suppose the current density $\mathbf{J}$ depends on time but slowly enough that we may approximate its change over a short time interval δt by the first two terms in the Taylor expansion,

$$\mathbf{J}(\mathbf{r}, t+\delta t) \;\approx\; \mathbf{J}(\mathbf{r}, t) \;+\; \delta t \dot{\mathbf{J}}(\mathbf{r}, t). \tag{4}$$

In particular,

$$\mathbf{J}(\mathbf{r}', t_{\rm ret}) \;\approx\; \mathbf{J}(\mathbf{r}', t) \;-\; (t - t_{\rm ret})\dot{\mathbf{J}}(\mathbf{r}', t). \tag{5}$$

Show that in this approximation, the Jefimenko equation for the magnetic field becomes the quasi-static Biot–Savart–Laplace field

$$\mathbf{B}(\mathbf{r}, t) \;\approx\; \frac{\mu_0}{4\pi} \iiint d^3\mathrm{Vol}' \, \frac{\mathbf{J}(\mathbf{r}', t) \times \mathbf{n}}{|\mathbf{r} - \mathbf{r}'|^3} \tag{6}$$

where the RHS is evaluated at the observer time t rather than the retarded time $t_{\rm ret}$.

This means that the quasi-static approximation is actually much better than we had any right to expect: the two errors involved (neglecting retardation and dropping the second term in the Jefimenko equation) cancel one another to the first order.

4. [20 points] An ideal electric dipole is stuck at the coordinate origin, but its dipole moment $\mathbf{p}(t)$ charges with time. The singular charge and current densities for this dipole are

$$\rho(\mathbf{r}, t) \;=\; -\mathbf{p}(t) \cdot \nabla \delta^{(3)}(\mathbf{r}), \qquad \mathbf{J}(\mathbf{r}, t) \;=\; \dot{\mathbf{p}}(t) \delta^{(3)}(\mathbf{r}). \tag{7}$$

Show that for this dipole

$$V(\mathbf{r}, t) \;=\; \frac{1}{4\pi\epsilon_0} \left[\frac{\mathbf{n} \cdot \mathbf{p}}{r^2} + \frac{\mathbf{n} \cdot \dot{\mathbf{p}}}{cr} \right]_{\rm ret}, \tag{8}$$

$$\mathbf{A}(\mathbf{r}, t) \;=\; \frac{\mu_0}{4\pi} \left[\frac{\dot{\mathbf{p}}}{r} \right]_{\rm ret}, \tag{9}$$

$$\mathbf{E}(\mathbf{r}, t) \;=\; -\frac{1}{4\pi\epsilon_0} \left[\frac{\mathbf{p} - 3(\mathbf{n} \cdot \mathbf{p})\mathbf{n}}{r^3} + \frac{\dot{\mathbf{p}} - 3(\mathbf{n} \cdot \dot{\mathbf{p}})\mathbf{n}}{cr^2} + \frac{\ddot{\mathbf{p}} - (\mathbf{n} \cdot \ddot{\mathbf{p}})\mathbf{n}}{c^2 r} \right]_{\rm ret}, \tag{10}$$

$$\mathbf{B}(\mathbf{r},t) \;=\; -\frac{\mu_0}{4\pi}\left[\frac{\mathbf{n}\times\dot{\mathbf{p}}}{r^2}+\frac{\mathbf{n}\times\ddot{\mathbf{p}}}{cr}\right]_{\rm ret}. \tag{11}$$

For simplicity, ignore the contact terms — *i.e.*, δ-like singularities at $\mathbf{r}=\mathbf{0}$ — in the potentials V and $\mathbf{A}$ and in the fields $\mathbf{E}$ and $\mathbf{B}$.

5. An expanding ball of radius $R=vt$ (constant v for $t>0$) maintains a uniform charge density $\rho(t)=Q/(4\pi R^3(t)/3)$ and a constant net charge Q. The observer located at the center of this ball sees the affective charge

$$Q_{\rm eff}(t) \;=\; \iiint \rho(\mathbf{r}',t_{\rm ret})\,d^3{\rm Vol}'. \tag{12}$$

Calculate this effective charge for $t>0$. (It is actually time-independent as long as $t>0$.) Also, show that for a slow expansion speed $v\ll c$, $Q_{\rm eff}\approx Q(1-\frac{3v}{4c})$, while for a barely-slower-than-light expansion $Q_{\rm eff}\to Q(3\ln(2)-\frac{3}{2}\approx 0.58)$.

6. A uniformly charged rod of length L and net charge Q slides along the x axis at constant speed v. As a function of the *observer time* t, the back end of the rod is at $x_1=vt$ while the frond end is at $x_2=vt+L$.

   (a) For an observer located at the coordinate origin at time $t>(L/c)$, find the retarded times at the two end of the rod and their locations at these retarded times.

   (b) Calculate the retarded potentials $V(0,t)$ and $\mathbf{A}(0,t)$ at the coordinate origin).

   (c) Take the limit of a short rod, $L\ll vt$, but don't assume $v\ll c$. Check that in that limit, the potentials you got in part (b) become the Liénard–Wiechert potentials of a point charge Q moving at constant velocity.

7. Verify that the Liénard–Wiechert potentials V and $\mathbf{A}$ of a point charge moving at constant velocity $\mathbf{v}$ obey the Landau gauge condition.